The Morse covering theorem and integration

Erik Talvila
University College of the Fraser Valley
Abbotsford, British Columbia, Canada
This is 100km east of Vancouver and the Pacific Ocean.

joint work with

Peter Loeb
University of Illinois at Urbana-Champaign

Offprints
Peter A. Loeb and Erik Talvila
Covering theorems and Lebesgue integration

Peter A. Loeb and Erik Talvila
Lusin’s theorem and Bochner integration
Setting for the story

\( f : \mathbb{R}^d \to Y \)
\( Y = \text{Banach space with norm } \| \cdot \| \)

\( \mu \) is a Radon measure
\( \Omega \subset \mathbb{R}^d \) is a measureable set
\( f = 0 \) on \( \mathbb{R}^d \setminus \Omega \)

\( f \) is \( \mu \)-measurable:
there is a sequence \( \{ f_m \} \) of simple functions
such that \( \lim_{n \to \infty} \| f_n(x) - f(x) \| = 0 \) for \( \mu \)-almost all \( x \in \mathbb{R}^d \).

Bochner integral
\( f \) is Bochner integrable if there is a sequence
of simple functions \( \{ f_n \} \) such that

\[
\lim_{n \to \infty} \int_{\Omega} \| f_n(x) - f(x) \| d\mu(x) = 0.
\]

In this case,

\[
\int_{\Omega} f \, d\mu = \lim_{n \to \infty} \int_{\Omega} f_n \, d\mu.
\]
Moral of the story: Integration using a covering theorem

For $\Omega \subset \mathbb{R}^d$ evaluate $\int_{\Omega} f \, d\mu$

**Idea:** Find a countable sequence of disjoint sets $\{S_n\}$ with $z_n \in S_n$ and $\mu(\Omega \setminus \bigcup S_n) = 0$

$z_n$ are “nice” points

$f(x) \approx f(z_n)$ on small enough “nice” set $S_n$

$$\int_{S_n} f \, d\mu \approx f(z_n)\mu(S_n) \text{ and } \int_{\Omega} f \, d\mu \approx \sum f(z_n)\mu(S_n)$$

**Continuity** at $z$: $(\forall \epsilon > 0) \ (\exists \delta > 0)$

$$(x \in B(z, \delta) \Rightarrow \|f(z) - f(x)\| < \epsilon)$$.

Then

$$\int_{B(z, \delta)} \|f(x) - f(z)\| \, d\mu(x) \leq \epsilon \mu(B(z, \delta))$$
The Magician: Approximate Continuity

**Approximate Continuity** at $z$:

\[(\forall \epsilon > 0) (\forall \eta > 0) (\exists R > 0)\]

\[(0 < r \leq R \Rightarrow \mu(E(z, r)) < \eta \mu(B(z, r)))\]

and $E(z, r) = \{x \in B(z, r) : \|f(z) - f(x)\| \geq \epsilon\}$.

**closed ball**

$B(z, r) = \{x \in \mathbb{R}^d : |z - x| \leq r\}$
The Prince: Morse sets

Fix $\rho \geq 1$. $S(a)$ is a Morse set if there is $r > 0$ such that

$S(a)$ is nearly spherical ($\rho$-regular)

$B(a, r) \subset S(a) \subset B(a, \rho r)$

starlike

For each $x \in S(a)$ and $y \in B(a, r)$ the line segment $\alpha y + (1 - \alpha)x$ ($0 \leq \alpha \leq 1$) is in $S(a)$
or, the convex hull of $\{x\} \cup B(a, r)$ is in $S(a)$. 
The Old King: Morse covering theorem

**Theorem:** [A.P. Morse (1947), P. Loeb and E.T. (2001)]

$A \subset \mathbb{R}^d$

With each $a \in A$ associate Morse set $a \subset \bar{S}(a)$ such that

$$\sup_{a \in A} \text{diam}\{S(a)\} < M$$

There is $N$ depending only on $d$ such that there are $A_1, A_2, \ldots, A_N$ subsets of $A$, $A_i \cap A_j = \emptyset$,

$$A \subset \bigcup_{m=1}^{M} \bigcup_{a \in A_m} \bar{S}(a)$$

and for each $1 \leq i \leq M$, $S(a) \cap S(b) = \emptyset$ for all $a, b \in A_i$.

If $\mu^*(A) < \infty$, take $A_j$ that maximizes $\sum_{a \in A_j} \mu(\bar{S}(a))$.

There is a finite subset $\tilde{A} \subset A_j$ such that

$$\sum_{a \in \tilde{A}} \mu(\bar{S}(a)) \geq \frac{\mu^*(A)}{2M}.$$
The Ghost: Fine Morse covers

$S$ is a fine Morse cover of $\Omega \subset \mathbb{R}^d$ if for each $a \in \Omega$ there is sequence $\{S_n(a)\} \subset S$ with diameter$(S_n(a)) \to 0$.

Lemma on Morse covers
Suppose $S$ is a fine Morse cover of $\Omega \subset \mathbb{R}^d$.

There is a sequence $\{S_n\} \subset S$ with
$S_m \cap S_n = \emptyset \ (m \neq n)$
$\mu(\Omega \setminus \bigcup S_n) = 0$ and $\mu(\bigcup S_n \setminus \Omega) < \epsilon$

if 1. $S$ consists of closed sets
or 2. $(\forall E \in S) \mu(\Omega \cap (\overline{E} \setminus E')) = 0$
or 3. $S$ is scaled, i.e., for each $S(a) \in S$ and
$0 < p \leq 1 \ \{a + px : x + a \in S(a)\} \in S.$
The Good Witch: The gauge

Gauge function $\delta : \mathbb{R}^d \to (0, \infty)$

$S(a)$ is $\delta$-fine if $S(a) \subset B(a, \delta(a))$. 
The Princess: Bochner integration using Riemann sums

Theorem:
Suppose $\int_{\Omega} \| f \| \, d\mu$ exists.
Let $\rho \geq 1$ and $\epsilon > 0$.
Let $S$ be a fine, $\rho$-regular Morse cover of $\Omega$.
There is a gauge function $\delta$ such that for any $\delta$-fine sequence $\{ S_n(a_n) \} \subset S$ with $S_m \cap S_n = \emptyset$ $(m \neq n)$, $\mu(\Omega \setminus \bigcup S_n) = 0$ and $\mu(\bigcup S_n \setminus \Omega) < \epsilon$ we have
$$\int_{\Omega} \left\| f(x) - \sum_n f(a_n) \chi_{S_n}(x) \right\| \, d\mu(x) < \epsilon$$
and
$$\sum_n \left\| \int_{S_n} f(x) \, d\mu(x) - f(a_n) \mu(S_n) \right\| < \epsilon.$$

Idea of Proof:
Make simple functions that approximate $\int_{\Omega} f \, d\mu$ by forming sums $\sum_n f(a_n) \chi_{S_n}$ where $a_n$ are points of approximate continuity.
Happily ever after

**Theorem:**
Let $S$ be a $\rho$-regular fine Morse cover of $\Omega$. Suppose $G : \{\text{measureable subsets of } \mathbb{R}^d\} \rightarrow Y$ is countably additive, $\mu(E) = 0 \Rightarrow G(E) = 0$ and

$$\sup_{\{S_n\} \subset S} \left\{ \sum_n \|G(S_n)\| : \mu(\Omega \setminus \cup S_n) = 0 \right\} < \infty.$$ 

If there is a gauge function $\delta$ such that for all $\delta$-fine $\{S_n\} \subset S$ with $\mu(\Omega \setminus \cup S_n) = 0$ we have

$$\sum_n \|f(a_n)\mu(S_n) - G(S_n)\| < \epsilon$$

then $\int_{\Omega} f \, d\mu = G(\Omega)$. 