Distributional integrals

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Abstract: One way of defining the integral is by properties of its primitive. For Lebesgue integrals the primitives are absolutely continuous functions. If we take the primitives to be continuous functions and use distributional derivatives we get an integral that includes those of Lebesgue and Henstock-Kurzweil but has a very simple definition. No measure theory is required to define this integral.

Preprint: The distributional Denjoy integral, to appear in Real Analysis Exchange
Integrals defined by their primitives

$f: \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue integrable if there is an absolutely continuous function $F: \mathbb{R} \rightarrow \mathbb{R}$ such that

1. $\lim_{x \rightarrow \infty} F(x), \lim_{x \rightarrow -\infty} F(x)$ exist in $\mathbb{R}$
2. $F' = f$ almost everywhere.

Then $\int_{a}^{b} f = F(b) - F(a)$ \quad $-\infty \leq a < b \leq \infty$.

Absolutely continuous (AC):

$(\forall \epsilon > 0)(\exists \delta > 0)(\forall (x_i, y_i) \subset \mathbb{R})$

$$\sum |x_i - y_i| < \delta \Rightarrow \sum |F(x_i) - F(y_i)| < \epsilon$$

For Henstock–Kurzweil integrals the primitives are $ACG^*$

$$C^1 \subsetneq AC \subsetneq ACG^* \subsetneq C^0$$

Example

$f(x) = \sin(x^2) \quad F(x) = \int_{-\infty}^{x} f$

$f \notin L^1 \quad F \in ACG^* \setminus L^1$
Test functions
\( \mathcal{D} = C_c^\infty(\mathbb{R}) = \) smooth functions of compact support

Convergence \( \phi_n \to 0 \)
There is compact \( K \) such that \( \text{supp}(\phi_n) \subset K \).
For each \( m \geq 0 \), \( \phi_n^{(m)} \to 0 \) uniformly on \( K \) as \( n \to \infty \).

Schwartz distributions
\( \mathcal{D}' = \) dual space of \( \mathcal{D} \)
If \( T \in \mathcal{D}' \) then \( T: \mathcal{D} \to \mathbb{R} \quad \langle T, \phi \rangle \in \mathbb{R} \) for \( \phi \in \mathcal{D} \)
continuous linear functionals on \( \mathcal{D} \)
\( (\forall \phi, \psi \in \mathcal{D}) (\forall a \in \mathbb{R}) \langle T, a\phi + \psi \rangle = a\langle T, \phi \rangle + \langle T, \psi \rangle \)
\( \phi_n \to 0 \) in \( \mathcal{D} \) \( \Rightarrow \langle T, \phi_n \rangle \to 0 \) in \( \mathbb{R} \)

Examples
If \( f \in L_{loc}^1 \) then \( \langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f \phi \) defines a distribution.

Dirac delta \( \langle \delta, \phi \rangle = \phi(0) \)

Distributional Derivative
\( \langle T', \phi \rangle = -\langle T, \phi' \rangle \)
Primitives
\[ \mathcal{B} = \{ F: \overline{\mathbb{R}} \to \mathbb{R} \mid F \in C^0(\overline{\mathbb{R}}), F(-\infty) = 0 \} \]
\[ \mathbb{R} = [\infty, \infty] \]

Distributional integral
\[ \mathcal{A} = \{ f \in \mathcal{D}' \mid f = F' \text{ for some } F \in \mathcal{B} \} \]

\[ \int_a^b f = F(b) - F(a) \quad -\infty \leq a < b \leq \infty \]

\[ \int_{-\infty}^{\infty} f \phi = \langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = - \int_{-\infty}^{\infty} F \phi' \]

\( \mathcal{B} \) is a Banach space under uniform norm \( \| \cdot \|_\infty \)
\( \mathcal{A} \) is a Banach space under Alexiewicz norm

\[ \| f \| = \| F \|_\infty = \sup_{x \in \mathbb{R}} |F(x)| = \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x f \right| \]

Linear isometry
\[ \mathcal{A} \leftrightarrow \mathcal{B} \quad f \leftrightarrow F \]
Examples

If \( \int_{-\infty}^{\infty} f \) exists as a Lebesgue or Henstock–Kurzweil integral then \( f \in \mathcal{A} \) and \( \int_{a}^{b} f = F(b) - F(a) \) since \( F \) is continuous.

Singular function:

\( F \) is continuous, \( F'(x) = 0 \) a.e., then \( \int_{a}^{b} F' = F(b) - F(a) \). And, \( (L) \int_{a}^{b} F'(x) \, dx = 0 \).

Weierstrass function:

\( f \in C^0 \), \( f'(x) \) exists for no \( x \in \mathbb{R} \)

\( \int_{a}^{b} f' = f(b) - f(a) \)

Note that \( \langle f', \phi \rangle = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} f \phi' \).
Regulated functions

$F: \mathbb{R} \to \mathbb{R}$ is regulated if

$$F(a-) := \lim_{x \to a-} F(x), \quad F(a+) := \lim_{x \to a+} F(x)$$

exist for all $a \in \mathbb{R}$ and

$$F(-\infty) := \lim_{x \to -\infty} F(x), \quad F(\infty) := \lim_{x \to \infty} F(x) \text{ exist.}$$

Left continuous $F(a-) = F(a)$
Right continuous $F(a+) = F(a)$

$\mathcal{B} := \{ F: \mathbb{R} \to \mathbb{R} | F \text{ is left continuous or right continuous at each } x \in \mathbb{R}; F(-\infty) = 0 \}$
Regulated primitive integrals

\[ A := \left\{ f \in D' \mid f = F' \text{ for some } F \in B \right\} \]

For all \( \phi \in D \)

\[ \langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x) \, dx \]

If \( f \in A, \ F \in B \) such that \( F' = f \) then for \(-\infty < a < b < \infty\)

\[ \int_{(a,b)} f = F(b-) - F(a+) \quad \int_{(a,b]} f = F(b+) - F(a+) \]

\[ \int_{[a,b]} f = F(b+) - F(a-) \quad \int_{[a,b)} f = F(b-) - F(a-) \]

If \( F \) is continuous then these four integrals agree.

Also,

\[ \int_{-\infty}^{\infty} f = F(\infty) - F(-\infty) = F(\infty) \]
Radon measure: finite Borel measure that is

- **inner regular:** \( \mu(E) = \sup_{K \subset E} \mu(K) \)

- **outer regular:** \( \mu(E) = \inf_{G \supseteq E} \mu(G) \)

If \( F(x) := \int_{-\infty}^{x} d\mu \) then \( F \in BV \subset B \).
If \( g \in BV \) then

\[
\sup_{\phi \in D} \int_{-\infty}^{\infty} g(x)\phi'(x) \, dx < \infty \\
\|\phi\|_{\infty} \leq 1
\]

and there is a Radon measure \( \mu_g \) such that \( g' = \mu_g \), i.e.,

\[
\langle g', \phi \rangle = -\langle g, \phi' \rangle = \langle \mu_g, \phi \rangle = \int_{-\infty}^{\infty} \phi \, d\mu_g
\]