Covering Theorems and Integration

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Offprints
Peter A. Loeb and Erik Talvila
*Covering theorems and Lebesgue integration*

Peter A. Loeb and Erik Talvila
*Lusin’s theorem and Bochner integration*

Decorations by:
Plan of the talk:

1. Covering theorems
   (a) Compactness
   (b) Vitali Covering Theorem
   (c) Other covering theorems

2. Integration
   (a) Riemann approach to integration
   (b) Partial covers
   (c) Morse sets
   (d) Morse covering theorem

Moral: Covering theorems give us a way to define integrals. It is only necessary to find the measure of small enough “nice” sets centred on points of approximate continuity.
Compactness

$E \subset \mathbb{R}^d$ is compact:
Every open cover has a finite subcover.

This says nothing about valence, i.e.,
the number of sets covering a given point.
Vitali Covering Theorem

Vitali covering of $E \subset \mathbb{R}^d$

There is a set of closed intervals $\sigma$ that form a regular, fine cover of $E$: For each $x \in E$

$$(\exists \rho > 0) \ (\forall \epsilon > 0) \ (\exists J = I_1 \times I_2 \times \cdots \times I_d \in \sigma)$$

$$x \in J \subset \mathbb{R}^d$$

$$0 < \frac{\max_{1 \leq i \leq d} \lambda(I_i)}{\min_{1 \leq j \leq d} \lambda(I_j)} < \rho$$

$$\max_{1 \leq i \leq d} \lambda(I_i) < \epsilon$$

Each interval $I_i \subset \mathbb{R} \ (1 \leq i \leq d)$

Vitali Covering Theorem (1908)

If $\sigma$ is a Vitali covering of $E$ then there is a sequence $\{S_n\} \subset \sigma$ such that

$$S_n \cap S_m = \emptyset \ (m \neq n)$$

$$\lambda(E \setminus \cup S_n) = 0.$$
Lebesgue differentiation theorem

Radon measure

$\mu$ is a measure such that

- Borel sets are measurable
- Compact sets have finite measure
- $\mu$ is inner regular: $\mu(E) = \sup_{K \subseteq E} \mu(K)$
- $\mu$ is outer regular: $\mu(E) = \inf_{G \supseteq E} \mu(G)$

Lebesgue differentiation theorem

$f \in L^1(\mu)$ if and only if for $\mu$-almost all $x \in \mathbb{R}^d$

$$\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(x) - f(y)|d\mu(y) = 0$$
Besicovitch covering theorem

\( A \subset \mathbb{R}^d. \)

For each \( x \in A \) there is closed ball \( B(x, r(x)) \) with \( 0 < r(x) < M. \)

There is a constant \( N, \) depending only on \( d, \) such that there are \( N \) families \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N \) each \( \mathcal{B}_n \subset \{(B(x, r(x)) : x \in A\} \) is a countable set of disjoint balls. And,

\[
A \subset \bigcup_{n=1}^{N} \bigcup_{B \in \mathcal{B}_n} B.
\]

The valence of each point is at most \( N. \)

**Corollary:**

\( A \subset \mathbb{R}^d, \mu(A) < \infty \)

\( \sigma \) is a collection of closed balls that is a fine cover of \( A. \) Then there is a countable disjoint family of balls \( \{S_i\} \) such that

\[
\mu(A \setminus \cup S_i) = 0.
\]
Names associated with covering theorems

G. Vitali (1908) – cubes
H. Lebesgue (1910) – sets regular with respect to cubes
N. Wiener (1939) – dilations of balls
H. Whitney (1939) – cubes that get small near a boundary
A.S. Besicovitch (1945) – balls, useful for Radon measures
A.P. Morse (1947) – starlike sets in finite-dimensional spaces
A. Denjoy (1950) – metric spaces
H. Federer (1969) – Riemannian manifolds
M. de Guzmán (1975) – Vitali extensions
Riemann approach to integration

\( f: \mathbb{R} \to \mathbb{R} \)

Partition domain into intervals

\(-\infty = x_0 < x_1 < x_2 < \ldots < x_N = \infty\)

Choose tag \( z_n \in [x_{n-1}, x_n] \) for each \( 1 \leq n \leq N \)

Approximate \( f \) by \( f(x) = f(z_n) \) for \( x \in [x_{n-1}, x_n] \)

\[
\int_{-\infty}^{\infty} f \, d\lambda \approx \sum_{n=1}^{N} f(z_n) [x_n - x_{n-1}]
\]

Convention: \( f(\pm\infty) := 0 \quad 0 \cdot \infty := 0 \)

**Riemann integral:** uniform partition of \([a, b]\)
choose \( z_n \in [x_{n-1}, x_n] \) arbitrarily
gauge $\gamma : \mathbb{R} \rightarrow \{\text{open intervals in } \mathbb{R}\}$
$\gamma(x) =$ open interval containing $x$

Demand: $z_n \in [x_{n-1}, x_n] \subset \gamma(z_n)$

Gives an integral that includes the Lebesgue integral on $\mathbb{R}^d$.
Integrates all derivatives
\[
\int_a^b f' \, d\lambda = f(b) - f(a)
\]
$\int_a^b f \, d\lambda$ can exist even if $\int_a^b |f| \, d\lambda$ does not exist.
Partial covers

Lebesgue integral [Ma, Lee, Chew RAE (1992/93)]
\[ \int_a^b f \, d\lambda = A \text{ if} \]

\[(\forall \epsilon > 0) (\exists \gamma) (\exists \eta > 0)\]

for any \( \gamma \)-fine partial partition \( \mathcal{P} = \{ (z_n, I_n) \}_{n=1}^N \)

with \( \sum_{n=1}^N \lambda(I_n) > b - a - \eta \)

\[ \left| \sum_{n=1}^N f(z_n) \lambda(I_n) - A \right| < \epsilon. \]
Henstock/Kurzweil integral [Chew and Lee NZJM (1994)]

\[ \int_{a}^{b} f \, d\lambda \] exists if there is a finitely additive interval function \( F : \{ \text{intervals} \} \to \mathbb{R} \) such that

\[ (\forall \epsilon > 0) \ (\exists \gamma) \]

for any \( \gamma \)-fine partial partition \( \mathcal{P} = \{(z_n, I_n)\}_{n=1}^{N} \)

\[ \sum_{n=1}^{N} |f(z_n)\lambda(I_n) - F(I_n)| < \epsilon. \]

Errors are absolutely summable even if

\[ \int_{a}^{b} |f| \, d\lambda = \infty. \]
Integration using a covering theorem

\[ \Omega \subset \mathbb{R}^d \quad \mu \text{ a Radon measure} \quad \int_\Omega f \, d\mu \]

**Idea:** Find a countable sequence of disjoint sets \( \{S_n\} \) with \( z_n \in S_n \) and \( \mu(\Omega \setminus \cup S_n) = 0 \). 

- \( z_n \) are “nice” points 
- \( f(x) \approx f(z_n) \) on small enough “nice” set \( S_n \) 
- \( \int_{S_n} f \, d\mu \approx f(z_n) \mu(S_n) \) and \( \int_\Omega f \, d\mu \approx \sum f(z_n) \mu(S_n) \)

**Continuity** at \( z \):

\[
(\forall \epsilon > 0) \quad (\exists \delta > 0) \quad (x \in B(z, \delta) \Rightarrow |f(z) - f(x)| < \epsilon).
\]

\[
\int_{B(z,\delta)} |f(x) - f(z)| \, d\mu(x) \leq \epsilon \mu(B(z,\delta))
\]

**Approximate Continuity** at \( z \):

\[
(\forall \epsilon > 0) \quad (\forall \eta > 0) \quad (\exists R > 0) \quad (0 < r \leq R \Rightarrow \mu(E(z,r)) < \eta \mu(B(z,r)))
\]

where \( E(z,r) = \{x \in B(z, r) : |f(z) - f(x)| \geq \epsilon\} \).
Morse sets

Fix $\rho \geq 1$.

$S(a)$ is a Morse set if there is $r > 0$ such that $S(a)$ is

**nearly spherical**

$B(a, r) \subset S(a) \subset B(a, \rho r)$

**starlike**

For each $x \in S(a)$ and $y \in B(a, r)$ the line segment $\alpha y + (1 - \alpha)x$ ($0 \leq \alpha \leq 1$) is in $S(a)$ or, the convex hull of $\{x\} \cup B(a, r)$ is in $S(a)$. 
**Morse covering theorem** [A.P. Morse (1947), P. Loeb and E.T. (2001)]

$A \subset \mathbb{R}^d$

With each $a \in A$ associate Morse set $a \subset \hat{S}(a)$ such that

$$\sup_{a \in A} \text{diam}\{S(a)\} < M$$

There is $N$ depending only on $d$ such that there are $A_1, A_2, \ldots, A_N$ subsets of $A$, $A_i \cap A_j = \emptyset$,

$$A \subset \bigcup_{m=1}^{M} \bigcup_{a \in A_m} \hat{S}(a)$$

and for each $1 \leq i \leq M$, $S(a) \cap S(b) = \emptyset$ for all $a, b \in A_i$.

If $\mu^*(A) < \infty$, take $A_j$ that maximizes $\sum_{a \in A_j} \mu(\hat{S}(a))$.

There is a finite subset $\tilde{A} \subset A_j$ such that

$$\sum_{a \in \tilde{A}} \mu(\hat{S}(a)) \geq \frac{\mu^*(A)}{2M}.$$
Lemma on Morse covers

Suppose $S$ is a fine Morse cover of $\Omega \subset \mathbb{R}^d$.

There is a sequence $\{S_n\} \subset S$ with

$S_m \cap S_n = \emptyset \ (m \neq n)$

$\mu(\Omega \setminus \bigcup S_n) = 0$ and $\mu(\bigcup S_n \setminus \Omega) < \epsilon$

if 1. $S$ consists of closed sets
or 2. $(\forall E \in S) \mu(\Omega \cap (\overline{E} \setminus E)) = 0$
or 3. $S$ is scaled, i.e., for each $S(a) \in S$ and $0 < p \leq 1 \ \{a + px : x + a \in S(a)\} \in S$. 
**Theorem:**

Suppose $f : \Omega \to [0, \infty)$ and $\int_{\Omega} f \, d\mu = F \in \mathbb{R}$.

Let $\rho \geq 1$ and $\epsilon > 0$. Let $S$ be a $\rho$-regular Morse cover of $\Omega$.

There is gauge $\gamma$ such that, whenever $\{S_n\}$ is a sequence in $S$ with

$$S_n(x) \subset \gamma(x) \quad \text{and} \quad \mu(\Omega \setminus \bigcup S_n(x)) = 0$$

we have

$$\left| \sum_{n} f(x_n) \mu(S_n) - F \right| < \epsilon.$$

**Proof:**

$A := \{x \in \Omega : f \text{ not approximately continuous at } x\}$.

$\mu(A) = 0 \Rightarrow \int_{A} f \, d\mu = 0$

$A_n := \{x \in A : n - 1 \leq f(x) < n\}$

Take $G_n \supset A_n$ with $\mu(G_n) < \frac{\epsilon}{n2^n}$.

Take $\gamma$ such that $x \in A_n \Rightarrow \gamma(x) \subset G_n$.

$$\sum_{x \in A} f(x) \mu(S) < \sum_{n=1}^{\infty} \left[ n \left( \sum_{x_i \in A_n} \mu(S_i) \right) \right] \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}.$$
\[
\left| \int_{\Omega} f \, d\mu - \sum_n f(x_n) \mu(S_n) \right|
\]
\[
= \left| \sum_n \int_{S_n} [f(y) - f(x)] \, d\mu(y) \right|
\]
\[
\leq \sum_n \int_{S_n \setminus E_n} |f(y) - f(x)| \, d\mu(y)
\]
\[
+ \sum_n \int_{E_n} f(y) \, d\mu(y) + \sum_n f(x_n) \int_{E_n} \, d\mu(y)
\]
\[
\leq \sum_n \epsilon_n \mu(S_n) + \int_{\cup E_n} f \, d\mu + \sum_n f(x_n) \eta_n \mu(S_n).
\]

Take \( \epsilon_n = \frac{\epsilon 2^{-n}}{1 + \mu(S_n)} \)

\( \eta_n \) small enough so that \( \mu(\cup E_n) = \sum \mu(E_n) \leq \sum \eta_n \mu(S_n) \) is small and \( \int_{\eta_n \mu(S_n)} f \, d\mu < \epsilon 2^{-n} \)

\( \eta_n \leq \frac{\epsilon 2^{-n}}{[1+f(x_n)][1+\mu(S_n)]} \)
Theorem:
If there is a gauge \( \gamma \) so that whenever \( S_n(x) \subset \gamma(x) \) and \( \mu(\Omega \setminus \cup S_n) = 0 \) we have
\[
\left| \sum_{n=1}^{\infty} f(x_n) \mu(S_n) - F \right| < \epsilon
\]
then \( \int_{\Omega} f \, d\mu = F \).

Proof:
We can assume all tags \( x_n \) are points of approximate continuity.
For each \( m \geq 1 \) take gauge \( \gamma_m \) such that when \( S^m(x^m_n) \subset \gamma_m(x^m_n) \) and \( \mu(\Omega \setminus \cup S(x^m_n)) = 0 \) we have
\[
\left| \sum_{n=1}^{\infty} f(x^m_n) \mu(S(x^m_n)) - F \right| < \frac{1}{m}.
\]
Define
\[
f_m(x) = \begin{cases} 
\max(f(x^m_n) - \eta^m_n, 0), & x \in S^m_n \setminus E^m_n \\
0, & \text{else}.
\end{cases}
\]
Bad set: \( E^m_n = \{ x \in S^m_n : |f(x) - f(x^m_n)| \geq \epsilon^m_n \} \)
\( \mu(E^m_n) \leq \eta^m_n \mu(S^m_n) \)

Good set: \( S^m_n \setminus E^m_n \)
Fatou’s lemma:

\[
\int_{\Omega} f \, d\mu \leq \liminf_m \int_{\Omega} f_m \, d\mu \\
\leq \liminf_m \sum_n f(x_n^m) \mu(S_n^m) \\
\leq \liminf_m \left( F + \frac{1}{m} \right) \\
= F.
\]

\[f_m \leq f\] so

\[
\int_{\Omega} f \, d\mu \geq \int_{\Omega} f_m \, d\mu \\
\geq \sum_n [f(x_n^m) - \eta_n^m] \mu(S_n^m \setminus E_n^m) \\
= \sum_n f(x_n^m) \mu(S_n^m) - \sum_n f(x_n^m) \mu(E_n^m) \\
\quad - \sum_m \eta_n^m \mu(S_n^m \setminus E_n^m) \\
\geq \left( F - \frac{1}{m} \right) - \frac{1}{m} - \frac{1}{m}.
\]

Therefore, \(\int_{\Omega} f \, d\mu = F\).
Erdős numbers

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