Henstock–Kurzweil Fourier transforms

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Fourier transform

$f : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} \widehat{f}(s) &= \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)\cos(sx)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx \end{aligned}$$

where $i^2 = -1$ and $s \in \mathbb{R}$.

Examples

1. $f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ $\qquad \widehat{f}(s) = \frac{1-e^{-ias}}{is}$

2. $f(x) = e^{-|x|}$ $\qquad \widehat{f}(s) = \frac{2}{1+s^2}$

3. $f(x) = \frac{\mathrm{sgn(x)}}{\sqrt{|x|}}$ $\qquad \widehat{f}(s) = \frac{\sqrt{2\pi}\,\mathrm{sgn}(s)}{\sqrt{|s|}}$

4. $f(x) = e^{ix^2}$ $\qquad \widehat{f}(s) = \sqrt{\pi}\, e^{i(\pi - s^2)/4}$

5. $f(x) = \frac{\sin(ax)}{x}$ $\qquad \widehat{f}(s) = i \log\left|\frac{s-a}{s+a}\right|$

Example

If $\lim\limits_{|x|\to\infty} f(x) = 0$ then $\widehat{f'} = is\widehat{f}$

$$\begin{aligned}\int\limits_{-\infty}^{\infty} e^{-isx} f'(x)\,dx &= \lim_{x\to\infty}\left[e^{-isx}f(x)\right] - \lim_{x\to-\infty}\left[e^{-isx}f(x)\right] \\ &\qquad + is\int\limits_{-\infty}^{\infty} e^{-isx} f(x)\,dx \\ &= is\widehat{f}(s)\end{aligned}$$

Solve the differential equation

$$y''(x) + ay'(x) + by(x) = f(x) \qquad -\infty < x < \infty$$
$$\lim_{|x|\to\infty} y'(x) = \lim_{|x|\to\infty} y(x) = 0$$

$$\left[(is)^2 + isa + b\right]\widehat{y}(s) = \widehat{f}(s)$$

Inversion

$$y(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{isx}\,\widehat{y}(s)\,ds$$

L^1 theory

$$\int_{-\infty}^{\infty} |f| < \infty$$

Basic properties

1. $f \in L^1$ is necessary and sufficient for existence of $\widehat{f}$ on $\mathbb{R}$ as a Lebesgue integral.

2. If $f \in L^1$ then

$$\begin{cases} \widehat{f} & \text{is uniformly continuous on } \mathbb{R} \\ \widehat{f}(s) \to 0 & \text{as } |s| \to \infty \quad \text{(Riemann-Lebesgue lemma)} \end{cases}$$

3. $\check{\widehat{f}} := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \widehat{f}(s)\, ds$ (inverse transform)

If $f, \widehat{f} \in L^1$ then $\check{\widehat{f}} = f$ almost everywhere on $\mathbb{R}$

Henstock–Kurzweil integration

The integral is defined in terms of Riemann sums.

Tagged partition of $[-\infty, \infty]$
$\mathcal{P} = \{(z_n, [x_{n-1}, x_n])\}_{n=1}^N$ and $z_n \in [x_{n-1}, x_n]$
$-\infty = x_0 < x_1 < \cdots < x_N = \infty$

Gauge
$\gamma : [-\infty, \infty] \to \{\text{open intervals in } \mathbb{R}\}$
$\gamma(x)$ is an open interval containing x
$\mathcal{P}$ is γ-fine if $\gamma(z_n) \supset [x_{n-1}, x_n]$

Henstock–Kurzweil integral
$\int_{-\infty}^{\infty} f = A \in \mathbb{R}$ if $\quad (\forall \epsilon > 0)\,(\exists \gamma)$

$$\mathcal{P} \text{ is } \gamma\text{-fine} \Longrightarrow \left| \sum_{n=1}^{N} f(z_n)(x_n - x_{n-1}) - A \right| < \epsilon$$

Convention $\quad 0 \cdot \infty = 0$

Properties of the Henstock–Kurzweil integral

$\mathcal{HK}$ = Henstock–Kurzweil integrable functions

- includes Lebesgue: $\mathcal{HK} \supsetneq L^1$
- nonabsolute: $f \in \mathcal{HK} \not\Rightarrow |f| \in \mathcal{HK}$
- integrates all derivatives: $\int_a^b f' = f(b) - f(a)$
- If $F(x) = \int_a^x f$ then $F'(x) = f(x)$ a.e. but F need not be AC
- multipliers are functions of bounded variation ($\mathcal{BV}$): If $f \in \mathcal{HK}$ and $g \in \mathcal{BV}$ then $fg \in \mathcal{HK}$

Fundamental Theorem of Calculus

The following theorem is false for Riemann and Lebesgue integrals:

Fundamental Theorem of Calculus (?)
Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then for all $x \in [a, b]$

$$\int_a^x F'(t)\, dt = F(x) - F(a).$$

Example

$$F(x) = \begin{cases} x^2 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$F'(x) = \begin{cases} 2x \sin(x^{-2}) - 2x^{-1} \cos(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\int_0^1 |F'| = \infty$ so $\int F'$ does not exist as a Riemann or Lebesgue integral.

Further properties of Henstock–Kurzweil integrals

There are no improper integrals:

$$\int_{-\infty}^{\infty} f = A \Longleftrightarrow \begin{cases} \int_{-\infty}^{x} f \quad \text{exists for each } -\infty < x < \infty \\ \text{and } \lim_{x\to\infty} \int_{-\infty}^{x} f = A \end{cases}$$

Bounded variation

$$Vf = \sup_{\phi \in C_c^1, |\phi| \leq 1} \int f\phi'$$

Convergence Theorem

Suppose $f \in \mathcal{HK}$, $Vg_n \leq M$ and $g_n \to g$.

Then $\int fg_n \to \int fg$.

Henstock–Kurzweil Fourier transforms

Existence

$\widehat{f}$ exists on $\mathbb{R}$

Necessary: f is locally integrable

Sufficient: $f \in L^1(\infty)$

or $\left\{ f(x) \downarrow 0 \text{ as } |x| \to \infty \right.$

Behaviour at infinity

If $\int_{-\infty}^{\infty} |f(x)|\, dx$ diverges but $\int_{-\infty}^{\infty} f(x)e^{-isx}\, dx$ converges then $\widehat{f}$ can grow arbitrarily rapidly, i.e.,

Given $a_n \uparrow \infty$ there is $f \in C$ such that $\widehat{f}$ exists on $\mathbb{R}$ and $\widehat{f}(n) \geq a_n$.

We can even take $f \in C^\infty$.

Theorem: If $a_n \uparrow \infty$ there is a continuous function f such that $\widehat{f}(n) \geq a_n$ for all $n \geq 1$.

Proof: Let $n, r_n > 0$. Define the step function

$$\widehat{f}(s) = \sum_{n=1}^{\infty} a_n \chi_{(n-r_n, n+r_n)}(s)$$

where $(n - r_n, n + r_n)$ are disjoint intervals and

$$\chi_{(a,b)}(x) = \begin{cases} 1, & x \in (a,b) \\ 0, & x \notin (a,b). \end{cases}$$

Note: $\widehat{f}(n) = a_n$

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{s=-\infty}^{\infty} e^{isx} \sum_{n=1}^{\infty} a_n \chi_{(n-r_n,n+r_n)}(s)\, ds \\ &= \frac{1}{2\pi} \sum_{n=1}^{\infty} a_n \int_{s=n-r_n}^{n+r_n} e^{isx}\, ds \\ &= \frac{1}{2\pi i x} \sum_{n=1}^{\infty} a_n \left[e^{i(n+r_n)x} - e^{i(n-r_n)x} \right] \\ &= \frac{1}{2\pi i x} \sum_{n=1}^{\infty} a_n e^{inx} \left[2i \sin(r_n x) \right] \\ &= \frac{1}{\pi x} \sum_{n=1}^{\infty} a_n e^{inx} \sin(r_n x). \end{aligned}$$

Since

$$|f(x)| \le \frac{1}{\pi} \sum_{n=1}^{\infty} a_n \left| \frac{\sin(r_n x)}{x} \right| \le \frac{1}{\pi} \sum_{n=1}^{\infty} a_n r_n,$$

f is continuous if $\sum_{n=1}^{\infty} a_n r_n < \infty$.

Example: Stationary Phase

Let $f(x) = x^\alpha$ for $x > 0$ and $f(x) = 0$ for $x \le 0$. Then

$$
\begin{aligned}
\hat{f}(s) &= \int_{x=0}^{\infty} x^\alpha e^{i(x^2 - sx)} dx \quad \text{where } 0 < \alpha < 1 \\
&= s^{\alpha+1} \int_{t=0}^{\infty} t^\alpha e^{is^2(t^2-t)} dt \quad (x = st) \\
&= s^{\alpha+1} e^{-is^2/4} \int_{t=0}^{\infty} t^\alpha e^{is^2(t-1/2)^2} dt \\
&\sim s^{\alpha+1} e^{-is^2/4} \int_{t=1/2-\epsilon}^{1/2+\epsilon} t^\alpha e^{is^2(t-1/2)^2} dt \\
&\sim \frac{s^{\alpha+1} e^{-is^2/4}}{2^\alpha} \int_{t=-\epsilon}^{\epsilon} e^{is^2t^2} dt \\
&\sim \frac{s^{\alpha+1} e^{-is^2/4}}{2^\alpha} \int_{t=-\infty}^{\infty} e^{is^2t^2} dt \\
&= \frac{s^{\alpha+1} e^{-is^2/4}}{2^\alpha s} \int_{u=-\infty}^{\infty} e^{iu^2} du \quad (t = u/s) \\
&= C_\alpha \, s^\alpha e^{-is^2/4} \quad (s \to \infty)
\end{aligned}
$$

Inverse transform

$$\breve{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} g(s)\, ds$$

Inversion Theorem:
Suppose $\widehat{f}$ exists almost everywhere. Define $F(x) = \int_{x_0}^{x} f$ for $x_0 \in \mathbb{R}$. If $F'(x_0) = f(x_0)$ and $\breve{\widehat{f}}$ exists at x_0 then $f(x_0) = \breve{\widehat{f}}(x_0)$. If $\breve{\widehat{f}}$ exists almost everywhere then $f = \breve{\widehat{f}}$ almost everywhere.

Proof:

Fix $x \in \mathbb{R}$ and $y > 0$. Write $z = x + iy$.
Put $\psi(s) = e^{-|s|}$ then $\widehat{\psi}(t) = \frac{2}{t^2+1}$.
Put $\phi_z(s) = e^{-y|s|}e^{isx}$. Then

$$\begin{aligned} \widehat{\phi}_z(t) &= \widehat{\psi}((t-x)/y)/y \\ &= \frac{2}{y\left[(x-t)^2 + y^2\right]}. \end{aligned}$$

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-y|s|}e^{isx}\widehat{f}(s)\,ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_z(s)\widehat{f}(s)\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\phi}_z(t)f(t)\,dt \\ &= P[f](x+iy) \end{aligned}$$

$P[f](x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{(x-t)^2+y^2}$ is the half plane Poisson integral and solves the Dirichlet problem:

$$\Delta u = 0, \quad Im(z) > 0$$
$$u(x) = f(x), \quad x \in \mathbb{R}$$

Under the condition $\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+1}$ exists we have

$P[f](x+iy) \to f(x)$ a.e. as $y \to 0^+$.

$$\lim_{y\to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y|s|} e^{isx} \widehat{f}(s)\, ds$$

$$= \check{\widehat{f}}(x)$$

$$= \lim_{y\to 0^+} P[f](x+iy)$$

$$= f(x) \quad \text{(for almost all) } x \in \mathbb{R}.$$

Summability kernels

Abel/Poisson

$$\begin{aligned}\psi(x) &= e^{-|x|}\\ \widehat{\psi}(s) &= \frac{2}{s^2+1}\end{aligned}$$

Cesàro/Fejér

$$\begin{aligned}\psi(x) &= (1-|x|)\chi_{[-1,1]}(x)\\ \widehat{\psi}(s) &= \left[\frac{\sin(s/2)}{s/2}\right]^2\end{aligned}$$

Gauss/Weierstrass

$$\begin{aligned}\psi(x) &= e^{-x^2}\\ \widehat{\psi}(s) &= \sqrt{\pi}e^{-s^2/4}\end{aligned}$$

Example of rapid growth

$$f(x) \;=\; x^{\alpha} e^{ix^{\nu}} \qquad \text{for } x > 0$$

$$\widehat{f}(s) \;\sim\; C_{\nu,\alpha} \left[e^{id_{\nu} s^{\nu/(\nu-1)}} \right] \left[s^{\frac{2\alpha+2-\nu}{2(\nu-1)}} \right] \qquad (s \to \infty).$$

If ν is close to 1 then $\widehat{f}$ grows rapidly as $s \to \infty$.

If $\nu > 2$ and $\nu/2 \le \alpha < \nu - 1$ then $\widehat{f}$ exists on $\mathbb{R}$ but $\check{\widehat{f}}$ does not exist.