A product convergence theorem for Henstock–Kurzweil integrals

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Abstract. Necessary and sufficient for $\int_a^b fg_n \to \int_a^b fg$ for all Henstock–Kurzweil integrable functions f is that g be of bounded variation, g_n be uniformly bounded and of uniform bounded variation and, on each compact interval in (a,b), $g_n \to g$ in measure or in the L^1 norm. The same conditions are necessary and sufficient for $\|f(g_n-g)\| \to 0$ for all Henstock–Kurzweil integrable functions f. If $g_n \to g$ a.e. then convergence $\|fg_n\| \to \|fg\|$ for all Henstock–Kurzweil integrable functions f is equivalent to $\|f(g_n-g)\| \to 0$. This extends a theorem due to Lee Peng-Yee.

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Let $-\infty \le a < b \le \infty$ and denote the Henstock–Kurzweil integrable functions on (a,b) by $\mathcal{HK}$. The Alexiewicz norm of $f \in \mathcal{HK}$ is $\|f\| = \sup_I |\int_I f|$ where the supremum is taken over all intervals $I \subset (a,b)$. If g is a real-valued function on $[a,b]$ we write $V_{[a,b]}g$ for the variation of g over $[a,b]$, dropping the subscript when the identity of $[a,b]$ is clear. The set of functions of normalised bounded variation, $\mathcal{NBV}$, consists of the functions on $[a,b]$ that are of bounded variation, are left continuous and vanish at a. It is known that the multipliers for $\mathcal{HK}$ are $\mathcal{NBV}$, i.e., $fg \in \mathcal{HK}$ for all $f \in \mathcal{HK}$ if and only if g is equivalent to a function in $\mathcal{NBV}$. This paper is concerned with necessary and sufficient conditions under which $\int_a^b fg_n \to \int_a^b fg$ for all $f \in \mathcal{HK}$. One such set of conditions was given by Lee Peng-Yee in [2, Theorem 12.11]. If g is of bounded variation, changing g on a countable set will make it an element of $\mathcal{NBV}$. With this observation, a minor modification of Lee's theorem produces the following result.

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Theorem 1 [2, Theorem 12.11] *Let* $-\infty < a < b < \infty$, *let* g_n *and* g *be real-valued functions on* $[a, b]$ *with* g *of bounded variation. In order for* $\int_a^b fg_n \to \int_a^b fg$ *for all* $f \in \mathcal{HK}$ *it is necessary and sufficient that*

$$\left.\begin{array}{l} \textit{for each interval } (c,d) \subset (a,b), \int_c^d g_n \to \int_c^d g \textit{ as } n \to \infty, \\ \textit{for each } n \geq 1, g_n \textit{ is equivalent to a function } h_n \in \mathcal{NBV}, \\ \textit{and there is } M \in [0,\infty) \textit{ such that } Vh_n \leq M \textit{ for all } n \geq 1. \end{array}\right\} \quad (1)$$

We extend this theorem to unbounded intervals, show that the condition $\int_c^d g_n \to \int_c^d g$ in (1) can be replaced by $g_n \to g$ on each compact interval in (a, b) either in measure or in the L^1 norm, and that this also lets us conclude $\|f(g_n - g)\| \to 0$. We also show that if $g_n \to g$ in measure or almost everywhere then $\|fg_n\| \to \|fg\|$ for all $f \in \mathcal{HK}$ if and only if $\|fg_n - fg\| \to 0$ for all $f \in \mathcal{HK}$.

One might think the conditions (1) imply $g_n \to g$ almost everywhere. This is not the case, as is illustrated by the following example [1, p. 61].

Example 2 Let $g_n = \chi_{(j2^{-k},(j+1)2^{-k}]}$ where $0 \leq j < 2^k$ and $n = j + 2^k$. Note that $\|g_n\|_\infty = 1$, $g_n \in \mathcal{NBV}$, $Vg_n \leq 2$, and $|\int_c^d g_n| \leq \|g_n\| = 2^{-k} < 2/n \to 0$, so that (1) is satisfied with $g = 0$. For each $x \in (0, 1]$ we have $\inf_n g_n(x) = 0$, $\sup_n g_n(x) = 1$, and for no $x \in (0, 1]$ does $g_n(x)$ have a limit. However, $g_n \to 0$ in measure since if $T_n = \{x \in [0, 1] : |g_n(x)| > \epsilon\}$ then for each $0 < \epsilon \leq 1$, we have $\lambda(T_n) < 2/n \to 0$ as $n \to \infty$ (λ is Lebesgue measure).

We have the following extension of Theorem 1.

Theorem 3 *Let* $[a, b]$ *be a compact interval in* $\mathbb{R}$, *let* g_n *and* g *be real-valued functions on* $[a, b]$ *with* g *of bounded variation. In order for* $\int_a^b fg_n \to \int_a^b fg$ *for all* $f \in \mathcal{HK}$ *it is necessary and sufficient that*

$$\left.\begin{array}{l} g_n \to g \textit{ in measure as } n \to \infty, \\ \textit{for each } n \geq 1,\ g_n \textit{ is equivalent to a function } h_n \in \mathcal{NBV}, \\ \textit{and there is } M \in [0,\infty) \textit{ such that } Vh_n \leq M \textit{ for all } n \geq 1. \end{array}\right\} \quad (2)$$

If $(a, b) \subset \mathbb{R}$ *is unbounded, then change the first line of* (2) *by requiring* $g_n\chi_I \to g\chi_I$ *in measure for each compact interval* $I \in (a, b)$.

Proof: By working with $g_n - g$ we can assume $g = 0$. First consider the case when (a, b) is a bounded interval.

If $\int_a^b fg_n \to 0$ for all $f \in \mathcal{HK}$, then using Theorem 1 and changing g_n on a countable set, we can assume $g_n \in \mathcal{NBV}$, $Vg_n \leq M$, $\|g_n\|_\infty \leq M$ and

$\int_c^d g_n \to 0$ for each interval $(c,d) \subset (a,b)$. Suppose g_n does not converge to 0 in measure. Then there are $\delta, \epsilon > 0$ and an infinite index set $\mathcal{J} \subset \mathbb{N}$ such that $\lambda(S_n) > \delta$ for each $n \in \mathcal{J}$, where $S_n = \{x \in (a,b) : g_n(x) > \epsilon\}$. (Or else there is a corresponding set on which $g_n(x) < -\epsilon$ for all $n \in \mathcal{J}$.) Now let $n \in \mathcal{J}$. Since g_n is left continuous, if $x \in S_n$ there is a number $c_{n,x} > 0$ such that $[x - c_{n,x}, x] \subset S_n$. Hence, $V_n := \{[c,x] : x \in S_n \text{ and } [c,x] \subset S_n\}$ is a Vitali cover of S_n. So there is a finite set of disjoint closed intervals, $\sigma_n \subset V_n$, with $\lambda(S_n \setminus \cup_{I \in \sigma_n} I) < \delta/2$. Write $(a,b) \setminus \cup_{I \in \sigma_n} I = \cup_{I \in \tau_n} I$ where τ_n is a set of disjoint open intervals with $\mathrm{card}(\tau_n) = \mathrm{card}(\sigma_n) + 1$. Let $P_n = \mathrm{card}(\{I \in \tau_n : g_n(x) \leq \epsilon/2 \text{ for some } x \in I\})$. Each interval $I \in \tau_n$ that does not have a or b as an endpoint has contiguous intervals on its left and right that are in σ_n (for each of which $g_n(x) > \epsilon/2$ for some x). The interval I then contributes more than $(\epsilon - \epsilon/2) + (\epsilon - \epsilon/2) = \epsilon$ to the variation of g_n. If I has a as an endpoint then, since $g_n(a) = 0$, I contributes more than ϵ to the variation of g_n. If I has b as an endpoint then I contributes more than $\epsilon/2$ to the variation of g_n. Hence, $Vg_n \geq (P_n - 1)\epsilon + \epsilon/2 = (P_n - 1/2)\epsilon$. (This inequality is still valid if $P_n = 1$.) But, $Vg_n \leq M$ so $P_n \leq P$ for all $n \in \mathcal{J}$ and some $P \in \mathbb{N}$. Then we have a set of intervals, U_n, formed by taking unions of intervals from σ_n and those intervals in τ_n on which $g_n > \epsilon/2$. Now, $\lambda(\cup_{I \in U_n} I) > \delta/2$, $\mathrm{card}(U_n) \leq P + 1$ and $g_n > \epsilon/2$ on each interval $I \in U_n$. Therefore, there is an interval $I_n \in U_n$ such that $\lambda(I_n) > \delta/[2(P+1)]$. The sequence of centres of intervals I_n has a convergent subsequence. There is then an infinite index set $\mathcal{J}' \subset \mathcal{J}$ with the property that for all $n \in \mathcal{J}'$ we have $g_n > \epsilon/2$ on an interval $I \subset (a,b)$ with $\lambda(I) > \delta/[3(P+1)]$. Hence, $\limsup_{n \geq 1} \int_I g_n > \delta\epsilon/[6(P+1)]$. This contradicts the fact that $\int_I g_n \to 0$, showing that indeed $g_n \to 0$ in measure.

Suppose (2) holds. As above, we can assume $g_n \in \mathcal{NBV}$, $Vg_n \leq M$, $\|g_n\|_\infty \leq M$ and $g_n \to 0$ in measure. Let $\epsilon > 0$. Define $T_n = \{x \in (a,b) : |g_n(x)| > \epsilon\}$. Then

$$\left| \int_a^b g_n \right| \leq \int_{T_n} |g_n| + \int_{(a,b)\setminus T_n} |g_n| \tag{3}$$

$$\leq M\lambda(T_n) + \epsilon(b-a). \tag{4}$$

Since $\lim \lambda(T_n) = 0$, it now follows that $\int_c^d g_n \to 0$ for each $(c,d) \subset (a,b)$. Theorem 1 now shows $\int_a^b f g_n \to 0$ for all $f \in \mathcal{HK}$.

Now consider integrals on $\mathbb{R}$. If $\int_{-\infty}^{\infty} f g_n \to 0$ for all $f \in \mathcal{HK}$ then it is necessary that $\int_a^b f g_n \to 0$ for each compact interval $[a,b]$. By the current theorem, $g_n \to g$ in measure on each $[a,b]$. And, it is necessary that $\int_1^\infty f g_n \to 0$. The change of variables $x \mapsto 1/x$ now shows it is necessary that g_n be equivalent to a function that is uniformly bounded and of uniform bounded variation on $[1, \infty]$. Similarly with $\int_{-\infty}^{1} fg_n \to 0$. Hence, it is necessary that g_n be uniformly bounded and of uniform bounded variation on $\mathbb{R}$.

Suppose (2) holds with $g_n \to g$ in measure on each compact interval in $\mathbb{R}$. Write $\int_{-\infty}^{\infty} fg_n = \int_{-\infty}^{a} fg_n + \int_{a}^{b} fg_n + \int_{b}^{\infty} fg_n$. Use Lemma 24 in [4] to write $|\int_{-\infty}^{a} fg_n| \leq \|f\chi_{(-\infty,a)}\| V_{[-\infty,a]} g_n \leq \|f\chi_{(-\infty,a)}\| M \to 0$ as $a \to -\infty$. We can then take a large enough interval $[a, b] \subset \mathbb{R}$ and apply the current theorem on $[a, b]$. Other unbounded intervals are handled in a similar manner. ■

Remark 4 If (2) holds then dominated convergence shows $\|g_n - g\|_1 \to 0$. And, convergence in $\|\cdot\|_1$ implies convergence in measure. Therefore, in the first statement of (2) and in the last statement of Theorem 3, 'convergence in measure' can be replaced with 'convergence in $\|\cdot\|_1$'. Similar remarks apply to Theorem 6.

Remark 5 The change of variables argument in the second last paragraph of Theorem 3 can be replaced with an appeal to the Banach–Steinhaus Theorem on unbounded intervals. See [3, Lemma 7]. Similarly in the proof of Theorem 8.

The sequence of Heaviside step functions $g_n = \chi_{(n,\infty]}$ shows (2) is not necessary to have $\int_{-\infty}^{\infty} fg_n \to 0$ for all $f \in \mathcal{HK}$. For then, $\int_{-\infty}^{\infty} fg_n = \int_{n}^{\infty} f \to 0$. In this case, $g_n \in \mathcal{NBV}$ and $Vg_n = 1$. However, $\lambda(T_n) = \infty$ for all $0 < \epsilon < 1$. Note that for each compact interval $[a, b]$ we have $\int_{a}^{b} g_n \to 0$ and $g_n \to 0$ in measure on $[a, b]$.

It is somewhat surprising that the conditions (2) are also necessary and sufficient to have $\|f(g_n - g)\| \to 0$ for all $f \in \mathcal{HK}$.

Theorem 6 *Let $[a, b]$ be a compact interval in $\mathbb{R}$, let g_n and g be real-valued functions on $[a, b]$ with g of bounded variation. In order for $\|f(g_n - g)\| \to 0$ for all $f \in \mathcal{HK}$ it is necessary and sufficient that*

$$\left.\begin{array}{l} g_n \to g \text{ in measure as } n \to \infty, \\ \text{for each } n \geq 1,\ g_n \text{ is equivalent to a function } h_n \in \mathcal{NBV}, \\ \text{and there is } M \in [0, \infty) \text{ such that } Vh_n \leq M \text{ for all } n \geq 1. \end{array}\right\} \quad (5)$$

If $(a, b) \subset \mathbb{R}$ is unbounded, then change the first line of (5) by requiring $g_n\chi_I \to g\chi_I$ in measure for each compact interval $I \in (a, b)$.

Proof: Certainly (5) is necessary in order for $\|f(g_n - g)\| \to 0$ for all $f \in \mathcal{HK}$.

If we have (5), let I_n be any sequence of intervals in (a, b). We can again assume $g = 0$. Write $\tilde{g}_n = g_n\chi_{I_n}$. Then $\|\tilde{g}_n\|_\infty \leq \|g_n\|_\infty$, $V\tilde{g}_n \leq Vg_n +$

$2\|g_n\|_\infty$ and $\tilde{g}_n \to 0$ in measure. The result now follows by applying Theorem 3 to $f\tilde{g}_n$.

Unbounded intervals are handled as in Theorem 3. ■

By combining Theorem 3 and Theorem 6 we have the following.

Theorem 7 *Let* $(a,b) \subset \mathbb{R}$ *then* $\int_a^b fg_n \to \int_a^b fg$ *for all* $f \in \mathcal{HK}$ *if and only if* $\|fg_n - fg\| \to 0$ *for all* $f \in \mathcal{HK}$.

Note that $\|f(g_n - g)\| \geq |\, \|fg_n\| - \|fg\| \,|$ so if $\|f(g_n - g)\| \to 0$ then $\|fg_n\| \to \|fg\|$. Thus, (5) is sufficient to have $\|fg_n\| \to \|fg\|$ for all $f \in \mathcal{HK}$. However, this condition is not necessary. For example, let $[a,b] = [0,1]$. Define $g_n(x) = (-1)^n$. Then $\|g_n\|_\infty = 1$ and $Vg_n = 0$. Let $g = g_1$. For no $x \in [-1,1]$ does the sequence $g_n(x)$ converge to $g(x)$. For no open interval $I \subset [0,1]$ do we have $\int_I (g_n - g) \to 0$. And, g_n does not converge to g in measure. However, let $f \in \mathcal{HK}$ with $\|f\| > 0$. Then $\|f(g_n - g)\| = 0$ when n is odd and when n is even, $\|f(g_n - g)\| = 2\|f\|$. And yet, for all n, $\|fg_n\| = \|f\| = \|fg\|$.

It is natural to ask what extra condition should be given so that $\|fg_n\| \to \|fg\|$ will imply $\|fg_n - fg\| \to 0$. We have the following.

Theorem 8 *Let* $g_n \to g$ *in measure or almost everywhere. Then* $\|fg_n\| \to \|fg\|$ *for all* $f \in \mathcal{HK}$ *if and only if* $\|fg_n - fg\| \to 0$ *for all* $f \in \mathcal{HK}$.

Proof: Let $[a,b]$ be a compact interval. If $\|fg_n\| \to \|fg\|$ then g is equivalent to $h \in \mathcal{NBV}$ [2, Theorem 12.9] and for each $f \in \mathcal{HK}$ there is a constant C_f such that $\|fg_n\| \leq C_f$. By the Banach–Steinhaus Theorem [2, Theorem 12.10], each g_n is equivalent to a function $h_n \in \mathcal{NBV}$ with $Vh_n \leq M$ and $\|h_n\|_\infty \leq M$. Let $(c,d) \subset (a,b)$. By dominated convergence, $\int_c^d g_n \to \int_c^d g$. It now follows from Theorem 1 that $\int_a^b fg_n \to \int_a^b fg$ for all $f \in \mathcal{HK}$. Hence, by Theorem 7, $\|fg_n - fg\| \to 0$ for all $f \in \mathcal{HK}$.

Now suppose $(a,b) = \mathbb{R}$ and $\|fg_n\| \to \|fg\|$ for all $f \in \mathcal{HK}$. The change of variables $x \mapsto 1/x$ shows the Banach–Steinhaus Theorem still holds on $\mathbb{R}$. We then have each g_n equivalent to $h_n \in \mathcal{NBV}$ with $Vh_n \leq M$ and $\|h_n\|_\infty \leq M$. As with the end of the proof of Theorem 3, given $\epsilon > 0$ we can find $c \in \mathbb{R}$ such that $|\int_{-\infty}^c fg_n| < \epsilon$ for all $n \geq 1$. The other cases are similar. ■

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