A product convergence theorem for Henstock–Kurzweil integrals

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Abstract. Necessary and sufficient for \( \int_a^b fg \to \int_a^b fg \) for all Henstock–Kurzweil integrable functions \( f \) is that \( g \) be of bounded variation, \( g_n \) be uniformly bounded and of uniform bounded variation and, on each compact interval in \((a,b)\), \( g_n \to g \) in measure or in the \( L^1 \) norm. The same conditions are necessary and sufficient for \( \|fg_n\| \to \|fg\| \) for all Henstock–Kurzweil integrable functions \( f \). If \( g_n \to g \) a.e. then convergence \( \|fg_n\| \to \|fg\| \) for all Henstock–Kurzweil integrable functions \( f \) is equivalent to \( \|f(g_n - g)\| \to 0 \). This extends a theorem due to Lee Peng-Yee.

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Let \(-\infty \leq a < b \leq \infty\) and denote the Henstock–Kurzweil integrable functions on \((a,b)\) by \( \mathcal{HK} \). The Alexiewicz norm of \( f \in \mathcal{HK} \) is \( \|f\| = \sup_I \left| \int_I f \right| \) where the supremum is taken over all intervals \( I \subset (a,b) \). If \( g \) is a real-valued function on \([a,b]\) we write \( V_{[a,b]}g \) for the variation of \( g \) over \([a,b]\), dropping the subscript when the identity of \([a,b]\) is clear. The set of functions of normalised bounded variation, \( \mathcal{NBV} \), consists of the functions on \([a,b]\) that are of bounded variation, are left continuous and vanish at \( a \). It is known that the multipliers for \( \mathcal{HK} \) are \( \mathcal{NBV} \), i.e., \( fg \in \mathcal{HK} \) for all \( f \in \mathcal{HK} \) if and only if \( g \) is equivalent to a function in \( \mathcal{NBV} \). This paper is concerned with necessary and sufficient conditions under which \( \int_a^b fg_n \to \int_a^b fg \) for all \( f \in \mathcal{HK} \). One such set of conditions was given by Lee Peng-Yee in [2, Theorem 12.11]. If \( g \) is of bounded variation, changing \( g \) on a countable set will make it an element of \( \mathcal{NBV} \). With this observation, a minor modification of Lee’s theorem produces the following result.

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Theorem 1 [2, Theorem 12.11] Let $-\infty < a < b < \infty$, let $g_n$ and $g$ be real-valued functions on $[a, b]$ with $g$ of bounded variation. In order for $\int_a^b fg_n \to \int_a^b fg$ for all $f \in \mathcal{H}^c$ it is necessary and sufficient that

\[
\begin{align*}
\text{for each interval } (c, d) \subset (a, b), & \int_c^d g_n \to \int_c^d g \text{ as } n \to \infty, \\
& \text{for each } n \geq 1, g_n \text{ is equivalent to a function } h_n \in \mathcal{BV}, \\
& \text{and there is } M \in [0, \infty) \text{ such that } Vh_n \leq M \text{ for all } n \geq 1.
\end{align*}
\]

(1)

We extend this theorem to unbounded intervals, show that the condition $\int_c^d g_n \to \int_c^d g$ in (1) can be replaced by $g_n \to g$ on each compact interval in $(a, b)$ either in measure or in the $L^1$ norm, and that this also lets us conclude $\|f(g_n-g)\| \to 0$. We also show that if $g_n \to g$ in measure or almost everywhere then $\|fg_n\| \to \|fg\|$ for all $f \in \mathcal{H}^c$ if and only if $\|fg_n - fg\| \to 0$ for all $f \in \mathcal{H}^c$.

One might think the conditions (1) imply $g_n \to g$ almost everywhere. This is not the case, as is illustrated by the following example [1, p. 61].

Example 2 Let $g_n = \chi_{(j2^{-k}, (j+1)2^{-k})}$ where $0 \leq j < 2^k$ and $n = j + 2^k$. Note that $\|g_n\|_\infty = 1$, $g_n \in \mathcal{BV}$, $Vg_n \leq 2$, and $\int_{a}^{b} g_n \leq \|g_n\| = 2^{-k} < 2/n \to 0$, so that (1) is satisfied with $g = 0$. For each $x \in (0, 1]$ we have $\inf_n g_n(x) = 0$, $\sup_n g_n(x) = 1$, and for no $x \in (0, 1]$ does $g_n(x)$ have a limit. However, $g_n \to 0$ in measure since if $T_n = \{ x \in [0, 1] : |g_n(x)| > \epsilon \}$ then for each $0 < \epsilon \leq 1$, we have $\lambda(T_n) < 2/n \to 0$ as $n \to \infty$ (\lambda is Lebesgue measure).

We have the following extension of Theorem 1.

Theorem 3 Let $[a, b]$ be a compact interval in $\mathbb{R}$, let $g_n$ and $g$ be real-valued functions on $[a, b]$ with $g$ of bounded variation. In order for $\int_a^b fg_n \to \int_a^b fg$ for all $f \in \mathcal{H}^c$ it is necessary and sufficient that

\[
\begin{align*}
g_n & \to g \text{ in measure as } n \to \infty, \\
& \text{for each } n \geq 1, g_n \text{ is equivalent to a function } h_n \in \mathcal{BV}, \\
& \text{and there is } M \in [0, \infty) \text{ such that } Vh_n \leq M \text{ for all } n \geq 1.
\end{align*}
\]

(2)

If $(a, b) \subset \mathbb{R}$ is unbounded, then change the first line of (2) by requiring $g_n \chi_I \to g \chi_I$ in measure for each compact interval $I \subset (a, b)$.

Proof: By working with $g_n - g$ we can assume $g = 0$. First consider the case when $(a, b)$ is a bounded interval.

If $\int_a^b fg_n \to 0$ for all $f \in \mathcal{H}^c$, then using Theorem 1 and changing $g_n$ on a countable set, we can assume $g_n \in \mathcal{BV}$, $Vg_n \leq M$, $\|g_n\|_\infty \leq M$ and
\[ \int_a^b g_n \to 0 \text{ for each interval } (c,d) \subset (a,b). \] Suppose \( g_n \) does not converge to 0 in measure. Then there are \( \delta, \epsilon > 0 \) and an infinite index set \( J \subset \mathbb{N} \) such that \( \lambda(S_n) > \delta \) for each \( n \in J \), where \( S_n = \{ x \in (a,b) : g_n(x) > \epsilon \} \). (Or else there is a corresponding set on which \( g_n(x) < -\epsilon \) for all \( n \in J \).) Now let \( n \in J \). Since \( g_n \) is left continuous, if \( x \in S_n \) there is a number \( c_{n,x} > 0 \) such that \( [x-c_{n,x},x] \subset S_n \). Hence, \( V_n := \{ [c,x] : x \in S_n \text{ and } [c,x] \subset S_n \} \) is a Vitali cover of \( S_n \). So there is a finite set of disjoint closed intervals, \( \sigma_n \subset V_n \), with \( \lambda(S_n \setminus \bigcup_{I \in \sigma_n} I) < \delta/2 \). Write \( (a,b) \setminus \bigcup_{I \in \sigma_n} I = \bigcup_{I \in \tau_n} I \) where \( \tau_n \) is a set of disjoint open intervals with \( \text{card}(\tau_n) = \text{card}(\sigma_n) + 1 \). Let \( P_n = \text{card}(\{ I \in \tau_n : g_n(x) \leq \epsilon/2 \text{ for some } x \in I \}) \). Each interval \( I \in \tau_n \) that does not have \( a \) or \( b \) as an endpoint has contiguous intervals on its left and right that are in \( \sigma_n \) (for each of which \( g_n(x) > \epsilon/2 \) for some \( x \)). The interval \( I \) then contributes more than \( (\epsilon - \epsilon/2) + (\epsilon - \epsilon/2) = \epsilon \) to the variation of \( g_n \). If \( I \) has \( a \) as an endpoint then, since \( g_n(a) = 0 \), \( I \) contributes more than \( \epsilon \) to the variation of \( g_n \). If \( I \) has \( b \) as an endpoint then \( I \) contributes more than \( \epsilon/2 \) to the variation of \( g_n \). Hence, \( V g_n \geq (P_n - 1)\epsilon + \epsilon/2 = (P_n - 1/2)\epsilon \). (This inequality is still valid if \( P_n = 1 \).) But, \( V g_n \leq M \) so \( P_n \leq P \) for all \( n \in J \) and some \( P \in \mathbb{N} \). Then we have a set of intervals, \( U_n \), formed by taking unions of intervals from \( \sigma_n \) and those intervals in \( \tau_n \) on which \( g_n > \epsilon/2 \). Now, \( \lambda(\bigcup_{I \in U_n} I) > \delta/2 \), \( \text{card}(U_n) \leq P + 1 \) and \( g_n > \epsilon/2 \) on each interval \( I \in U_n \). Therefore, there is an interval \( I_n \in U_n \) such that \( \lambda(I_n) > \delta/[2(P + 1)] \). The sequence of centres of intervals \( I_n \) has a convergent subsequence. There is then an infinite index set \( J' \subset J \) with the property that for all \( n \in J' \) we have \( g_n > \epsilon/2 \) on an interval \( I \subset (a,b) \) with \( \lambda(I) > \delta/[3(P + 1)] \). Hence, \( \limsup_{n \to \infty} \int_{I_n} g_n > \delta \epsilon/[6(P + 1)] \). This contradicts the fact that \( \int_I g_n \to 0 \), showing that indeed \( g_n \to 0 \) in measure.

Suppose (2) holds. As above, we can assume \( g_n \in \mathcal{NBV} \), \( V g_n \leq M \), \( \| g_n \|_{\infty} \leq M \) and \( g_n \to 0 \) in measure. Let \( \epsilon > 0 \). Define \( T_n = \{ x \in (a,b) : |g_n(x)| > \epsilon \} \). Then

\[
\left| \int_a^b g_n \right| \leq \int_{T_n} |g_n| + \int_{(a,b) \setminus T_n} |g_n| \leq M \lambda(T_n) + \epsilon(b-a).
\]

Since \( \lim \lambda(T_n) = 0 \), it now follows that \( \int_a^d g_n \to 0 \) for each \( (c,d) \subset (a,b) \).

Theorem 1 now shows \( \int_a^b f g_n \to 0 \) for all \( f \in \mathcal{HK} \).

Now consider integrals on \( \mathbb{R} \). If \( \int_{-\infty}^a f g_n \to 0 \) for all \( f \in \mathcal{HK} \) then it is necessary that \( \int_a^b f g_n \to 0 \) for each compact intervals \( [a,b] \). By the current theorem, \( g_n \to g \) in measure on each \( [a,b] \). And, it is necessary that \( \int_1^\infty f g_n \to 0 \). The change of variables \( x \to 1/x \) now shows it is necessary that \( g_n \) be...
equivalent to a function that is uniformly bounded and of uniform bounded variation on $[1, \infty)$. Similarly with $\int_{-\infty}^{1} f g_n \to 0$. Hence, it is necessary that $g_n$ be uniformly bounded and of uniform bounded variation on $\mathbb{R}$.

Suppose (2) holds with $g_n \to g$ in measure on each compact interval in $\mathbb{R}$. Write $\int_{-\infty}^{\infty} f g_n = \int_{a}^{b} f g_n + \int_{-\infty}^{a} f g_n + \int_{b}^{\infty} f g_n$. Use Lemma 24 in [4] to write $| \int_{-\infty}^{\infty} f g_n | \leq \| f \chi_{(-\infty,a)} \| V_{[-\infty,a]} g_n \leq \| f \chi_{(-\infty,a)} \| M \to 0$ as $a \to -\infty$. We can then take a large enough interval $[a, b] \subset \mathbb{R}$ and apply the current theorem on $[a, b]$. Other unbounded intervals are handled in a similar manner. ■

**Remark 4** If (2) holds then dominated convergence shows $\| g_n - g \|_1 \to 0$. And, convergence in $\| \cdot \|_1$ implies convergence in measure. Therefore, in the first statement of (2) and in the last statement of Theorem 3, ‘convergence in measure’ can be replaced with ‘convergence in $\| \cdot \|_1$’. Similar remarks apply to Theorem 6.

**Remark 5** The change of variables argument in the second last paragraph of Theorem 3 can be replaced with an appeal to the Banach–Steinhaus Theorem on unbounded intervals. See [3, Lemma 7]. Similarly in the proof of Theorem 8.

The sequence of Heaviside step functions $g_n = \chi_{(n, \infty]}$ shows (2) is not necessary to have $\int_{-\infty}^{\infty} f g_n \to 0$ for all $f \in \mathcal{HK}$. For then, $\int_{-\infty}^{\infty} f g_n = \int_{n}^{\infty} f \to 0$. In this case, $g_n \in \mathcal{NBV}$ and $V g_n = 1$. However, $\lambda(T_n) = \infty$ for all $0 < \epsilon < 1$. Note that for each compact interval $[a, b]$ we have $\int_{a}^{b} g_n \to 0$ and $g_n \to 0$ in measure on $[a, b]$.

It is somewhat surprising that the conditions (2) are also necessary and sufficient to have $\| f(g_n - g) \| \to 0$ for all $f \in \mathcal{HK}$.

**Theorem 6** Let $[a, b]$ be a compact interval in $\mathbb{R}$, let $g_n$ and $g$ be real-valued functions on $[a, b]$ with $g$ of bounded variation. In order for $\| f(g_n - g) \| \to 0$ for all $f \in \mathcal{HK}$ it is necessary and sufficient that

$$\begin{cases} 
  g_n \to g \text{ in measure as } n \to \infty, \\
  \text{for each } n \geq 1, g_n \text{ is equivalent to a function } h_n \in \mathcal{NBV}, \\
  \text{and there is } M \in [0, \infty) \text{ such that } V h_n \leq M \text{ for all } n \geq 1.
\end{cases}$$

(5)

If $(a, b) \subset \mathbb{R}$ is unbounded, then change the first line of (5) by requiring $g_n \chi_{I_n} \to g \chi_{I}$ in measure for each compact interval $I \in (a, b)$.

**Proof:** Certainly (5) is necessary in order for $\| f(g_n - g) \| \to 0$ for all $f \in \mathcal{HK}$.

If we have (5), let $I_n$ be any sequence of intervals in $(a, b)$. We can again assume $g = 0$. Write $\tilde{g}_n = g_n \chi_{I_n}$. Then $\| \tilde{g}_n \|_{\infty} \leq \| g_n \|_{\infty}$, $V \tilde{g}_n \leq V g_n +$
2\|g_n\|_\infty and \tilde{g}_n \to 0 in measure. The result now follows by applying Theorem 3 to \(f\tilde{g}_n\).

Unbounded intervals are handled as in Theorem 3. \(\blacksquare\)

By combining Theorem 3 and Theorem 6 we have the following.

**Theorem 7** Let \((a, b) \subset \mathbb{R}\) then \(\int_a^b fg_n \to \int_a^b fg\) for all \(f \in \mathcal{H}K\) if and only if \(\|fg_n - fg\| \to 0\) for all \(f \in \mathcal{H}K\).

Note that \(\|f(g_n - g)\| \geq \|fg_n\| - \|fg\|\) so if \(\|f(g_n - g)\| \to 0\) then \(\|fg_n\| \to \|fg\|\). Thus, (5) is sufficient to have \(\|fg_n\| \to \|fg\|\) for all \(f \in \mathcal{H}K\). However, this condition is not necessary. For example, let \([a, b] = [0, 1]\). Define \(g_n(x) = (-1)^n\). Then \(g_n\|_{\infty} = 1\) and \(Vg_n = 0\). Let \(g = g_1\). For no \(x \in [-1, 1]\) does the sequence \(g_n(x)\) converge to \(g(x)\). For no open interval \(I \subset [0, 1]\) do we have \(f_n(g_n - g) \to 0\). And, \(g_n\) does not converge to \(g\) in measure. However, let \(f \in \mathcal{H}K\) with \(\|f\| > 0\). Then \(\|f(g_n - g)\| = 0\) when \(n\) is odd and when \(n\) is even, \(\|f(g_n - g)\| = 2\|f\|\). And yet, for all \(n\), \(\|fg_n\| = \|f\| = \|fg\|\).

It is natural to ask what extra condition should be given so that \(\|fg_n\| \to \|fg\|\) will imply \(\|fg_n - fg\| \to 0\). We have the following.

**Theorem 8** Let \(g_n \to g\) in measure or almost everywhere. Then \(\|fg_n\| \to \|fg\|\) for all \(f \in \mathcal{H}K\) if and only if \(\|fg_n - fg\| \to 0\) for all \(f \in \mathcal{H}K\).

**Proof:** Let \([a, b]\) be a compact interval. If \(\|fg_n\| \to \|fg\|\) then \(g\) is equivalent to \(h \in NBV \ [2, \text{Theorem 12.9}]\) and for each \(f \in \mathcal{H}K\) there is a constant \(C_f\) such that \(\|fg_n\| \leq C_f\). By the Banach–Steinhaus Theorem [2, Theorem 12.10], each \(g_n\) is equivalent to a function \(h_n \in NBV\) with \(Vh_n \leq M\) and \(\|h_n\|_{\infty} \leq M\). Let \((c, d) \subset (a, b)\). By dominated convergence, \(\int_c^d g_n \to \int_c^d g\). It now follows from Theorem 1 that \(\int_a^b fg_n \to \int_a^b fg\) for all \(f \in \mathcal{H}K\). Hence, by Theorem 7, \(\|fg_n - fg\| \to 0\) for all \(f \in \mathcal{H}K\).

Now suppose \((a, b) = \mathbb{R}\) and \(\|fg_n\| \to \|fg\|\) for all \(f \in \mathcal{H}K\). The change of variables \(x \mapsto 1/x\) shows the Banach–Steinhaus Theorem still holds on \(\mathbb{R}\). We then have each \(g_n\) equivalent to \(h_n \in NBV\) with \(Vh_n \leq M\) and \(\|h_n\|_{\infty} \leq M\). As with the end of the proof of Theorem 3, given \(\epsilon > 0\) we can find \(c \in \mathbb{R}\) such that \(\int_{-\infty}^c fg_n < \epsilon\) for all \(n \geq 1\). The other cases are similar. \(\blacksquare\)

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**References**

