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A PRODUCT CONVERGENCE THEOREM FOR HENSTOCK-KURZWEIL INTEGRALS

Abstract

Necessary and sufficient for \( \int_a^b f g_n \to \int_a^b f g \) for all Henstock–Kurzweil integrable functions \( f \) is that \( g \) be of bounded variation, \( g_n \) be uniformly bounded and of uniform bounded variation and, on each compact interval in \((a, b)\), \( g_n \to g \) in measure or in the \( L^1 \) norm. The same conditions are necessary and sufficient for \( \|f(g_n - g)\| \to 0 \) for all Henstock–Kurzweil integrable functions \( f \). If \( g_n \to g \) a.e., then convergence \( \|f g_n\| \to \|f g\| \) for all Henstock–Kurzweil integrable functions \( f \) is equivalent to \( \|f(g_n - g)\| \to 0 \). This extends a theorem due to Lee Peng-Yee.

Let \(-\infty \leq a < b \leq \infty \) and denote the Henstock–Kurzweil integrable functions on \((a, b)\) by \( \mathcal{H}K \). The Alexiewicz norm of \( f \in \mathcal{H}K \) is \( \|f\| = \sup_I |\int_I f| \) where the supremum is taken over all intervals \( I \subset (a, b) \). If \( g \) is a real-valued function on \([a, b]\), we write \( V_{[a, b]} g \) for the variation of \( g \) over \([a, b]\), dropping the subscript when the identity of \([a, b]\) is clear. The set of functions of normalized bounded variation, \( \mathcal{N}BV \), consists of the functions on \([a, b]\) that are of bounded variation, are left continuous and vanish at \( a \). It is known that the multipliers for \( \mathcal{H}K \) are \( \mathcal{N}BV \); i.e., \( f g \in \mathcal{H}K \) for all \( f \in \mathcal{H}K \) if and only if \( g \) is equivalent to a function in \( \mathcal{N}BV \). This paper is concerned with necessary and sufficient conditions under which \( \int_a^b f g_n \to \int_a^b f g \) for all \( f \in \mathcal{H}K \). One such set of conditions was given by Lee Peng-Yee in [2, Theorem 12.11]. If \( g \) is of bounded variation, changing \( g \) on a countable set will make it an element of \( \mathcal{N}BV \). With this observation, a minor modification of Lee’s theorem produces the following result.

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Theorem 1. [2, Theorem 12.11] Let $-\infty < a < b < \infty$, let $g_n$ and $g$ be real-valued functions on $[a, b]$ with $g$ of bounded variation. In order for $\int_a^b fg_n \to \int_a^b fg$ for all $f \in HK$ it is necessary and sufficient that

\[
\begin{align*}
&\text{for each interval } (c, d) \subset (a, b), \int_c^d g_n \to \int_c^d g \text{ as } n \to \infty, \\
&\text{for each } n \geq 1, g_n \text{ is equivalent to a function } h_n \in NBV, \\
&\text{and there is } Vh_n \leq M \text{ for all } n \geq 1.
\end{align*}
\]

(1)

We extend this theorem to unbounded intervals, show that the condition $\int_a^d g_n \to \int_a^d g$ in (1) can be replaced by $g_n \to g$ on each compact interval in $(a, b)$ either in measure or in the $L^1$ norm, and that this also lets us conclude $\|f(g_n - g)\| \to 0$. We also show that if $g_n \to g$ in measure or almost everywhere, then $\|fg_n\| \to \|fg\|$ for all $f \in HK$ if and only if $\|fg_n - fg\| \to 0$ for all $f \in HK$.

One might think the conditions (1) imply $g_n \to g$ almost everywhere. This is not the case, as is illustrated by the following example [1, p. 61].

Example 2. Let $g_n = \chi_{(j2^{-k}, (j+1)2^{-k})}$ where $0 \leq j < 2^k$ and $n = j + 2^k$. Note that $\|g_n\|_\infty = 1$, $g_n \in NBV$, $Vg_n \leq 2$, and $\int_a^d g_n \leq \|g_n\| = 2^{-k} < 2/n \to 0$, so that (1) is satisfied with $g = 0$. For each $x \in (0, 1)$ we have $\inf_n g_n(x) = 0$, $\sup_n g_n(x) = 1$, and for no $x \in [0, 1]$ does $g_n(x)$ have a limit. However, $g_n \to 0$ in measure since if $T_n = \{x \in [0, 1] : |g_n(x)| > \epsilon\}$, then for each $0 < \epsilon \leq 1$, we have $\lambda(T_n) < 2/n \to 0$ as $n \to \infty$ ($\lambda$ is Lebesgue measure).

We have the following extension of Theorem 1.

Theorem 3. Let $[a, b]$ be a compact interval in $\mathbb{R}$, let $g_n$ and $g$ be real-valued functions on $[a, b]$ with $g$ of bounded variation. In order for $\int_a^b fg_n \to \int_a^b fg$ for all $f \in HK$ it is necessary and sufficient that

\[
\begin{align*}
&g_n \to g \text{ in measure as } n \to \infty, \\
&\text{for each } n \geq 1, g_n \text{ is equivalent to a function } h_n \in NBV, \\
&\text{and there is } M \in [0, \infty) \text{ such that } Vh_n \leq M \text{ for all } n \geq 1.
\end{align*}
\]

(2)

If $(a, b) \subset \mathbb{R}$ is unbounded, then change the first line of (2) by requiring $g_n \chi_I \to g\chi_I$ in measure for each compact interval $I \in (a, b)$.

Proof. By working with $g_n - g$ we can assume $g = 0$. First consider the case when $(a, b)$ is a bounded interval. If $\int_a^b fg_n \to 0$ for all $f \in HK$, then using Theorem 1 and changing $g_n$ on a countable set, we can assume $g_n \in NBV$, $Vg_n \leq M$, $\|g_n\|_\infty \leq M$ and $\int_a^d g_n \to 0$ for each interval $(c, d) \subset (a, b)$. Suppose $g_n$ does not converge to 0 in measure. Then there are $\delta, \epsilon > 0$ and
an infinite index set $\mathcal{J} \subset \mathbb{N}$ such that $\lambda(S_n) > \delta$ for each $n \in \mathcal{J}$, where $S_n = \{x \in (a, b) : g_n(x) > \epsilon\}$. (Or else there is a corresponding set on which $g_n(x) < -\epsilon$ for all $n \in \mathcal{J}$.) Now let $n \in \mathcal{J}$. Since $g_n$ is left continuous, if $x \in S_n$, there is a number $c_{n,x} > 0$ such that $[x - c_{n,x}, x] \subset S_n$. Hence, $V_n := \{[c, x] : x \in S_n \text{ and } [c, x] \subset S_n\}$ is a Vitali cover of $S_n$. So there is a finite set of disjoint closed intervals $W$ such that $\lambda(V_n \setminus \bigcup_{I \in \sigma_n} I) < \delta/2$. Write $(a, b) \setminus \bigcup_{I \in \sigma_n} I = \bigcup_{I \in \tau_n} I$ where $\tau_n$ is a set of disjoint open intervals with $\text{card}(\tau_n) = \text{card}(\sigma_n) + 1$. Let

$$P_n = \text{card}\{(I \in \tau_n : g_n(x) \leq \epsilon/2 \text{ for some } x \in I)\}.$$  

Each interval $I$ in $\tau_n$ that does not have $a$ or $b$ as an endpoint has contiguous intervals on its left and right that are in $\sigma_n$ (for each of which $g_n > \epsilon$). The interval $I$ then contributes more than $(\epsilon - \epsilon/2) + (\epsilon - \epsilon/2) = \epsilon$ to the variation of $g_n$. If $I$ has $a$ as an endpoint, then since $g_n(a) = 0$, $I$ contributes more than $\epsilon$ to the variation of $g_n$. If $I$ has $b$ as an endpoint, then $I$ contributes more than $\epsilon/2$ to the variation of $g_n$. Hence,

$$Vg_n \geq (P_n - 1)\epsilon + \epsilon/2 = (P_n - 1/2)\epsilon.$$  

(This inequality is still valid if $P_n = 1$.) But, $Vg_n \leq M$; so $P_n \leq P$ for all $n \in \mathcal{J}$ and some $P \in \mathbb{N}$. Then we have a set of intervals, $U_n$, formed by taking unions of intervals from $\sigma_n$ and those intervals in $\tau_n$ on which $g_n > \epsilon/2$. Now, $\lambda(\bigcup_{I \in U_n} I) > \delta/2$, $\text{card}(U_n) \leq P + 1$ and $g_n > \epsilon/2$ on each interval $I \in U_n$. Therefore, there is an interval $I_n \in U_n$ such that $\lambda(I_n) > \delta/[2(P + 1)]$. The sequence of centers of intervals $I_n$ has a convergent subsequence. There is then an infinite index set $\mathcal{J}' \subset \mathcal{J}$ with the property that for all $n \in \mathcal{J}'$ we have $g_n > \epsilon/2$ on an interval $I \subset (a, b)$ with $\lambda(I) > \delta/[3(P + 1)]$. Hence, $\limsup_{n \to \infty} \int_I g_n \geq \delta\epsilon/[6(P + 1)]$. This contradicts the fact that $\int_I g_n \to 0$, showing that indeed $g_n \to 0$ in measure.

Suppose (2) holds. As above, we can assume $g_n \in \mathcal{N}BV$, $Vg_n \leq M$, $\|g_n\|_{\infty} \leq M$ and $g_n \to 0$ in measure. Let $\epsilon > 0$. Define

$$T_n = \{x \in (a, b) : |g_n(x)| > \epsilon\}.$$  

Then

$$\int_a^b g_n \leq \int_{T_n} |g_n| + \int_{(a, b) \setminus T_n} |g_n| \leq M\lambda(T_n) + \epsilon(b - a).$$  

Since $\lim \lambda(T_n) = 0$, it now follows that $\int_c^d g_n \to 0$ for each $(c, d) \subset (a, b)$. Theorem 1 now shows $\int_a^b f g_n \to 0$ for all $f \in \mathcal{H}K$. 

Now consider integrals on \( \mathbb{R} \). If \( \int_{-\infty}^{\infty} fg_n \to 0 \) for all \( f \in \mathcal{H} \mathcal{K} \), then it is necessary that \( \int_{a}^{b} fg_n \to 0 \) for each compact interval \([a, b]\). By the current theorem, \( g_n \to g \) in measure on each \([a, b]\). And, it is necessary that \( \int_{-\infty}^{\infty} fg_n \to 0 \). The change of variables \( x \mapsto 1/x \) now shows it is necessary that \( g_n \) be equivalent to a function that is uniformly bounded and of uniform bounded variation on \([1, \infty]\). Similarly with \( \int_{-\infty}^{1} fg_n \to 0 \). Hence, it is necessary that \( g_n \) be uniformly bounded and of uniform bounded variation on \( \mathbb{R} \).

Suppose \( (2) \) holds with \( g_n \to g \) in measure on each compact interval \([a, b]\). Write \( \mathcal{L}_{\infty} \mathcal{K} = \int_{a}^{\infty} f g_n + \int_{-\infty}^{b} f g_n + \int_{b}^{\infty} f g_n \). Use Lemma 24 in [4] to write \( |\int_{-\infty}^{\infty} f g_n| \leq \| f \chi_{(-\infty, a]} \| V_{[-\infty, a]} g_n \leq \| f \chi_{(-\infty, a]} \| M \to 0 \) as \( a \to -\infty \). We can then take a large enough interval \([a, b] \subset \mathbb{R} \) and apply the current theorem on \([a, b]\). Other unbounded intervals are handled in a similar manner.

Remark 4. If \((2)\) holds, then dominated convergence shows \( \| g_n - g \|_{1} \to 0 \). And, convergence in \( \| \cdot \|_{1} \) implies convergence in measure. Therefore, in the first statement of \((2)\) and in the last statement of Theorem 3, ‘convergence in measure’ can be replaced with ‘convergence in \( \| \cdot \|_{1} \)’. Similar remarks apply to Theorem 6.

Remark 5. The change of variables argument in the second last paragraph of Theorem 3 can be replaced with an appeal to the Banach–Steinhaus Theorem on unbounded intervals. See [3, Lemma 7]. A similar remark holds for the proof of Theorem 8.

The sequence of Heaviside functions \( g_n = \chi_{(n, \infty]} \) shows \((2)\) is not necessary to have \( \int_{-\infty}^{\infty} fg_n \to 0 \) for all \( f \in \mathcal{H} \mathcal{K} \). For then, \( \mathcal{L}_{\infty} \mathcal{K} = \mathcal{L}^{\infty} f \to 0 \). In this case, \( g_n \in \mathcal{N} \mathcal{B} \mathcal{V} \) and \( V g_n = 1 \). However, \( \lambda(T_n) = \infty \) for all \( 0 < \epsilon < 1 \). Note that for each compact interval \([a, b]\) we have \( \int_{a}^{b} g_n \to 0 \) and \( g_n \to 0 \) in measure on \([a, b]\).

It is somewhat surprising that condition \((2)\) is also necessary and sufficient to have \( \| f(g_n - g) \| \to 0 \) for all \( f \in \mathcal{H} \mathcal{K} \).

**Theorem 6.** Let \([a, b]\) be a compact interval in \( \mathbb{R} \), let \( g_n \) and \( g \) be real-valued functions on \([a, b]\) with \( g \) of bounded variation. In order for \( \| f(g_n - g) \| \to 0 \) for all \( f \in \mathcal{H} \mathcal{K} \) it is necessary and sufficient that

\[
\begin{align*}
g_n \to g & \text{ in measure as } n \to \infty, \\
\text{for each } n \geq 1, g_n & \text{ is equivalent to a function } h_n \in \mathcal{N} \mathcal{B} \mathcal{V}, \\
\text{and there is } M & \in [0, \infty) \text{ such that } V h_n \leq M \text{ for all } n \geq 1.
\end{align*}
\]

If \((a, b) \subset \mathbb{R} \) is unbounded, then change the first line of \((3)\) by requiring \( g_n \chi_{I} \to g \chi_{I} \) in measure for each compact interval \( I \in (a, b) \).
PROOF. Certainly (3) is necessary in order for \( \| f(g_n - g) \| \to 0 \) for all \( f \in \mathcal{HK} \).

If we have (3), let \( I_n \) be any sequence of intervals in \((a, b)\). We can again assume \( g = 0 \). Write \( \tilde{g}_n = g_n \chi_{I_n} \). Then

\[
\| \tilde{g}_n \|_\infty \leq \| g_n \|_\infty, \quad V \tilde{g}_n \leq V g_n + 2 \| g_n \|_\infty \quad \text{and} \quad \tilde{g}_n \to 0 \quad \text{in measure}.
\]

The result now follows by applying Theorem 3 to \( f \tilde{g}_n \).

Unbounded intervals are handled as in Theorem 3.

By combining Theorem 3 and Theorem 6 we have the following.

**Theorem 7.** Let \((a, b) \subset \mathbb{R}\). Then \( \int_a^b f g_n \to \int_a^b f g \) for all \( f \in \mathcal{HK} \) if and only if \( \| f(g_n - g) \| \to 0 \) for all \( f \in \mathcal{HK} \).

Note that \( \| f(g_n - g) \| \geq \| f g_n \| - \| f g \| \); so if \( \| f(g_n - g) \| \to 0 \), then \( \| f g_n \| \to \| f g \| \). Thus, (3) is sufficient to have \( \| f g_n \| \to \| f g \| \) for all \( f \in \mathcal{HK} \).

However, this condition is not necessary. For example, let \([a, b] = [0, 1]\). Define \( g_n(x) = (-1)^n \). Then \( \| g_n \|_\infty = 1 \) and \( V g_n = 0 \). Let \( g = g_1 \). For no \( x \in [-1, 1] \) does the sequence \( g_n(x) \) converge to \( g(x) \). For no open interval \( I \subset [0, 1] \) do we have \( \int_I (g_n - g) \to 0 \). And, \( g_n \) does not converge to \( g \) in measure. However, let \( f \in \mathcal{HK} \) with \( \| f \| > 0 \). Then \( \| f(g_n - g) \| = 0 \) when \( n \) is odd and when \( n \) is even, \( \| f(g_n - g) \| = \| f \| \). And yet, for all \( n \), \( \| f g_n \| = \| f \| = \| f g \| \).

It is natural to ask what extra condition should be given so that \( \| f g_n \| \to \| f g \| \) will imply \( \| f g_n - f g \| \to 0 \). We have the following.

**Theorem 8.** Let \( g_n \to g \) in measure or almost everywhere. Then \( \| f g_n \| \to \| f g \| \) for all \( f \in \mathcal{HK} \) if and only if \( \| f g_n - f g \| \to 0 \) for all \( f \in \mathcal{HK} \).

**Proof.** Let \([a, b]\) be a compact interval. If \( \| f g_n \| \to \| f g \| \), then \( g \) is equivalent to \( h \in \mathcal{NBV} \) [2, Theorem 12.9] and for each \( f \in \mathcal{HK} \) there is a constant \( C_f \) such that \( \| f g_n \| \leq C_f \). By the Banach–Steinhaus Theorem [2, Theorem 12.10], each \( g_n \) is equivalent to a function \( h_n \in \mathcal{NBV} \) with \( V h_n \leq M \) and \( \| h_n \|_\infty \leq M \). Let \((c, d) \subset (a, b)\). By dominated convergence, \( \int_c^d g_n \to \int_c^d g \). It now follows from Theorem 1 that \( \int_c^d f g_n \to \int_c^d f g \) for all \( f \in \mathcal{HK} \). Hence, by Theorem 7, \( \| f g_n - f g \| \to 0 \) for all \( f \in \mathcal{HK} \).

Now suppose \((a, b) = \mathbb{R}\) and \( \| f g_n \| \to \| f g \| \) for all \( f \in \mathcal{HK} \). The change of variables \( x \mapsto 1/x \) shows the Banach–Steinhaus Theorem still holds on \( \mathbb{R} \). We then have each \( g_n \) equivalent to \( h_n \in \mathcal{NBV} \) with \( V h_n \leq M \) and \( \| h_n \|_\infty \leq M \). As with the end of the proof of Theorem 3, given \( \epsilon > 0 \) we can find \( c \in \mathbb{R} \) such that \( \int_{-\infty}^c |f g_n| < \epsilon \) for all \( n \geq 1 \). The other cases are similar.

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References


