Convolutions with the continuous primitive integral

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Abstract. If $F$ is a continuous function on the real line and $f = F'$ is its distributional derivative then the continuous primitive integral of distribution $f$ is $\int_a^b f = F(b) - F(a)$. This integral contains the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals. Under the Alexiewicz norm the space of integrable distributions is a Banach space. We define the convolution $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$ for $f$ an integrable distribution and $g$ a function of bounded variation or an $L^1$ function. Usual properties of convolutions are shown to hold: commutativity, associativity, commutation with translation. For $g$ of bounded variation, $f * g$ is uniformly continuous and we have the estimate $\|f * g\|_\infty \leq \|f\|\|g\|_{BV}$ where $\|f\| = \sup_I |\int_I f|$ is the Alexiewicz norm. This supremum is taken over all intervals $I \subset \mathbb{R}$. When $g \in L^1$ the estimate is $\|f * g\| \leq \|f\|\|g\|_1$. There are results on differentiation and integration of convolutions. A type of Fubini theorem is proved for the continuous primitive integral.

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1 Introduction and notation

The convolution of two functions $f$ and $g$ on the real line is $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$. Convolutions play an important role in pure and applied mathematics in Fourier analysis, approximation theory, differential equations, integral equations and many other areas. In this paper we consider convolutions for the continuous primitive integral. This integral extends the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals on the real line and has a very simple definition in terms of distributional derivatives.

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Some of the main results for Lebesgue integral convolutions are that the convolution defines a Banach algebra on $L^1$ and $*: L^1 \times L^1 \to L^1$ such that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. The convolution is commutative, associative and commutes with translations. If $f \in L^1$ and $g \in C^n$ then $f * g \in C^n$ and $(f * g)^{(n)}(x) = f * g^{(n)}(x)$. Convolutions also have the approximation property that if $f \in L^p'$ (1 ≤ $p < \infty$) and $g \in L^1$ then $\|f * g_t - af\|_p \to 0$ as $t \to 0$, where $g_t(x) = g(x/t)/t$ and $a = \int_{-\infty}^{\infty} g$. When $f$ is bounded and continuous, there is a similar result for $p = \infty$. For these results see, for example, [4]. See [8] for related results with the Henstock–Kurzweil integral. Using the Alexiewicz norm, all of these results have generalisations to continuous primitive integrals that are proven below.

We now define the continuous primitive integral. For this we need some notation for distributions. The space of test functions is $D = C_c^\infty(\mathbb{R}) = \{\phi: \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty(\mathbb{R})$ and $\text{supp}(\phi)$ is compact$\}$. The support of function $\phi$ is the closure of the set on which $\phi$ does not vanish and is denoted $\text{supp}(\phi)$. Under usual pointwise operations $D$ is a linear space over field $\mathbb{R}$. In $D$ we have a notion of convergence. If $\{\phi_n\} \subset D$ then $\phi_n \to 0$ as $n \to \infty$ if there is a compact set $K \subset \mathbb{R}$ such that for each $n$, $\text{supp}(\phi_n) \subset K$, and for each $m \geq 0$ we have $\phi_n^{(m)} \to 0$ uniformly on $K$ as $n \to \infty$. The distributions are denoted $D'$ and are the continuous linear functionals on $D$. For $T \in D'$ and $\phi \in D$ we write $\langle T, \phi \rangle \in \mathbb{R}$. For $\phi, \psi \in D$ and $a, b \in \mathbb{R}$ we have $\langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle$. And, if $\phi_n \to 0$ in $D$ then $\langle T, \phi_n \rangle \to 0$ in $\mathbb{R}$. Linear operations are defined in $D'$ by $\langle aS + bT, \phi \rangle = a\langle S, \phi \rangle + b\langle T, \phi \rangle$ for $S, T \in D'$; $a, b \in \mathbb{R}$ and $\phi \in D$. If $f \in L^1_{\text{loc}}$ then $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x) \, dx$ defines a distribution $T_f \in D'$. The integral exists as a Lebesgue integral. All distributions have derivatives of all orders that are themselves distributions. For $T \in D'$ and $\phi \in D$ the distributional derivative of $T$ is $T'$ where $\langle T', \phi \rangle = -\langle T, \phi' \rangle$. This is also called the weak derivative. If $p: \mathbb{R} \to \mathbb{R}$ is a function that is differentiable in the pointwise sense at $x \in \mathbb{R}$ then we write its derivative as $p'(x)$. If $p$ is a $C^\infty$ bijection such that $p'(x) \neq 0$ for any $x \in \mathbb{R}$ then the composition with distribution $T$ is defined by $\langle T \circ p, \phi \rangle = \langle T, \frac{\phi \circ p^{-1}}{p' \circ p^{-1}} \rangle$ for all $\phi \in D$. Translations are a special case. For $x \in \mathbb{R}$ define the translation $\tau_x$ on distribution $T \in D'$ by $\langle \tau_x T, \phi \rangle = \langle T, \tau_{-x} \phi \rangle$ for test function $\phi \in D$ where $\tau_x \phi(y) = \phi(y - x)$. All of the results on distributions we use can be found in [5].

The following Banach space will be of importance: $\mathcal{B}_C = \{F: \mathbb{R} \to \mathbb{R} \mid F \in C^0(\mathbb{R}), F(-\infty) = 0, F(\infty) \in \mathbb{R}\}$. We use the notation $F(-\infty) =$
\[
\lim_{x \to -\infty} F(x) \text{ and } F(\infty) = \lim_{x \to \infty} F(x). \]
The extended real line is denoted \( \mathbb{R} = [-\infty, \infty] \). The space \( \mathcal{B}_C \) then consists of functions continuous on \( \mathbb{R} \) with a limit of 0 at \( -\infty \). We denote the functions that are continuous on \( \mathbb{R} \) that have real limits at \( \pm \infty \) by \( C^0(\mathbb{R}) \). Hence, \( \mathcal{B}_C \) is properly contained in \( C^0(\mathbb{R}) \), which is itself properly contained in the space of uniformly continuous functions on \( \mathbb{R} \). The space \( \mathcal{B}_C \) is a Banach space under the uniform norm;

\[
\|F\|_\infty = \sup_{x \in \mathbb{R}} |F(x)| = \max_{x \in \mathbb{R}} |F(x)| \quad \text{for } F \in \mathcal{B}_C.
\]
The \textit{continuous primitive integral} is defined by taking \( \mathcal{B}_C \) as the space of primitives. The space of integrable distributions is \( \mathcal{A}_C = \{ f \in \mathcal{D}' \mid f = F' \text{ for } F \in \mathcal{B}_C \} \). If \( f \in \mathcal{A}_C \)
then \( \int_a^b f = F(b) - F(a) \) for \( a, b \in \mathbb{R} \). The distributional differential equation \( T' = 0 \) has only constant solutions so the primitive \( F \in \mathcal{B}_C \) satisfying \( F' = f \) is unique. Integrable distributions are then tempered and of order one. This integral, including a discussion of extensions to \( \mathbb{R}^n \), is described in [9]. A more general integral is obtained by taking the primitives to be regulated functions, i.e., functions with a left and right limit at each point. See [10].

Examples of distributions in \( \mathcal{A}_C \) are \( T_f \) for functions \( f \) that have a finite Lebesgue, Henstock–Kurzweil or wide Denjoy integral. We identify function \( f \) with the distribution \( T_f \). Pointwise function values can be recovered from \( T_f \) at points of continuity of \( f \) by evaluating the limit \( \langle T_f, \phi_n \rangle \) for a \textit{delta sequence} converging to \( x \in \mathbb{R} \). This is a sequence of test functions \( \{ \phi_n \} \subset \mathcal{D} \) such that for each \( n, \phi_n \geq 0, \int_{-\infty}^{\infty} \phi_n = 1 \), and the support of \( \phi_n \) tends to \( \{x\} \) as \( n \to \infty \). Note that if \( F \in C^0(\mathbb{R}) \) is an increasing function with \( F'(x) = 0 \) for almost all \( x \in \mathbb{R} \) then the Lebesgue integral \( \int_a^b F'(x) \, dx = 0 \) but \( F' \in \mathcal{A}_C \)
and \( \int_a^b F' = F(b) - F(a) \). For another example of a distribution in \( \mathcal{A}_C \), let \( F \in C^0(\mathbb{R}) \) be continuous and nowhere differentiable in the pointwise sense. Then \( F' \in \mathcal{A}_C \) and \( \int_a^b F' = F(b) - F(a) \) for all \( a, b \in \mathbb{R} \).

The space \( \mathcal{A}_C \) is a Banach space under the \textit{Alexiewicz norm};

\[
\|f\|_A = \sup_{I \subset \mathbb{R}} |\int_I f| \quad \text{where the supremum is taken over all intervals } I \subset \mathbb{R}.
\]
An equivalent norm is \( \|f\|'_A = \sup_{x \in \mathbb{R}} |\int_{-\infty}^x f| \). The continuous primitive integral contains the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals since their primitives are continuous functions. These three spaces of functions are not complete under the Alexiewicz norm and in fact \( \mathcal{A}_C \) is their completion. The lack of a Banach space has hampered application of the Henstock–Kurzweil integral to problems outside of real analysis. As we will see below, the Banach space \( \mathcal{A}_C \) is a suitable setting for applications of non-absolute integration.

We will also need to use functions of bounded variation. Let \( g : \mathbb{R} \to \mathbb{R} \).
Convolutions for continuous primitive integral

The variation of \( g \) is \( Vg = \sup \sum |g(x_i) - g(y_i)| \) where the supremum is taken over all disjoint intervals \( \{(x_i, y_i)\} \). The functions of bounded variation are denoted \( BV = \{g: \mathbb{R} \to \mathbb{R} \mid Vg < \infty\} \). This is a Banach space under the norm \( \|g\|_{BV} = |g(-\infty)| + Vg \). Equivalent norms are \( \|g\|_\infty + Vg \) and \( |g(a)| + Vg \) for each \( a \in \mathbb{R} \). Functions of bounded variation have a left and right limit at each point in \( \mathbb{R} \) and limits at \( \pm \infty \) so as above we will define \( g(\pm \infty) = \lim_{x \to \pm \infty} g(x) \).

If \( g \in L_{loc}^1 \) then the essential variation of \( g \) is \( ess\, var\, g = \sup \int_{-\infty}^\infty g \phi' \) where the supremum is taken over all \( \phi \in \mathcal{D} \) with \( \|\phi\|_\infty \leq 1 \). Then \( EBV = \{g \in L_{loc}^1 \mid ess\, var\, g < \infty\} \). This is a Banach space under the norm \( \|g\|_{EBV} = ess\, sup |g| + ess\, var\, g \). Let \( 0 \leq \gamma \leq 1 \). For \( g: \mathbb{R} \to \mathbb{R} \) define \( g_\gamma(x) = (1 - \gamma)g(x^{-}) + \gamma g(x^{+}) \). For left continuity, \( \gamma = 0 \) and for right continuity \( \gamma = 1 \). The functions of normalised bounded variation are \( NBV_{\gamma} = \{g \mid g \in BV\} \). If \( g \in EBV \) then \( ess\, var\, g = \inf Vh \) such that \( h = g \) almost everywhere. For each \( 0 \leq \gamma \leq 1 \) there is exactly one function \( h \in NBV_{\gamma} \) such that \( g = h \) almost everywhere. In this case \( ess\, var\, g = Vh \). Changing \( g \) on a set of measure zero does not affect its essential variation. Each function of essential bounded variation has a distributional derivative that is a signed Radon measure. This will be denoted \( \mu_g \) where \( \langle g', \phi \rangle = -\langle g, \phi' \rangle = -\int_{-\infty}^\infty g \phi' = \int_{-\infty}^\infty \phi \, d\mu_g \) for all \( \phi \in \mathcal{D} \).

We will see that \( *: \mathcal{A}_C \times BV \to C^0(\mathbb{R}) \) and that \( \|f * g\|_\infty \leq \|f\| \|g\|_{BV} \). Similarly for \( g \in EBV \). Convolutions for \( f \in \mathcal{A}_C \) and \( g \in L^1 \) will be defined using sequences in \( BV \cap L^1 \) that converge to \( g \) in the \( L^1 \) norm. It will be shown that \( *: \mathcal{A}_C \times L^1 \to \mathcal{A}_C \) and that \( \|f * g\| \leq \|f\| \|g\|_1 \).

Convolutions can be defined for distributions in several different ways.

**Definition 1** Let \( S, T \in \mathcal{D}' \) and \( \phi, \psi \in \mathcal{D} \). Define \( \tilde{\phi}(x) = \phi(-x) \). (i) \( \langle T * \psi, \phi \rangle = \langle T, \phi * \tilde{\psi} \rangle \). (ii) For each \( x \in \mathbb{R} \), let \( T * \psi(x) = \langle T, \tau_x \tilde{\psi} \rangle \). (iii) \( \langle S * T, \phi \rangle = \langle S(x), \langle T(y), \phi(x + y) \rangle \rangle \). In (i), \( *: \mathcal{D}' \times \mathcal{D} \to \mathcal{D}' \). This definition also applies to other spaces of test functions and their duals, such as the Schwartz space of rapidly decreasing functions or the compactly supported distributions. In (ii), \( *: \mathcal{D}' \times \mathcal{D} \to C^\infty \). In [4] it is shown that definitions (i) and (ii) are equivalent. In (iii), \( *: \mathcal{D}' \times \mathcal{D}' \to \mathcal{D}' \). However, this definition requires restrictions on the supports of \( S \) and \( T \). It suffices that one of these distributions have compact support. Other conditions on the supports can be imposed. See [5] and [11]. This definition is an instance of the tensor product, \( \langle S \otimes T, \Phi \rangle = \langle S(x), \langle T(y), \Phi(x, y) \rangle \rangle \) where now \( \Phi \in \mathcal{D}(\mathbb{R}^2) \).
Under (i), \( T \ast \psi \) is in \( C^\infty \). It satisfies \( (T \ast \psi) \ast \phi = T \ast (\psi \ast \phi) \), \( \tau_x (T \ast \psi) = (\tau_x T) \ast \psi = T \ast (\tau_x \psi) \), and \( (T \ast \psi)^{(n)} = T \ast \psi^{(n)} = T^{(n)} \ast \psi \). Under (iii), with appropriate support restrictions, \( S \ast T \) is in \( \mathcal{D}' \). It is commutative and associative, commutes with translations, and satisfies \( (S \ast T)^{(n)} = S^{(n)} \ast T = S \ast T^{(n)} \). It is weakly continuous in \( \mathcal{D}' \), i.e., if \( T_n \to T \) in \( \mathcal{D}' \) then \( T_n \ast \psi \to T \ast \psi \) in \( \mathcal{D}' \). See [4], [5], [7] and [11] for additional properties of convolutions of distributions.

Although elements of \( A_\mathcal{C} \) are distributions, we show in this paper that their behaviour as convolutions is more like that of integrable functions.

An appendix contains the proof of a type of Fubini theorem.

\section{Convolution in \( A_\mathcal{C} \times \mathcal{B} \mathcal{V} \)}

In this section we prove basic results for the convolution when \( f \in A_\mathcal{C} \) and \( g \in \mathcal{B} \mathcal{V} \). Under these conditions \( f \ast g \) is commutative, continuous on \( \mathbb{R} \) and commutes with translations. It can be estimated in the uniform norm in terms of the Alexiewicz and \( \mathcal{B} \mathcal{V} \) norms. There is also an associative property. We first need the result that \( \mathcal{B} \mathcal{V} \) forms the space of multipliers for \( A_\mathcal{C} \), i.e., if \( f \in A_\mathcal{C} \) then \( fg \in A_\mathcal{C} \) for all \( g \in \mathcal{B} \mathcal{V} \). The integral \( \int f g \) is defined using the integration by parts formula in the Appendix. The Hölder inequality (14) shows that \( \mathcal{B} \mathcal{V} \) is the dual space of \( A_\mathcal{C} \).

We define the convolution of \( f \in A_\mathcal{C} \) and \( g \in \mathcal{B} \mathcal{V} \) as \( f \ast g(x) = \int_{-\infty}^{\infty} (f \circ \tau_x) g \) where \( \tau_x(t) = x - t \). We write this as \( f \ast g(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dy \).

**Theorem 2** Let \( f \in A_\mathcal{C} \) and let \( g \in \mathcal{B} \mathcal{V} \). Then

\begin{itemize}
  \item[(a)] \( f \ast g \) exists on \( \mathbb{R} \)
  \item[(b)] \( f \ast g = g \ast f \)
  \item[(c)] \( \|f \ast g\|_\infty \leq \|f\| \cdot \|g\|_{\mathcal{B} \mathcal{V}} \)
  \item[(d)] \( f \ast g \in C^0(\mathbb{R}) \), \( \lim_{x \to \pm\infty} f \ast g(x) = g(\pm\infty) \int_{-\infty}^{\infty} f \)
  \item[(e)] If \( h \in L^1 \) then \( f \ast (g \ast h) = (f \ast g) \ast h \in C^0(\mathbb{R}) \).
  \item[(f)] Let \( x, z \in \mathbb{R} \). Then \( \tau_z (f \ast g)(x) = (\tau_z f) \ast g(x) = (f \ast \tau_z g)(x) \).
  \item[(g)] For each \( f \in A_\mathcal{C} \) define \( \Phi_f : \mathcal{B} \mathcal{V} \to C^0(\mathbb{R}) \) by \( \Phi_f[g] = f \ast g \). Then \( \Phi_f \) is a bounded linear operator and \( \|\Phi_f\| \leq \|f\| \).
  \item[(h)] There exists a nonzero distribution \( f \in A_\mathcal{C} \) such that \( \|\Phi_f\| = \|f\| \).
  \item[(i)] For each \( g \in \mathcal{B} \mathcal{V} \) define \( \Psi_g : A_\mathcal{C} \to C^0(\mathbb{R}) \) by \( \Psi_g[f] = f \ast g \). Then \( \Psi_g \) is a bounded linear operator and \( \|\Psi_g\| \leq \|g\|_{\mathcal{B} \mathcal{V}} \).
  \item[(j)] There exists a nonzero function \( g \in \mathcal{B} \mathcal{V} \) such that \( \|\Psi_g\| = \|g\|_{\mathcal{B} \mathcal{V}} \).
  \item[(k)] \( \text{supp}(f \ast g) \subset \text{supp}(f) + \text{supp}(g) \).
\end{itemize}

**Proof:** (a) Existence is given via the integration by parts formula (13) in the Appendix. (b) See [9, Theorem 11] for a change of variables theorem.
that can be used with $y \mapsto x - y$. (c) This inequality follows from the Hölder inequality (14). (d) Let $x, t \in \mathbb{R}$. From (c) we have

$$|f * g(t) - f * g(x)| \leq \|f(t - \cdot) - f(x - \cdot)\|_{BV} g$$

$$= \|f(t - x - \cdot) - f(\cdot)\|_{BV} g$$

$$\to 0 \text{ as } t \to x.$$

The last line follows from continuity in the Alexiewicz norm [9, Theorem 22]. Hence, $f * g$ is uniformly continuous on $\mathbb{R}$. And, $\lim_{x \to -\infty} \int_{-\infty}^{\infty} f(y)g(x-y) \, dy = \int_{-\infty}^{\infty} f(y) \lim_{x \to -\infty} g(x-y) \, dy = g(\infty) \int_{-\infty}^{\infty} f$. The limit $x \to \infty$ can be taken under the integral sign since $g(x-y)$ is of uniform bounded variation, i.e., $V_{y \in \mathbb{R}} g(x-y) = Vg$. Theorem 22 in [9] then applies. Similarly as $x \to -\infty$.

(e) First show $g * h \in BV$. Let $\{(s_i, t_i)\}$ be disjoint intervals in $\mathbb{R}$. Then

$$\sum |g * h(s_i) - g * h(t_i)| \leq \sum \int_{-\infty}^{\infty} |g(s_i - y) - g(t_i - y)||h(y)| \, dy$$

$$= \int_{-\infty}^{\infty} \sum |g(s_i - y) - g(t_i - y)||h(y)| \, dy.$$

Hence, $V(g * h) \leq Vg \|h\|_1$. The interchange of sum and integral follows from the Fubini–Tonelli theorem. Now (d) shows $f * (g * h) \in C^0(\mathbb{R})$. Write

$$f * (g * h)(x) = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x-y-z)h(z) \, dz \, dy$$

$$= \int_{-\infty}^{\infty} h(z) \int_{-\infty}^{\infty} f(y)g(x-y-z) \, dy \, dz$$

$$= (f * g) * h(x).$$

We can interchange orders of integration using Proposition 17. For (ii) in Proposition 17, the function $z \mapsto \sup_{y \in \mathbb{R}} g(x-y-z)h(z) = Vg h(z)$ is in $L^1$ for each fixed $x \in \mathbb{R}$. Since $g$ is of bounded variation it is bounded so $|g(x-y-z)h(z)| \leq \|g\|_\infty |h(z)|$ and condition (iii) is satisfied. (f) This follows from a linear change of variables as in (a). (g) From (c) we have

$$\|f\| = \sup_{\|g\|_{BV} = 1} \|f * g\|_{\infty} \leq \sup_{\|g\|_{BV} = 1} \|f\| \|g\|_{BV} = \|f\|. \text{ Let } f > 0 \text{ be in } L^1. \text{ If } g = 1 \text{ then } \|g\|_{BV} = 1 \text{ and } f * g(x) = \int_{-\infty}^{\infty} f$$

$$\|f\|_1. \text{ To prove } \|\Phi_g\| \leq \|g\|_{BV}, \text{ note that } \|\Phi_g\| = \sup_{\|f\|_1 = 1} \|f * g\|_{\infty} \leq \sup_{\|f\|_1 = 1} \|f\| \|g\|_{BV} = \|g\|_{BV}. \text{ Let } g = \chi_{(0,\infty)}. \text{ Then } \|\Phi_g\| = \sup_{\|f\|_1 = 1} \|f * g\|_{\infty} = \sup_{\|f\|_1 = 1} \sup_{x \in \mathbb{R}} |\int_{-\infty}^{x} f| = 1 = \|g\|_{BV}. \text{ (h) Suppose } x \notin \supp(f) +
supp(\(g\)). Note that we can write \(f * g(x) = \int_{-\infty}^{\infty} g(x-y) \, dF(y)\), in terms of a Henstock–Stieltjes integral. See [9] for details. This integral is approximated by Riemann sums \(\sum_{n=1}^{N} g(x - z_n)[F(t_n) - F(t_{n-1})]\) where \(z_n \in [t_{n-1}, t_n]\), \(-\infty = t_0 < t_1 < \ldots < t_N = \infty\) and there is a gauge function \(\gamma\) mapping \(\mathbb{R}\) to the open intervals in \(\mathbb{R}\) such that \([t_{n-1}, t_n] \subset \gamma(z_n)\). If \(z_n \notin supp(f)\) then since \(\mathbb{R} \setminus supp(f)\) is open there is an open interval \(z_n \subset I \subset \mathbb{R} \setminus supp(f)\). We can take \(\gamma\) such that \([t_{n-1}, t_n] \subset I\) for all \(1 \leq n \leq N\). And, \(F\) is constant on each interval in \(\mathbb{R} \setminus supp(f)\). Therefore, \(g(x - z_n)[F(t_n) - F(t_{n-1})] = 0\) and only tags \(z_n \in supp(f)\) can contribute to the Riemann sum. But for all \(z_n \in supp(f)\) we have \(x - z_n \notin supp(g)\) so \(g(x - z_n)[F(t_n) - F(t_{n-1})] = 0\). It follows that \(f * g(x) = 0\). \(\blacksquare\)

Similar results are proven for \(f \in L^p\) in [4, § 8.2].

If we use the equivalent norm \(\|f\|' = \sup_{x \in \mathbb{R}} |\int_{-\infty}^{x} f|\) then \(\|\Phi_f\| = \|f\|'\). For, integration by parts gives \(\|\Phi_f\| \leq \|f\|'\). Now given \(f \in \mathcal{A}_C\) let \(g = \chi_{(0,\infty)}\). Then \(\|g\|_{EBV} = 1\). And, \(f * g(x) = \int_{-\infty}^{x} f\). Hence, \(\|f * g\| = \|f\|'\) and \(\|\Phi_f\| = \|f\|'\). We can have strict inequality in \(\|\Psi_g\| \leq \|g\|_{EBV}\). For example, let \(g = \chi_{(0)}\). Then \(\|g\|_{EBV} = 2\) but integration by parts shows \(f * g = 0\) for each \(f \in \mathcal{A}_C\).

**Remark 3** If \(f \in \mathcal{A}_C\) and \(g \in \mathcal{EBV}\) we can use Definition 16 to define \(f * g(x) = f * g_\gamma(x)\) where \(g_\gamma = g\) almost everywhere and \(g_\gamma \in \mathcal{NBV}_\gamma\). All of the results in Theorem 2 and the rest of the paper have analogues. Note that \(f * g(x) = F(\infty)g_\gamma(-\infty) + F * \mu_g\).

**Proposition 4** The three definitions of convolution for distributions in Definition 1 are compatible with \(f * g\) for \(f \in \mathcal{A}_C\) and \(g \in \mathcal{BV}\).

**Proof:** Let \(f \in \mathcal{A}_C\), \(g \in \mathcal{BV}\) and \(\phi, \psi \in \mathcal{D}\). Definition 1(i) gives

\[
\langle f, \tilde{\psi} * \phi \rangle = \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} \psi(y-x) \phi(y) \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \psi(y-x) \phi(y) \, dx \, dy
\]

\[
= \langle f * \psi, \phi \rangle.
\]

Since \(\psi \in \mathcal{BV}\) and \(\phi \in L^1\), Proposition 17 justifies the interchange of integrals. Definition 1(ii) gives

\[
\langle f, \tau_x \tilde{\psi} \rangle = \int_{-\infty}^{\infty} f(y) \psi(x-y) \, dy = f * \psi(x).
\]
Definition 1(iii) gives
\[
\langle f(y), \langle g(x), \phi(x + y) \rangle \rangle = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x)\phi(x + y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x - y)\phi(x) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\infty} f(y)g(x - y) \, dx \, dy
\]
\[
= \langle f \ast g, \phi \rangle.
\]

The interchange of integrals is accomplished using Proposition 17 since \(g \in BV\) and \(\phi \in L^1\).

The locally integrable distributions are defined as \(AC_{\text{loc}} = \{f \in D' \mid f = F'\text{ for some } F \in C^0(\mathbb{R})\}\). Let \(f \in AC_{\text{loc}}\) and let \(g \in BV\) with support in the compact interval \([a, b]\). By the Hake theorem [9, Theorem 25], \(f \ast g(x)\) exists if and only if the limits of \(\int_{\alpha}^{\beta} f(x - y)g(y) \, dy\) exist as \(\alpha \to -\infty\) and \(\beta \to \infty\). This gives
\[
f \ast g(x) = \int_{a}^{b} f(x - y)g(y) \, dy = \int_{x - a}^{x - b} f(y)g(x - y) \, dy.
\]

There are analogues of the results in Theorem 2. For example, \(|f \ast g(x)| \leq |\int_{x - a}^{x - b} f| \inf_{[a, b]} |g| + \|f \chi_{[x - b, x - a]}\|_{V[a, b]} g\). There are also versions where the supports are taken to be semi-infinite intervals.

We can also define the distributions with bounded primitive as \(AC_{\text{bd}} = \{f \in D' \mid f = F'\text{ for some bounded } F \in C^0(\mathbb{R}) \text{ with } F(0) = 0\}\). Let \(f \in AC_{\text{bd}}\) and let \(F\) be its unique primitive. If \(g \in BV\) such that \(g(\pm \infty) = 0\) then
\[
f \ast g = \lim_{\alpha \to -\infty} \lim_{\beta \to \infty} \int_{\alpha}^{\beta} f(x - y)g(y) \, dy
\]
\[
= \lim_{\alpha \to -\infty} \lim_{\beta \to \infty} \left[ F(x - \alpha)g(\alpha) - F(x - \beta)g(\beta) + \int_{\alpha}^{\beta} F(x - y) \, dg(y) \right]
\]
\[
= \int_{-\infty}^{\infty} F(x - y) \, dg(y) = \int_{-\infty}^{\infty} F(y) \, dg(x - y).
\]

It follows that \(\|f \ast g\|_{\infty} \leq \|F\|_{\infty} Vg\).
It is possible to formulate other existence criteria. For example, if \( f(x) = \log |x| \sin(x) \) and \( g(x) = |x| - \alpha \) for some \( 0 < \alpha < 1 \) then \( f \) and \( g \) are not in \( A_C, BV \) or \( L^p \) for any \( 1 \leq p \leq \infty \) but \( f \ast g \) exists on \( \mathbb{R} \) because \( f, g \in L^1_{loc} \) and if \( F(x) = \int_0^x f \) then \( \lim_{|x| \to \infty} F(x)g(x) = 0 \).

The following example shows that \( f \ast g \) need not be of bounded variation and hence not absolutely continuous. Let \( g = \chi_{(0, \infty)} \). For \( f \in A_C \) we have \( f \ast g(x) = \int_{-\infty}^x f = F(x) \) where \( F \in B_C \) is the primitive of \( f \). But \( F \) need not be of bounded variation or even of local bounded variation. For example, let \( f(x) = \sin(x^2) - 2x^2 \cos(x^2) \) and let \( F \) be its primitive in \( B_C \). Finally, although \( f \ast g \) is continuous it need not be integrable over \( \mathbb{R} \). For example, let \( g = 1 \) then \( f \ast g(x) = \int_{-\infty}^\infty f \) and \( \int_{-\infty}^\infty f \ast g \) only exists if \( \int_{-\infty}^\infty f = 0 \).

## 3 Convolution in \( A_C \times L^1 \)

We now extend the convolution \( f \ast g \) to \( f \in A_C \) and \( g \in L^1 \). Since there are functions in \( L^1 \) that are not of bounded variation, there are distributions \( f \in A_C \) and functions \( g \in L^1 \) such that the integral \( \int_{-\infty}^\infty f(x - y)g(y) \, dy \) does not exist. The convolution is then defined as the limit in \( \| \cdot \| \) of a sequence \( f \ast g_n \) for \( g_n \in BV \cap L^1 \) such that \( g_n \to g \) in the \( L^1 \) norm. This is possible since \( BV \cap L^1 \) is dense in \( L^1 \). We also give an equivalent definition using the fact that \( L^1 \) is dense in \( A_C \). Take a sequence \( \{f_n\} \subset L^1 \) such that \( \|f_n - f\| \to 0 \). Then \( f \ast g \) is the limit in \( \| \cdot \| \) of \( f_n \ast g \). In this more general setting of convolution defined in \( A_C \times L^1 \) we now have an Alexiewicz norm estimate for \( f \ast g \) in terms of estimates of \( f \) in the Alexiewicz norm and \( g \) in the \( L^1 \) norm. There is associativity with \( L^1 \) functions and commutativity with translations.

**Definition 5** Let \( f \in A_C \) and let \( g \in L^1 \). Let \( \{g_n\} \subset BV \cap L^1 \) such that \( \|g_n - g\|_1 \to 0 \). Define \( f \ast g \) as the unique element in \( A_C \) such that \( \|f \ast g_n - f \ast g\| \to 0 \).

To see that the definition makes sense, first note that \( BV \cap L^1 \) is dense in \( L^1 \) since step functions are dense in \( L^1 \). Hence, the required sequence \( \{g_n\} \) exists. Let \( [\alpha, \beta] \subset \mathbb{R} \) be a compact interval. Let \( F \in B_C \) be the primitive of
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Then

\[
\int_\alpha^\beta f * g_n(x) \, dx = \int_\alpha^\beta \int_{-\infty}^\infty f(y) g_n(x - y) \, dy \, dx
\]

\[
= \int_{-\infty}^\infty f(y) \int_\alpha^{\beta-y} g_n(x) \, dx \, dy
\]

\[
= -\int_{-\infty}^\infty F(y) \frac{d}{dy} \left[ \int_\alpha^{\beta-y} g_n \right] \]

\[
= \int_{-\infty}^\infty F(y) [g_n(\beta - y) - g_n(\alpha - y)] \, dy
\]

\[
= \int_{-\infty}^\infty \left( \int_\alpha^{\beta-y} f \right) g_n(y) \, dy.
\]

The interchange of orders of integration in (1) is accomplished with Proposition 17 using $g(x, y) = g_n(x - y) \chi_{[\alpha, \beta]}(x)$. Integration by parts gives (2) since \( \lim_{y \to \infty} \int_\alpha^{\beta-y} g_n = 0 \). As \( F \) is continuous and the function \( y \mapsto \int_\alpha^{\beta-y} g_n \) is absolutely continuous we get (3). Taking the supremum over \( \alpha, \beta \in \mathbb{R} \) gives

\[
\|f * g_n\| \leq \|f\|\|g_n\|_1.
\]

We now have

\[
\|f * g_m - f * g_n\| = \|f * (g_m - g_n)\| \leq \|f\|\|g_m - g_n\|_1
\]

and \( \{f * g_n\} \) is a Cauchy sequence in \( A_C \). Since \( A_C \) is complete this sequence has a limit in \( A_C \) which we denote \( f * g \). The definition does not depend on the choice of sequence \( \{g_n\} \), for if \( \{h_n\} \subset BV \cap L^1 \) such that \( \|h_n - g\|_1 \to 0 \) then \( \|f * g_n - f * h_n\| \leq \|f\|\|g_n - g\|_1 + \|h_n - g\|_1 \to 0 \) as \( n \to \infty \). The above calculation also shows that if \( g \in BV \cap L^1 \) then the integral definition \( f * g(x) = \int_{-\infty}^\infty f(x - y)g(y) \, dy \) and the limit definition agree.

**Definition 6** Let \( f \in A_C \) and let \( g \in L^1 \). Let \( \{f_n\} \subset L^1 \) such that \( \|f_n - f\| \to 0 \). Define \( f * g \) as the unique element in \( A_C \) such that \( \|f_n * g - f * g\| \to 0 \).

To show this definition makes sense, first show \( L^1 \) is dense in \( A_C \).
Proposition 7 \( L^1 \) is dense in \( \mathcal{A}_C \).

**Proof:** Let \( AC(\mathbb{R}) \) be the functions that are absolutely continuous on each compact interval and which are of bounded variation on the real line. Then \( f \in L^1 \) if and only if there exists \( F \in AC(\mathbb{R}) \) such that \( F'(x) = f(x) \) for almost all \( x \in \mathbb{R} \). Let \( f \in \mathcal{A}_C \) be given. Let \( F \in \mathcal{B}_C \) be its primitive. For \( \epsilon > 0 \), take \( M > 0 \) such that \( |F(x)| < \epsilon \) for \( x < -M \) and \( |F(x) - F(\infty)| < \epsilon \) for \( x > M \). Due to the Weierstrass approximation theorem there is a continuous function \( P : \mathbb{R} \to \mathbb{R} \) such that \( P(x) = F(-M) \) for \( x \leq -M \), \( P(x) = F(M) \) for \( x \geq M \), \( |P(x) - F(x)| < \epsilon \) for \( |x| \leq M \) and \( P \) is a polynomial on \([-M, M]\). Hence, \( P \in AC(\mathbb{R}) \) and \( \|P' - f\| < 3\epsilon \). \( \blacksquare \)

In Definition 6, the required sequence \( \{f_n\} \subset L^1 \) exists. Let \( [\alpha, \beta] \subset \mathbb{R} \) be a compact interval. Then, by the usual Fubini–Tonelli theorem in \( L^1 \),

\[
\int_{\alpha}^{\beta} f_n \ast g(x) \, dx = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f_n(x - y)g(y) \, dy \, dx
\]

\[
= \int_{-\infty}^{\infty} g(y) \int_{\alpha}^{\beta} f_n(x - y) \, dx \, dy.
\]

Take the supremum over \( \alpha, \beta \in \mathbb{R} \) and use the \( L^1 - L^\infty \) Hölder inequality to get

\[
\|f_n \ast g\| \leq \|f_n\| \|g\|_1.
\]

It now follows that \( \{f_n \ast g\} \) is a Cauchy sequence. It then converges to an element of \( \mathcal{A}_C \). Inequality (5) also shows this limit is independent of the choice of \( \{f_n\} \). To see that Definition 5 and Definition 6 agree, take \( \{f_n\} \subset L^1 \) with \( \|f_n - f\| \to 0 \) and \( \{g_n\} \subset BV \cap L^1 \) with \( \|g_n - g\|_1 \to 0 \). Then

\[
\|f_n \ast g - f \ast g_n\| = \|(f_n - f) \ast g - f \ast (g_n - g)\|
\]

\[
\leq \|(f_n - f) \ast g\| + \|f \ast (g_n - g)\|
\]

\[
\leq \|f_n - f\| \|g\|_1 + \|f\| \|g_n - g\|_1.
\]

Letting \( n \to \infty \) shows the limits of \( f_n \ast g \) in Definition 6 and \( f \ast g_n \) in Definition 5 are the same.

**Theorem 8** Let \( f \in \mathcal{A}_C \) and \( g \in L^1 \). Define \( f \ast g \) as in Definition 5. Then

(a) \( \|f \ast g\| \leq \|f\| \|g\|_1 \).

(b) Let \( h \in L^1 \). Then \( (f \ast g) \ast h = f \ast (g \ast h) \in \mathcal{A}_C \).
(c) For each \( z \in \mathbb{R} \), \( \tau_z(f \ast g) = (\tau_z f) \ast g = (f \ast \tau_z g) \). (d) For each \( f \in A_C \)
define \( \Phi_f : L^1 \to A_C \) by \( \Phi_f[g] = f \ast g \). Then \( \Phi_f \) is a bounded linear operator
and \( \| \Phi_f \| \leq \| f \|. \) There exists a nonzero distribution \( f \in A_C \) such that
\( \| \Phi_f \| = \| f \|. \) For each \( g \in L^1 \) define \( \Psi_g : A_C \to A_C \) by \( \Psi_g[f] = f \ast g \). Then \( \Psi_g \) is a bounded linear operator and \( \| \Psi_g \| \leq \| g \|_1 \). There exists a nonzero function \( g \in L^1 \) such that \( \| \Psi_g \| = \| g \|_{BV} \). (e) Define \( g_t(x) = g(x/t)/t \) for \( t > 0 \). Let \( a = \int_{-\infty}^{\infty} g_t(x) \, dx = \int_{-\infty}^{\infty} g \). Then \( \| f \ast g_t - af \| \to 0 \) as \( t \to 0 \). (f)
supp\((f \ast g)\) \( \subset \text{supp}(f) + \text{supp}(g) \).

**Proof:** Let \( \{g_n\} \) be as in Definition 5. (a) Since \( \| f \ast g_n \| \to \| f \ast g \| \),
equation (4) shows \( \| f \ast g \| \leq \| f \| g \|_1 \). (b) Let \( \{h_n\} \subset BV \cap L^1 \) such that

\( \| h_n - h \|_1 \to 0 \). Then \( (f \ast g) \ast h := \xi \in A_C \) such that \( \| (f \ast g) \ast h - \xi \| \to 0 \).

Since \( g \ast h \in L^1 \) there is \( \{p_n\} \subset BV \cap L^1 \) such that \( \| p_n - g \ast h \|_1 \to 0 \). Then \( f \ast (g \ast h) := \eta \in A_C \) such that \( \| f \ast p_n - \eta \| \to 0 \). Now,

\[
\| \xi - \eta \| \leq \| (f \ast g) \ast h_n - \xi \| + \| f \ast p_n - \eta \|
+ \| (f \ast g) \ast h_n - (f \ast g_n) \ast h_n \| + \| (f \ast g_n) \ast h_n - f \ast p_n \|.
\]

Using (4),

\[
\| (f \ast g) \ast h_n - (f \ast g_n) \ast h_n \| = \| (f \ast (g - g_n)) \ast h_n \|
\leq \| f \| \| g_n - g \|_1 \| h_n \|_1
\to 0 \text{ as } n \to \infty.
\]

Finally, use Theorem 2(e) and (4) to write

\[
\| (f \ast g_n) \ast h_n - f \ast p_n \| = \| f \ast (g_n \ast h_n - p_n) \|
\leq \| f \| \| g_n - g \|_1 \| h_n \|_1 + \| g \|_1 \| h_n - h \|_1
+ \| p_n - g \ast h \|_1
\to 0 \text{ as } n \to \infty.
\]

(c) The Alexiewicz norm is invariant under translation [9, Theorem 28] so \( \tau_z(f \ast g) \in A_C \). Use Theorem 2(f) to write \( \| \tau_z(f \ast g) - \tau_z(f \ast g_n) \| = \| f \ast g - f \ast g_n \| = \| \tau_z(f \ast g) - (\tau_z f) \ast g_n \| = \| \tau_z(f \ast g) - f \ast (\tau_z g_n) \|. \)

Translation invariance of the \( L^1 \) norm completes the proof. (d) From (a) we have \( \| \Phi_f \| = \sup_{\| g \|_1 = 1} \| f \ast g \| \leq \sup_{\| g \|_1 = 1} \| f \| \| g \|_1 = \| f \|. \) We get equality
by considering \( f \) and \( g \) to be positive functions in \( L^1 \). To prove \( \| \Psi_g \| \leq \| g \|_1 \),
note that \( \|\Psi_g\| = \sup_{\|f\|_1} \|f \ast g\| \leq \sup_{\|f\|_1} \|f\| \|g\|_1 = \|g\|_1 \). We get equality by considering \( f \) and \( g \) to be positive functions in \( L^1 \). (e) First consider \( g \in BV \cap L^1 \). We have
\[
f \ast g_t(x) = \int_{-\infty}^{\infty} f(x-y) g\left(\frac{y}{t}\right) \frac{dy}{t} = \int_{-\infty}^{\infty} f(x-ty)g(y) \, dy.
\]
For \(-\infty < \alpha < \beta < \infty\),
\[
\left| \int_{\alpha}^{\beta} [f \ast g_t(x) - a \, f(x)] \, dx \right| = \left| \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} [f(x-ty) - f(x)] \, g(y) \, dy \, dx \right|
\leq \int_{-\infty}^{\infty} \sup_{x/t \geq \alpha} |\tau_y f - f| \|g(y)\| \, dy
\leq 2 \|f\| \|g\|_1.
\]
By dominated convergence we can take the limit \( t \to 0 \) inside the integral (7). Continuity of \( f \) in the Alexiewicz norm then shows \( \|f \ast g_t - a \, f\| \to 0 \) as \( t \to 0 \).

Now take a sequence \( \{g^{(n)}\} \subset BV \cap L^1 \) such that \( \|g^{(n)} - g\|_1 \to 0 \). Define\( g_t^{(n)}(x) = g^{(n)}(x/t)/t \) and \( a^{(n)} = \int_{-\infty}^{\infty} g^{(n)}(x) \, dx \). We have
\[
\|f \ast g_t - a \, f\| \leq \|f \ast g_t^{(n)} - a^{(n)} f\| + \|f \ast g_t^{(n)} - f \ast g_t\| + \|a^{(n)} f - a \, f\|. \tag{8}
\]
By the inequality in (a), \( \|f \ast g_t^{(n)} - f \ast g_t\| \leq \|f\| \|g_t^{(n)} - g_t\|_1 \). Whereas,
\[
\|g_t^{(n)} - g_t\|_1 = \int_{-\infty}^{\infty} \left| g^{(n)}\left(\frac{x}{t}\right) - g\left(\frac{x}{t}\right) \right| \frac{dx}{t} = \|g^{(n)} - g\|_1 \to 0 \text{ as } n \to \infty.
\]
And, \( \|a^{(n)} f - a \, f\| = \|a^{(n)} - a\| \|f\| = \|g^{(n)} - g\|_1 \|f\| \). Given \( \epsilon > 0 \) fix \( n \) large enough so that \( \|f \ast g_t^{(n)} - f \ast g_t\| + \|a^{(n)} f - a \, f\| < \epsilon \). Now let \( t \to 0 \) in (8).

The interchange of order of integration in (6) is justified as follows. A change of variables and Proposition 17 give
\[
\int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f(x-ty)g(y) \, dy \, dx = \int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f(y)g\left(\frac{x-y}{t}\right) \frac{dy}{t} \, dx
= \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} f(y)g\left(\frac{x-y}{t}\right) \frac{dx}{t} \, dy.
\]
And,
\[
\int_{-\infty}^{\infty} \int_{\alpha}^{\beta} f(x-ty)g(y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \, \frac{g(y)}{t} \chi_{(\alpha-y,\beta-y)}(x) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \, \frac{g(y)}{t} \chi_{(\alpha-y,\beta-y)}(x) \, dy \, dx
\]
\[
= \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} f(x) \, \frac{g(y-x)}{t} \, dy \, dx.
\]

Note that \(\int_{\alpha}^{\beta} \int_{-\infty}^{\infty} f(x)g(y) \, dy \, dx = \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} f(x)g(y) \, dx \, dy\) by Corollary 18.

Young’s inequality states that \(\|f * g\|_p \leq \|f\|_p \|g\|_1\) when \(f \in L^p\) for some \(1 \leq p \leq \infty\) and \(g \in L^1\). Part (a) of Theorem 8 extends this to \(f \in \mathcal{A}_C\). See [4] for other results when \(f \in L^p\).

The fact that convolution is linear in both arguments, together with (b), shows that \(\mathcal{A}_C\) is an \(L^1\)-module over the \(L^1\) convolution algebra. See [3] for the definition. It does not appear that \(\mathcal{A}_C\) is a Banach algebra under convolution.

We now show that Definition 1(iii) and the above definitions agree.

**Proposition 9** Let \(f \in \mathcal{A}_C\), \(g \in L^1\) and \(\phi \in \mathcal{D}\). Define \(F(y) = \int_{-\infty}^{y} f\) and \(G(x) = \int_{-\infty}^{x} g\). Definition 1 and Definition 5 both give
\[
\langle f * g, \phi \rangle = \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x) \phi(x+y) \, dx \, dy \quad (9)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y) G(x) \phi''(x+y) \, dx \, dy. \quad (10)
\]

**Proof:** Let \(\Phi(y) = \int_{-\infty}^{\infty} g(x) \phi(x+y) \, dx\). Then \(\Phi \in C^\infty(\mathbb{R})\) and \(\Phi'(y) = \int_{-\infty}^{\infty} g(x) \phi'(x+y) \, dx\). And, \(\int_{-\infty}^{\infty} |\Phi'(y)| \, dy \leq \int_{-\infty}^{\infty} |g(x)| \int_{-\infty}^{\infty} |\phi'(x+y)| \, dy \, dx \leq \|g\|_{1} \|\phi'\|_{1}\) so \(\Phi \in AC(\overline{\mathbb{R}})\). Dominated convergence then shows \(\lim_{|y| \to \infty} \Phi(y) = 0\). Integration by parts now gives (9) and (10).
Therefore, by Theorem 2(d),
implies convergence in $D$, 
Convolutions for continuous primitive integral

If $g$ is not continuous at 0. If 1

For $0 < x < 1$ we have $f \ast g(x) = \int_0^y y^{-\alpha} - x^{-\alpha} dy = x^{1-2\alpha} \int_0^1 y^{-\alpha} (1-y)^{-\alpha} dy = x^{1-2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(2-2\alpha)}$. Hence, $f \ast g$ is not continuous at 0. If $1/2 < \alpha < 1$ then $f \ast g$ is unbounded at 0.

As another example, consider $f(x) = \sin(\pi x)/|x|$ and $g(x) = \chi_{(0,1)}(x)$. Then $f \in A_C$ and for each $1 \leq p < \infty$ we have $g \in BV \cap L^p$. And,

Therefore, by Theorem 2(d), $f \ast g \in C^0(\mathbb{R})$ and $\lim_{|x| \to \infty} f \ast g(x) = 0$ but for each $1 \leq p < \infty$ we have $f \ast g \not\in L^p$.

4 Differentiation and integration

If $g$ is sufficiently smooth then the pointwise derivative is $(f \ast g)'(x) = f \ast g'(x)$. Recall the definition $AC(\mathbb{R})$ of primitives of $L^1$ functions given in the proof
of Proposition 7. In the following theorem we require pointwise derivatives of \( g \) to exist at each point in \( \mathbb{R} \).

**Theorem 10** Let \( f \in \mathcal{A}_C \), \( n \in \mathbb{N} \) and \( g^{(k)} \in AC(\mathbb{R}) \) for each \( 0 \leq k \leq n \). Then \( f * g \in C^n(\mathbb{R}) \) and \( (f * g)^{(n)}(x) = f * g^{(n)}(x) \) for each \( x \in \mathbb{R} \).

**Proof:** First consider \( n = 1 \). Let \( x \in \mathbb{R} \). Then

\[
(f * g)'(x) = \lim_{h \to 0} \int_{-\infty}^{\infty} f(y) \left[ \frac{g(x + h - y) - g(x - y)}{h} \right] dy. \tag{11}
\]

To take the limit inside the integral we can show that the bracketed term in the integrand is of uniform bounded variation for \( 0 < |h| \leq 1 \). Let \( h \neq 0 \). Since \( g \in AC(\mathbb{R}) \) it follows that the variation is given by the Lebesgue integrals

\[
V_{y \in \mathbb{R}} \left[ \frac{g(x + h - y) - g(x - y)}{h} \right] = \int_{-\infty}^{\infty} \left| \frac{g'(x + h - y) - g'(x - y)}{h} \right| dy
\]

with

\[
\leq \int_{-\infty}^{\infty} |g''(y)| dy + \int_{-\infty}^{\infty} \left| \frac{g'(x + h - y) - g'(x - y)}{h} - g''(x - y) \right| dy. \tag{12}
\]

Since \( g' \in AC(\mathbb{R}) \) we have \( g'' \in L^1 \). The second integral on the right of (12) gives the \( L^1 \) derivative of \( g' \) in the limit \( h \to 0 \). See [4, p. 246]. Hence, in (11) we can use Theorem 22 in [9] to take the limit under the integral sign. This then gives \( (f * g)'(x) = f * g'(x) \). Theorem 2(d) now shows \( (f * g)' \in C^0(\mathbb{R}) \). Induction on \( n \) completes the proof. \( \blacksquare \)

Note that \( g' \in AC(\mathbb{R}) \) does not imply \( g \in AC(\mathbb{R}) \). For example, \( g(x) = x \). The conditions \( g^{(k)} \in BV \) for \( 0 \leq k \leq n+1 \) imply those in Theorem 10. To see this it suffices to consider \( n = 1 \). If \( g', g'' \in BV \) then \( g'' \) exists at each point and is bounded. Hence, the Lebesgue integral \( g'(x) = g'(0) + \int_0^x g''(y) dy \) exists for each \( x \in \mathbb{R} \) and \( g' \) is absolutely continuous. Since \( g' \in BV \) we then have \( g' \in AC(\mathbb{R}) \). Similarly for \( n > 1 \). The example \( g(x) = |x|^{1.5} \sin(1/[1 + x^2]) \) shows the \( AC(\mathbb{R}) \) condition in the theorem is weaker than the above \( BV \) condition since \( g, g' \in AC(\mathbb{R}) \) but \( g''(0) \) does not exist so \( g'' \notin BV \).

We found that when \( g \in BV \cap L^1 \) then \( f * g \in \mathcal{A}_C \). We can compute the the distributional derivative \( (F * g)' = f * g \) where \( F \) is a primitive of \( f \).

**Proposition 11** Let \( F \in C^0(\mathbb{R}) \) and write \( f = F' \in \mathcal{A}_C \). Let \( g \in BV \cap L^1 \). Then \( F * g \in C^0(\mathbb{R}) \) and \( (F * g)' = f * g \in \mathcal{A}_C \).
Proof: Let \( x, t \in \mathbb{R} \). Then by the usual Hölder inequality,

\[
|F * g(x) - F * g(t)| = \left| \int_{-\infty}^{\infty} [F(x - y) - F(t - y)] g(y) dy \right| 
\leq \|F(x - \cdot) - F(t - \cdot)\|_{\infty} \|g\|_1
\to 0 \quad \text{as } t \to x \quad \text{since } F \text{ is uniformly continuous on } \mathbb{R}.
\]

Hence, \( F * g \) is continuous on \( \mathbb{R} \). Dominated convergence shows that \( \lim_{x \to \pm\infty} F * g(x) = F(\pm\infty) \int_{-\infty}^{\infty} g \). Therefore, \( F * g \in C^0(\mathbb{R}) \).

Let \( \phi \in D \). Then

\[
\langle (F * g)', \phi \rangle = -\langle F * g, \phi' \rangle = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x - y) g(y) \phi'(x) dy dx
= -\int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} F(x - y) \phi'(x) dx dy \quad \text{(Fubini–Tonelli theorem)}.
\]

Integrate by parts and use the change of variables \( x \mapsto x + y \) to get

\[
\langle (F * g)', \phi \rangle = \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(x) \phi(x + y) dx dy
= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(y) \phi(x + y) dy dx \quad \text{(by Proposition 17)}
= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(y - x) \phi(y) dy dx
= \int_{-\infty}^{\infty} \phi(y) \int_{-\infty}^{\infty} f(x) g(y - x) dx dy \quad \text{(by Proposition 17)}
= \langle f * g, \phi \rangle.
\]

This gives an alternate definition of \( f * g \) for \( f \in A_C \) and \( g \in L^1 \).

**Theorem 12** Let \( f \in A_C \), let \( F \in B_C \) be the primitive of \( f \) and let \( g \in L^1 \). Define \( f * g \) as in Definition 5. Then \( (F * g)' = f * g \in A_C \).

**Proof:** Let \(-\infty < \alpha < \beta < \infty\). Let \( \{g_n\} \subset BV \cap L^1 \) such that \( \|g_n - g\|_1 \to 0 \). By Proposition 11 we have

\[
\int_{\alpha}^{\beta} (F * g)' = F * g(\beta) - F * g(\alpha) = \int_{-\infty}^{\infty} F(y) [g(\beta - y) - g(\alpha - y)] dy.
\]
As in (3), \( \int_\alpha^\beta f * g_n = \int_{-\infty}^\infty F(y) [g_n(\beta - y) - g_n(\alpha - y)] \, dy \). Hence,

\[
\left| \int_\alpha^\beta [(F * g)' - f * g_n] \right|
\leq \|F\|_\infty (\|g(\beta - \cdot) - g_n(\beta - \cdot)\|_1 + \|g(\alpha - \cdot) - g_n(\alpha - \cdot)\|_1)
\leq 2\|f\|\|g_n - g\|_1.
\]

Therefore, \( \| (F * g)' - f * g_n \| \leq 2\|f\|\|g_n - g\|_1 \to 0 \) as \( n \to \infty \).

The next theorem and its corollary give results on integrating convolutions.

**Theorem 13** Let \( f \in A_C \) and let \( g \in L^1 \). Define \( F(x) = \int_{-\infty}^x f \) and \( G(x) = \int_{-\infty}^x g \). Then \( f * G \in C^0(\mathbb{R}) \) and \( f * G(x) = F * g(x) \) for all \( x \in \mathbb{R} \).

**Proof:** Since \( G \in AC(\mathbb{R}) \), Theorem 2(d) shows \( f * G \in C^0(\mathbb{R}) \). We have

\[
 f * G(x) = \int_{-\infty}^x f(y) \int_{-\infty}^{x-y} g(z) \, dz \, dy \\
= \int_{-\infty}^\infty \int_{-\infty}^{\infty} f(y) \chi_{(-\infty,x-y)}(z) g(z) \, dz \, dy \\
= \int_{-\infty}^\infty \int_{-\infty}^{\infty} f(y) \chi_{(-\infty,x-y)}(z) g(z) \, dy \, dz \\
= \int_{-\infty}^\infty g(z) \int_{-\infty}^{x-z} f(y) \, dy \, dz \\
= F * g(x).
\]

Proposition 17 justifies the interchange of orders of integration. ■

**Corollary 14** (a) \( f * g = (F * g)' = (f * G)' \) (b) For all \( -\infty \leq \alpha < \beta \leq \infty \) we have \( \int_\alpha^\beta f * g = F * g(\beta) - F * g(\alpha) = f * G(\beta) - f * G(\alpha) \).

Hence, the convolution \( f * g \) can be evaluated by taking the distributional derivative of the Lebesgue integral \( F * g \). Since \( f * G \in C^0(\mathbb{R}) \) when \( f \in A_C \) and \( G \in BV \) we can use the equation \( f * g = (f * G)' \) to define \( f * g \) for \( f \in A_C \) and \( g = G' \) for \( G \in BV \). In this case, \( g \) will be a signed Radon measure. As \( G(x) = \int_{-\infty}^x g \) and this integral is a regulated primitive integral \([10]\), we will save this case for discussion elsewhere.
5 Appendix

The integration by parts formula is as follows. If \( f \in A_C \) and \( g \in BV \) it gives the integral of \( fg \) in terms of a Henstock–Stieltjes integral:

\[
\int_{-\infty}^{\infty} fg = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F dg.
\] (13)

See [9] and [6, p. 199].

We have the following corollary for functions of essential bounded variation.

**Corollary 15** Let \( F \in C^0(\mathbb{R}) \). Let \( g \in \mathcal{EBV} \). Fix \( 0 \leq \gamma \leq 1 \). Take \( g_\gamma \in NBV_\gamma \) such that \( g_\gamma = g \) almost everywhere. Let \( \mu_g \) be the signed Radon measure given by \( g' \). Then \( \int_{-\infty}^{\infty} F dg_\gamma = \int_{-\infty}^{\infty} F d\mu_g \).

**Proof:** The distributional derivative of \( g \) is \( \langle g', \phi \rangle = -\langle g, \phi' \rangle = -\int_{-\infty}^{\infty} g \phi' = \int_{-\infty}^{\infty} \phi d\mu_g \) for all \( \phi \in \mathcal{D} \). Note that \( g_\gamma \) is unique and \( \mu_g = \mu_{g_\gamma} \). Suppose \( \phi \in \mathcal{D} \) with \( \text{supp}(\phi) \subset [A, B] \subset \mathbb{R} \). Then, using integration by parts for the Henstock–Stieltjes integral,

\[
\langle g_\gamma, \phi' \rangle = \int_{A}^{B} g_\gamma \phi' = g_\gamma(B)\phi(B) - g_\gamma(A)\phi(A) - \int_{A}^{B} \phi dg_\gamma = -\int_{A}^{B} \phi d\mu_{g_\gamma} = -\int_{-\infty}^{\infty} \phi d\mu_g.
\]

Let \( F \in C^0(\mathbb{R}) \). There is a uniformly bounded sequence \( \{\phi_n\} \subset \mathcal{D} \) such that \( \phi_n \to F \) pointwise on \( \mathbb{R} \). By dominated convergence,

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} \phi_n dg_\gamma = \int_{-\infty}^{\infty} F dg_\gamma = \lim_{n \to \infty} \int_{-\infty}^{\infty} \phi_n d\mu_g = \int_{-\infty}^{\infty} F d\mu_g. \blacksquare
\]

Corollary 15 now justifies the following definition.

**Definition 16** Let \( f \in A_C \) and let \( F \in B_C \) be its primitive. Let \( g \in \mathcal{EBV} \). Fix \( 0 \leq \gamma \leq 1 \) and take \( g_\gamma \in NBV_\gamma \) such that \( g_\gamma = g \) almost everywhere. Define

\[
\int_{-\infty}^{\infty} fg = g_\gamma(\infty)F(\infty) - \int_{-\infty}^{\infty} F d\mu_g = \int_{-\infty}^{\infty} fg_\gamma.
\]
Since limits at infinity are not affected by the choice of $\gamma$, the definition is independent of $\gamma$.

The Hölder inequality is
\[
\left| \int_{-\infty}^{\infty} f g \right| \leq \left| \int_{-\infty}^{\infty} f \right| \inf_R |g| + \|f\| V g \leq \|f\| \|g\|_{BV}
\] (14)
and is valid for all $f \in A_C$ and $g \in BV$. For $g \in EBV$ we replace $g$ with $g_{\gamma}$. This gives
\[
\left| \int_{-\infty}^{\infty} f g \right| \leq \left| \int_{-\infty}^{\infty} f \right| \inf_R |g_{\gamma}| + \|f\| V g_{\gamma} \leq \|f\| \|g\|_{EBV}.
\] (15)

See [8, Lemma 24] for a proof using the Henstock–Kurzweil integral. The same proof works for the continuous primitive integral.

A Fubini theorem has been established in [1] for the continuous primitive integral on compact intervals. This says that if a double integral exists in the plane then the two iterated integrals exist and are equal. Of more utility for the case at hand is to show directly that iterated integrals are equal without resorting to the double integral. The following theorem extends a type of Fubini theorem proved on page 58 in [2] for the wide Denjoy integral on compact intervals.

**Proposition 17** Let $f \in A_C$. Let $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable. Assume (i) for each $x \in \mathbb{R}$ the function $y \mapsto g(x, y)$ is in $BV$; (ii) the function $x \mapsto V_{y \in \mathbb{R}} g(x, y)$ is in $L^1$; (iii) there is $M \in L^1$ such that for each $y \in \mathbb{R}$ we have $|g(x, y)| \leq M(x)$. Then the iterated integrals exist and are equal,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x, y) \, dy \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x, y) \, dx \, dy.
\]

**Proof:** Let $F \in B_C$ be the primitive of $f$. For each $x \in \mathbb{R}$ we have
\[
\int_{-\infty}^{\infty} f(y)g(x, y) \, dy = F(\infty) g(x, \infty) - \int_{-\infty}^{\infty} F(y) \, d_2 g(x, y)
\]
where $d_2(x, y)$ indicates a Henstock–Stieltjes integral with respect to $y$. Then,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x, y) \, dy \, dx = F(\infty) \int_{-\infty}^{\infty} g(x, \infty) \, dx
\]
\[
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y) \, d_2 g(x, y) \, dx.
\] (16)
The integral \( \int_{-\infty}^{\infty} g(x, \infty) \, dx \) exists due to condition (iii). The iterated integral in (16) converges absolutely since

\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y) \, d_2 g(x, y) \, dx \right| \leq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F(y) \, d_2 g(x, y) \right| \, dx \\
\leq \|F\| \int_{-\infty}^{\infty} V_{y \in \mathbb{R}} g(x, y) \, dx.
\]

Now, show the function \( y \mapsto \int_{-\infty}^{\infty} g(x, y) \, dx \) is in \( BV \). Let \( \{(s_i, t_i)\}_{i=1}^{n} \) be disjoint intervals in \( \mathbb{R} \). Then

\[
\sum_{i=1}^{n} \left| \int_{-\infty}^{\infty} g(x, s_i) \, dx - \int_{-\infty}^{\infty} g(x, t_i) \, dx \right| \leq \sum_{i=1}^{n} \int_{-\infty}^{\infty} |g(x, s_i) - g(x, t_i)| \, dx \\
= \int_{-\infty}^{\infty} \sum_{i=1}^{n} |g(x, s_i) - g(x, t_i)| \, dx \\
\leq \int_{-\infty}^{\infty} V_{y \in \mathbb{R}} g(x, y) \, dx.
\]

The interchange of summation and integration follows from condition (ii) and the usual Fubini–Tonelli theorem. Hence, the function \( y \mapsto \int_{-\infty}^{\infty} g(x, y) \, dx \) is in \( BV \) and the iterated integral \( \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x, y) \, dx \, dy \) exists.

Integrate by parts,

\[
\int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} g(x, y) \, dx \, dy = F(\infty) \int_{-\infty}^{\infty} g(x, \infty) \, dx \\
- \int_{-\infty}^{\infty} F(y) \, d \left[ \int_{-\infty}^{\infty} g(x, y) \, dx \right]. \tag{17}
\]

In (17), we have \( \lim_{y \to \infty} \int_{-\infty}^{\infty} g(x, y) \, dx = \int_{-\infty}^{\infty} g(x, \infty) \, dx \) due to dominated convergence and condition (iii). To complete the proof we need to show the integrals in (16) and (18) are equal. First consider the case when \( F = \chi_{(a,b)} \) for an interval \( (a, b) \subset \mathbb{R} \). Then (16) becomes \( \int_{-\infty}^{\infty} \int_{a}^{b} d_2 g(x, y) \, dx = \int_{-\infty}^{\infty} [g(x, b) - g(x, a)] \, dx \). And now (18) becomes \( \int_{a}^{b} d \left[ \int_{-\infty}^{\infty} g(x, y) \, dx \right] = \int_{a}^{b} g(x, b) \, dx - \int_{a}^{b} g(x, a) \, dx \). Hence, when \( F \) is a step function, \( F(y) = \sum_{i=1}^{n} c_i \chi_{I_i}(y) \) for some \( n \in \mathbb{N} \), disjoint intervals \( \{I_i\}_{i=1}^{n} \) and real numbers \( \{c_i\}_{i=1}^{n} \), we have the desired equality of (16) and (18). But \( F \in \mathcal{B}_C \) is uniformly continuous on \( \overline{\mathbb{R}} \), i.e., for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that for
all $0 \leq |x - y| < \delta$ we have $|F(x) - F(y)| < \epsilon$, for all $x < -1/\delta$ we have $|F(x)| < \epsilon$ and for all $x > 1/\delta$ we have $|F(x) - F(\infty)| < \epsilon$. It then follows from the compactness of $\mathbb{R}$ that the step functions are dense in $B_C$. Hence, there is a sequence of step functions $\{\sigma_N\}$ such that $\|F - \sigma_N\|_\infty \to 0$. In (16) we have

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma_N(y) \ d_2g(x, y) \ dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y) \ d_2g(x, y) \ dx.$$ 

The $N$ limit can be brought inside the $x$ integral using dominated convergence and (ii) since $|\int_{-\infty}^{\infty} \sigma_N(y) \ d_2g(x, y)| \leq (\|F\|_\infty + 1)V_{y \in \mathbb{R}} g(x, y)$ for large enough $N$. The $N$ limit can be brought inside the $y$ integral using dominated convergence since $|\sigma_N(y)| \leq (\|F\|_\infty + 1)$ for large enough $N$. In (18) we have

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \sigma_N(y) \ d \left[ \int_{-\infty}^{\infty} g(x, y) \ dx \right] = \int_{-\infty}^{\infty} F(y) \ d \left[ \int_{-\infty}^{\infty} g(x, y) \ dx \right].$$

The $N$ limit can be brought inside the $y$ integral since $\{\sigma_N\}$ converges to $F$ uniformly on $\mathbb{R}$ and $d \left[ \int_{-\infty}^{\infty} g(x, y) \ dx \right]$ is a finite signed measure. □

**Corollary 18** If $f$ has compact support we can replace (iii) with: (iv) for each $y \in \text{supp}(f)$ the function $x \mapsto g(x, y)$ is in $L^1$.

**References**


