Distributional integrals on the real line

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E.T., *The distributional Denjoy integral*

Plan of the talk:

1. Need for a new integral

2. Distributions

3. Distributional Denjoy integral

   (a) definition

   (b) examples

   (c) Alexiewicz norm

   (d) integration by parts

   (e) change of variables

   (f) Banach lattice

Need for a new integral

Denjoy integral (1912)
Let $f : \mathbb{R} \to \mathbb{R}$. Then f is integrable in the Denjoy sense if there is an $ACG*$ function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ a.e. Then $\int_a^b f = F(b) - F(a)$.

F **is** $AC*$ **on** $E \subset \mathbb{R}$
$(\forall \epsilon > 0)(\exists \delta > 0)$ (if $x_i, y_i \in E$ with $\sum |x_i - y_i| < \delta$ then $\sum_i \sup_{(x,y)\subseteq(x_i,y_i)} |F(x) - F(y)| < \epsilon$)

F **is** $ACG*$ **on** $\mathbb{R}$
$\mathbb{R} = \cup E_i$ and F is $AC*$ on each E_i

Distributions

Test functions

$\mathcal{D} = C_c^\infty$

Convergence $\quad \phi_n \to 0$

There is compact K such that supp$(\phi_n) \subset K$.

For each $m \geq 0$, $\phi_n^{(m)} \to 0$ uniformly on K as $n \to \infty$.

Schwartz distributions

$\mathcal{D}' =$ dual space of $\mathcal{D}$

If $T \in \mathcal{D}'$ then $T\colon \mathcal{D} \to \mathbb{R} \quad \langle T, \phi \rangle \in \mathbb{R}$

continuous linear functionals on $\mathcal{D}$

$(\forall \phi, \psi \in \mathcal{D})(\forall a \in \mathbb{R})\ \langle T, a\phi + \psi \rangle = a\langle T, \phi \rangle + \langle T, \psi \rangle$

$\phi_n \to 0$ in $\mathcal{D} \Rightarrow \langle T, \phi_n \rangle \to 0$ in $\mathbb{R}$

If $f \in L^p$ $(1 \leq p \leq \infty)$ then $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f\phi$ defines a distribution

Derivative

$\langle T', \phi \rangle = -\langle T, \phi' \rangle$

Distributional Denjoy integral

Continuous functions on extended real line
$\tilde{C} = \{F \in C^0(\mathbb{R}) \mid \lim_{-\infty} F = 0, \lim_{\infty} F \in \mathbb{R}\}$
If $F \in \tilde{C}$ define $F(\pm\infty) = \lim_{\pm} F$

If $f \in \mathcal{D}'$ then f is integrable
if there is $F \in \tilde{C}$ such that $F' = f$. Then
$\int_{-\infty}^{\infty} f = F(\infty)$.

Note: $F' = f$ means that for all $\phi \in \mathcal{D}$

$$\langle f, \phi\rangle = \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_{-\infty}^{\infty} F\phi'$$

Since $AC \subset ACG* \subset ACG \subset C^0(\overline{\mathbb{R}})$ this
includes Lebesgue and Denjoy integrals.

References

P. Mikusiński and K. Ostaszewski,
"RAE" (1988-89)
D.D. Ang, K. Schmitt, L.K. Vy,
"Bulletin of Belgian Math. Soc." (1997)

Linear distributional derivative is linear

Unique $T' = 0$ in $\mathcal{D}' \Rightarrow T =$ constant

Subintervals $\int_a^b f = F(b) - F(a)$

$\mathcal{A}$ = integrable distributions

Examples

1. Let $F \in C^0([0,1])$ be differentiable nowhere.
Then $F' \in \mathcal{A}$ and $\int_0^1 F' = F(1) - F(0)$

2. Let F be a singular function.
Then $F' \in L^1$ and $(L)\int_0^1 F' = 0$.
And, $F' \in \mathcal{A}$ with $\int_0^1 F' = F(1) - F(0)$.

Banach space under Alexiewicz norm

If $f \in \mathcal{A}$ and F is its primmitive in $\widetilde{C}$ then

$$\|f\| = \|F\|_\infty = \sup_{x \in \mathbb{R}} |F(x)| = \max_{x \in \overline{\mathbb{R}}} \left| \int_{-\infty}^x f \right|$$

Integration by parts

Let $f \in \mathcal{A}$ and $g \in \mathcal{BV}$. Then $fg \in \mathcal{A}$.

$$\int_{-\infty}^{x} fg = F(x)g(x) - \int_{-\infty}^{x} F\,dg$$

where $F' = f$ and $F \in \widetilde{C}$

Hölder inequality

For $f \in \mathcal{A}$ and $g \in \mathcal{BV}$

$$\left|\int_{-\infty}^{\infty} fg\right| \leq \|f\| \left[\inf |g| + Vg\right]$$

Change of variables

Derivative of composition of continuous functions

Let $F, g \in C^0(\overline{\mathbb{R}})$. Then $(F' \circ g)g' := (F \circ g)'$, i.e., for all $\phi \in \mathcal{D}$,

$$\begin{aligned} \langle (F' \circ g)g', \phi \rangle &= \langle (F \circ g)', \phi \rangle = -\langle F \circ g, \phi' \rangle \\ &= -\int_{-\infty}^{\infty} (F \circ g)(t)\, \phi'(t)\, dt. \end{aligned}$$

Change of variables

Suppose $f \in \mathcal{A}$ and $F' = f$ where $F \in C^0(\overline{\mathbb{R}})$. Let $-\infty \leq a < b \leq \infty$. If $g \in C^0([a,b])$ then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g)g' = (F \circ g)(b) - (F \circ g)(a).$$

If $g \in C^0((a,b))$ and $\lim_{t \to a^+} g(t) = -\infty$ and $\lim_{t \to b^-} g(t) = \infty$ then

$$\int_{-\infty}^{\infty} f = \int_a^b (f \circ g)g' = F(\infty) - F(-\infty).$$

Banach lattice

partial order in $\widetilde{C}$
$F \le G$ if $F(x) \le G(x)$ for all $x \in \overline{\mathbb{R}}$

Reflexive $\quad F \le F$
Antisymmetric $\quad (F \le G$ and $G \le F) \Rightarrow F = G$
Transitive $\quad (F \le G$ and $G \le H) \Rightarrow F \le H$

This is a Banach lattice.
$\widetilde{C}$ is closed under
$F \vee G = \sup(F, G)$ and $F \wedge G = \inf(F, G)$
1. $F \le G \Rightarrow F + H \le G + H$ for all $H \in \widetilde{C}$
2. $F \le G \Rightarrow aF \le aG$ for all $a \ge 0$ $(a \in \mathbb{R})$
3. $|F| \le |G| \Rightarrow \|F\|_\infty \le \|G\|_\infty$

Here, $|F| = F \vee -F$.

For $f, g \in \mathcal{A}$ define $f \le g$ if $F \le G$, i.e.,

$$f \le g \Leftrightarrow \int_{-\infty}^{x} f \le \int_{-\infty}^{x} g \text{ for all } x \in \overline{\mathbb{R}}$$

Example

$$f(t) = \begin{cases} \sin(t)/t, & t > 0 \\ 0, & t \le 0 \end{cases}$$

Then $f \geq 0$ in $\mathcal{A}$.