

Estimates of Henstock/Kurzweil Poisson integrals

Erik Talvila¹

Department of Mathematical and Statistical Sciences
 University of Alberta
 Edmonton AB Canada T6G 2E2
 etalvila@math.ualberta.ca

Abstract. If f is a real-valued function on $[-\pi, \pi]$ that is Henstock/Kurzweil integrable, let $u_r(\theta)$ be its Poisson integral. It is shown that $\|u_r\|_p = o(1/(1-r))$ as $r \rightarrow 1$ and this estimate is sharp for $1 \leq p \leq \infty$. The Alexiewicz norm estimates $\|u_r\| \leq \|f\|$ ($0 \leq r < 1$) and $\|u_r - f\| \rightarrow 0$ ($r \rightarrow 1$) hold. These estimates lead to two uniqueness theorems for the Dirichlet problem in the unit disc with Henstock/Kurzweil integrable boundary data. There are similar growth estimates when u is in the harmonic Hardy space associated with the Alexiewicz norm and when f is of bounded variation.

AMS (MOS) subject classification: 26A39, 31A20

1 Introduction

In this paper we consider estimates of Poisson integrals on the unit circle with respect to Alexiewicz and L^p norms. Define the open disk in \mathbb{R}^2 as $D := \{re^{i\theta} \mid 0 \leq r < 1, -\pi \leq \theta < \pi\}$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic. The Poisson kernel is $\Phi_r(\theta) := (1-r^2)/[2\pi(1-2r \cos \theta + r^2)]$. The Poisson integral of f is

$$P[f](re^{i\theta}) := \int_{-\pi}^{\pi} f(\phi)\Phi_r(\phi - \theta) d\phi.$$

Since ∂D has no end points, an appropriate form of the Alexiewicz norm of f is $\|f\| := \sup_{I \subset \mathbb{R}} \left| \int_I f \right|$ where I is an interval in \mathbb{R} of length not exceeding 2π . Let \mathcal{HK} denote the 2π -periodic functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with finite Alexiewicz

¹Research partially supported by NSERC.

norm over one period. Of course, with the same periodicity convention, $L^p \subsetneq \mathcal{HK}$ for all $1 \leq p \leq \infty$. Write $\|f\|_A$ for the Alexiewicz norm over set A . The Alexiewicz norm is discussed in [6]. The variation of f over one period is denoted Vf . For a function $u: D \rightarrow \mathbb{R}$ we write $u_r(\theta) = u(re^{i\theta})$.

The following results are well known [1]. Suppose that $1 \leq p \leq \infty$ and $f \in L^p$. If $|\theta_0| \leq \pi$ and $z \in D$, we say that $z \rightarrow e^{i\theta_0}$ nontangentially if there is $0 \leq \alpha < \pi/2$ such that $z \rightarrow e^{i\theta_0}$ within the sector $\{\zeta \in D : |\arg(\zeta - e^{i\theta_0})| < \alpha\}$. Write $u_r(\theta) = P[f](re^{i\theta})$. Then

$$u_r \text{ is harmonic in } D \tag{1}$$

$$\|u_r\|_p \leq \|f\|_p \text{ for all } 0 \leq r < 1 \tag{2}$$

$$\text{If } 1 \leq p < \infty \text{ then } \|u_r - f\|_p \rightarrow 0 \text{ as } r \rightarrow 1 \tag{3}$$

$$u(re^{i\theta}) \rightarrow f(\theta_0) \text{ for almost all } \theta_0 \text{ as } z \rightarrow e^{i\theta_0} \text{ nontangentially in } D. \tag{4}$$

We examine analogues of these results when f is Henstock/Kurzweil integrable (Theorem 6). We also prove that the growth estimate $\|u_r\|_p = o(1/(1-r))$ is sharp for $f \in \mathcal{HK}$ and $1 \leq p \leq \infty$ (Theorem 1). The Poisson integral of a function in \mathcal{HK} need not be the difference of two positive harmonic functions (Corollary 3). There are similar growth estimates when u is in $h^{\mathcal{HK}}$, the harmonic Hardy space associated with the Alexiewicz norm (Theorem 5). The Poisson integral provides an isometry from \mathcal{HK} into (but not onto) $h^{\mathcal{HK}}$ (Theorem 8). In Theorem 9 we consider the above results for functions of bounded variation. Theorem 10 and Theorem 11 establish uniqueness conditions for the Dirichlet problem using the Alexiewicz norm. Examples 12 show the applicability of the uniqueness theorems. All the results also hold when we use the wide Denjoy integral.

Since Φ_r and $1/\Phi_r$ are of bounded variation on ∂D , necessary and sufficient for the existence of $P[f]$ on D is that f be integrable, i.e., the Henstock/Kurzweil integral $\int_{-\pi}^{\pi} f$ is finite. In [2], integration by parts was used to show that we can differentiate under the integral sign. This in turn shows that $P[f]$ is harmonic in D and that $P[f] \rightarrow f$ nontangentially, almost everywhere in ∂D . In Theorem 4, p. 238 of [3], necessary and sufficient conditions were given for determining when a function that is harmonic on D is the Poisson integral of an \mathcal{HK} function. Corresponding results when $\|u_r\|_p$ are uniformly bounded have been known for some time ([1], Theorem 6.13).

2 Growth estimates

Our first result is to show that for $1 \leq p \leq \infty$, we have $\|u_r\|_p = o(1/(1-r))$ and this estimate is sharp. That is, $(1-r)\|u_r\|_p \rightarrow 0$ as $r \rightarrow 1$ ($1 \leq p < \infty$) and $\sup_{\theta \in [-\pi, \pi]} (1-r)|P[f](re^{i\theta})| \rightarrow 0$ as $r \rightarrow 1$ ($p = \infty$). Thus, for $p = \infty$, the manner of approach to the boundary is unrestricted. This same estimate for $p = \infty$ was obtained for L^1 functions in [8]. We show these estimates are the best possible under our minimal existence hypothesis. The proof uses the inequality

$$\left| \int_{-\pi}^{\pi} fg \right| \leq \|f\| \left(\inf_{[-\pi, \pi]} |g| + Vg \right), \quad (5)$$

which is valid for all $f \in \mathcal{HK}$ and g of bounded variation on $[-\pi, \pi]$. This was proved in [7, Lemma 24].

Theorem 1 *Let $f \in \mathcal{HK}$. For $re^{i\theta} \in D$ let $u_r(\theta) = P[f](re^{i\theta})$.*

- (a) *We have $|P[f](re^{i\theta})| = o(1/(1-r))$ and this estimate is sharp in the sense that if $\psi : D \rightarrow \mathbb{R}$ and $\psi(re^{i\theta}) = o(1/(1-r))$ then there is a function $f \in \mathcal{HK}$ such that $P[f] \neq o(\psi)$.*
- (b) *Let $1 \leq p < \infty$. Then $\|u_r\|_p = o(1/(1-r))$ and this estimate is sharp in the sense that if $\psi : [0, 1) \rightarrow \mathbb{R}$ and $\psi(r) = o(1/(1-r))$ then there is a function $f \in \mathcal{HK}$ such that $\|u_r\|_p \neq o(\psi)$.*

Proof: (a) Let $\Psi_r(\phi) := (1-r)^2/(1-2r\cos\phi+r^2)$ with $\Psi_1(0) := 1$. Let $0 < \delta < \pi$. Then

$$\frac{2\pi(1-r)P[f](re^{i\theta})}{1+r} = \int_{|\phi-\theta|<\delta} f(\phi)\Psi_r(\phi-\theta) d\phi + \int_{\delta<|\phi-\theta|<\pi} f(\phi)\Psi_r(\phi-\theta) d\phi.$$

Given $\epsilon > 0$, take δ small enough so that $\|f\|_{[\theta-\delta, \theta+\delta]} < \epsilon$ for all θ . Using (5),

$$\left| \int_{|\phi-\theta|<\delta} f(\phi)\Psi_r(\phi-\theta) d\phi \right| \leq 2\|f\|_{[\theta-\delta, \theta+\delta]}.$$

And,

$$\left| \int_{\theta+\delta}^{\theta-\delta+2\pi} f(\phi)\Psi_r(\phi-\theta) d\phi \right| \leq \|f\| \left[\frac{2(1-r)^2}{1-2r\cos\delta+r^2} - \frac{(1-r)^2}{(1+r)^2} \right] \\ \rightarrow 0 \quad \text{as } r \rightarrow 1.$$

To prove this estimate is sharp, suppose $\psi: D \rightarrow \mathbb{R}$ is given. It suffices to show that $P[f](r_n e^{i\theta_n}) \neq o(\psi(r_n e^{i\theta_n}))$ for some sequence $\{r_n e^{i\theta_n}\} \in D$ with $r_n \rightarrow 1$. Take $r_n e^{i\theta_n} \rightarrow 1$ and $\theta_n \downarrow 0$. Let $a_n = |\psi(r_n e^{i\theta_n})|$ and let $\{\alpha_n\}$ and $\{f_n\}$ be sequences of positive numbers. Define

$$f(\phi) = \begin{cases} f_n, & |\phi - \theta_n| < \alpha_n \text{ for some } n \\ 0, & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ take $0 < \alpha_n \leq \pi - \theta_n$ and small enough so that the intervals $(\theta_n - \alpha_n, \theta_n + \alpha_n)$ are disjoint. It suffices to take $\alpha_n \leq \frac{1}{2} \min(\theta_{n-1} - \theta_n, \theta_n - \theta_{n+1})$ ($\theta_0 := \pi$). Now,

$$\begin{aligned} 2\pi P[f](r_n e^{i\theta_n}) &= (1 - r_n^2) \sum_{k=1}^{\infty} f_k \int_{\theta_k - \alpha_k}^{\theta_k + \alpha_k} \frac{d\phi}{r_n^2 - 2r_n \cos(\theta_n - \phi) + 1} \\ &\geq \frac{2(1 - r_n^2) f_n \alpha_n}{r_n^2 - 2r_n \cos(\alpha_n) + 1} \\ &\geq \frac{2(1 + r_n)(1 - r_n) f_n \alpha_n}{(1 - r_n)^2 + r_n \alpha_n^2}. \end{aligned}$$

Hence, taking $\alpha_n = \min(\pi - \theta_n, \frac{1}{2}(\theta_{n-1} - \theta_n), \frac{1}{2}(\theta_n - \theta_{n+1}), 1 - r_n)$ and $f_n = \pi(1 - r_n)a_n/\alpha_n$ gives $P[f](r_n e^{i\theta_n}) \geq a_n$. And, $f \in L^1$ if $\sum f_k \alpha_k = \pi \sum (1 - r_k)a_k < \infty$. Since $(1 - r_k)a_k \rightarrow 0$ there is a subsequence $\{(1 - r_n)a_n\}_{n \in I}$ defined by an unbounded index set $I \subset \mathbb{N}$ such that $\sum_{k \in I} (1 - r_k)a_k < \infty$. Now take $f(\phi) = f_n$ when $|\phi - \theta_n| < \alpha_n$ for some $n \in I$ and $f(\phi) = 0$, otherwise. Then, $f \in L^1$ and $P[f](r_n e^{i\theta_n}) \geq |\psi(r_n e^{i\theta_n})|$ for all $n \in I$.

(b) Suppose $1 \leq p < \infty$. From part (a), we can write $u_r(\theta) = w_r(\theta)/(1 - r)$ where $\sup_{\theta \in [-\pi, \pi]} |w_r(\theta)| \rightarrow 0$ as $r \rightarrow 1$. And, w_r is periodic and real analytic on $[-\pi, \pi]$ for each $0 \leq r < 1$. Let $1 \leq p < \infty$. Then

$$\begin{aligned} \|u_r\|_p &= \frac{1}{1 - r} \left[\int_{-\pi}^{\pi} |w_r(\theta)|^p d\theta \right]^{1/p} \\ &\leq \frac{(2\pi)^{1/p}}{1 - r} \sup_{\theta \in [-\pi, \pi]} |w_r(\theta)|. \end{aligned}$$

Hence, $\|u_r\|_p = o(1/(1 - r))$ as $r \rightarrow 1$.

To prove this estimate is sharp, first consider $p = 1$. Let $\psi: [0, 1) \rightarrow (0, \infty)$ with $\psi(r) = o(1/(1 - r))$ be given. Although \mathcal{HK} is not complete it is

barrelled [6]. The Uniform Boundedness Principle [5] applies and this shows the existence of $f \in \mathcal{HK}$ such that $\|u_r\|_1 \neq o(\psi(r))$.

Define $r_n = 1 - 1/n$ for $n \in \mathbb{N}$. Let $f_n(\theta) = \psi(r_n) \sin(n\theta)$. Then

$$\begin{aligned} \|f_n\| &= \psi(r_n) \int_0^{\pi/n} \sin(n\theta) d\theta \\ &= 2\psi(r_n)/n \\ &= 2(1 - r_n)\psi(r_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

For $0 \leq r < 1$, define $S_r : \mathcal{HK} \rightarrow L^1$ by $S_r[f](\theta) = P[f](re^{i\theta})/\psi(r)$ for each $f \in \mathcal{HK}$. Let $f \in \mathcal{HK}$ and write $u_r(\theta) = P[f](re^{i\theta})$. Then

$$\begin{aligned} \|u_r\|_1 &= \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f(\phi) \Phi_r(\phi - \theta) d\phi \right| d\theta \\ &\leq 2\pi \|f\| [\inf \Phi_r + V\Phi_r] \\ &= \|f\| \left(\frac{1 + 6r + r^2}{1 - r^2} \right). \end{aligned} \quad (6)$$

Therefore, $\|S_r\| \leq \frac{1+6r+r^2}{\psi(r)(1-r^2)}$ and, for each $0 \leq r < 1$, S_r is a bounded linear operator from \mathcal{HK} to L^1 .

We have $S_r[f_n](\theta) = r^n \sin(n\theta)/\psi(r)$ so that

$$\begin{aligned} \|S_{r_n}[f_n]\|_1 &= r_n^n \int_{-\pi}^{\pi} |\sin(n\theta)| d\theta \\ &= 4(1 - 1/n)^n \\ &\rightarrow 4/e \text{ as } n \rightarrow \infty. \end{aligned} \quad (7)$$

It follows that $\{S_{r_n}\}$ is not equicontinuous. The Uniform Boundedness Principle [5, Theorem 11, p. 299] now shows that $\{S_{r_n}\}$ is not pointwise bounded on \mathcal{HK} . Hence, there exists $f \in \mathcal{HK}$ such that $\sup_{0 \leq r < 1} \|u_r\|_1/\psi(r) = \infty$.

The case $p > 1$ is similar. In place of (6), we have $\|u_r\|_p \leq (2\pi)^{1/p} \|f\| (1 + 6r + r^2)/(1 - r^2)$. And, in place of (7),

$$\|S_{r_n}[f_n]\|_p = (1 - 1/n)^n \left[\frac{2\sqrt{\pi} \Gamma((1+p)/2)}{\Gamma(1+p/2)} \right]^{1/p}. \quad \blacksquare$$

Remarks 2 The little *oh* order relation in Theorem 1 is false for measures. If μ is a finite Borel measure on $[-\pi, \pi)$ then $\|P[\mu]_r\|_{\infty} \leq \Phi_r(0)\mu([-\pi, \pi)) = O(1/(1-r))$. Jensen's inequality shows this same order relation holds for $1 \leq p < \infty$. And, the Dirac measure shows it is sharp for $1 \leq p \leq \infty$. \blacksquare

Several results follow immediately from these estimates. For $1 \leq p < \infty$, denote the harmonic Hardy spaces by $h^p := \{u : D \rightarrow \mathbb{R} \mid \Delta u = 0 \text{ in } D, \sup_{0 \leq r < 1} \|u_r\|_p < \infty\}$. And, h^∞ are the bounded harmonic functions on D . The harmonic Hardy space associated with the Alexiewicz norm is defined

$$h^{\mathcal{HK}} := \{u : D \rightarrow \mathbb{R} \mid \Delta u = 0, \sup_{0 \leq r < 1} \|u_r\| < \infty\}.$$

This is a normed linear space under the norm $\|u\|_{hk} := \sup_{0 \leq r < 1} \|u_r\|$.

Corollary 3 *There is a function $f \in \mathcal{HK}$ such that $P[f]$ is not the difference of two positive harmonic functions.*

Proof: Functions in h^1 are characterised as being the difference of two positive harmonic functions. See [1, Exercise 6.9]. ■

Corollary 4 *For $1 \leq p \leq \infty$ we have $h^p \subsetneq h^{\mathcal{HK}}$.*

Proof: We have $h^q \subset h^p$ for all $1 \leq p < q \leq \infty$. And, there is $f \in \mathcal{HK}$ with $u_r(\theta) := P[f](re^{i\theta})$ and $\|u_r\|_1 \neq O(1)$. ■

When $u \in h^{\mathcal{HK}}$ we can get slightly different estimates than in Theorem 1. (cf. [1, Proposition 6.16]).

Theorem 5 *Let $1 \leq p \leq \infty$. If $u \in h^{\mathcal{HK}}$ then $\|u_r\|_p \leq (2\pi)^{1/p} \frac{2r\|u\|_{hk}}{\pi(1-r)}$ for $1/2 \leq r < 1$ and $\|u_r\|_p \leq (2\pi)^{1/p} \frac{2\|u\|_{hk}}{\pi(1-r)}$ for $0 \leq r \leq 1/2$. (Replace the term $(2\pi)^{1/p}$ by 1 when $p = \infty$.) The order relations are sharp as $r \rightarrow 1$.*

Proof: Fix $z = re^{i\theta} \in D$ and $0 < t < 1 - r$. If $t \leq r$ then, by the Mean Value Property for harmonic functions, $u(z) = (\pi t^2)^{-1} \int_{r-t}^{r+t} \int_{\theta-\theta_0}^{\theta+\theta_0} u(\rho e^{i\phi}) d\phi \rho d\rho$, where $\theta_0 = \arccos[(r^2 + \rho^2 - t^2)/(2r\rho)]$ and $0 \leq \theta_0 < \pi/2$. Hence,

$$|u(z)| \leq \frac{1}{\pi t^2} \int_{r-t}^{r+t} \rho d\rho \sup_{|\rho-r|<t} \left| \int_{\theta-\theta_0}^{\theta+\theta_0} u(\rho e^{i\phi}) d\phi \right| \leq \frac{2r}{\pi t} \|u\|_{hk}.$$

Now let $t \rightarrow 1 - r$ when $1/2 \leq r < 1$ and let $t \rightarrow r$ when $0 \leq r \leq 1/2$. This establishes the estimates for $p = \infty$. The estimates for $1 \leq p < \infty$ follow from this.

Note that $\Phi_r \in h^{\mathcal{HK}}$ and $\|\Phi_r\|_\infty = (1+r)/[2\pi(1-r)]$. So, the order relation for $\|u_r\|_\infty$ is sharp as $r \rightarrow 1$. The cases $1 \leq p < \infty$ follow from Jensen's inequality. ■

Now consider the analogues of (2) and (3) for the Alexiewicz norm.

Theorem 6 Let $f \in \mathcal{HK}$. For $re^{i\theta} \in D$ define $u_r(\theta) := P[f](re^{i\theta})$. Then

- (a) $\|u_r\| \leq \|f\|$ for all $0 \leq r < 1$
- (b) $\|u_r - f\| \rightarrow 0$ as $r \rightarrow 1$
- (c) In (b), the decay of $\|u_r - f\|$ to 0 can be arbitrarily slow.

Proof: (a) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. Then, by Theorem 57 (p. 58) or Theorem 58 (p. 60) in [3], we can interchange the orders of integration to compute

$$\int_{\alpha}^{\beta} u_r = \int_{-\pi}^{\pi} f(\phi) v_r(\phi) d\phi,$$

where $v_r(\theta) = P[\chi_{[\alpha, \beta]}](re^{i\theta})$.

If $\beta - \alpha = 2\pi$ then $v_r = 1$ and the result is immediate. Now assume $0 < \beta - \alpha < 2\pi$. For fixed r the function v_r has one maximum, at $\phi_1 := (\alpha + \beta)/2$, and one minimum, at $\phi_2 := \phi_1 + \pi$. Use the Bonnet form of the Second Mean Value Theorem for integrals ([3], p. 34) to write

$$\begin{aligned} \int_{\alpha}^{\beta} u_r &= \int_{\phi_1}^{\phi_2} f(\phi) v_r(\phi) d\phi + \int_{\phi_2}^{\phi_1+2\pi} f(\phi) v_r(\phi) d\phi \\ &= v_r(\phi_1) \int_{\phi_1}^{\xi_1} f(\phi) d\phi + v_r(\phi_1) \int_{\xi_2}^{\phi_1+2\pi} f(\phi) d\phi \\ &= v_r(\phi_1) \int_{\xi_2-2\pi}^{\xi_1} f(\phi) d\phi \end{aligned}$$

where $\phi_1 < \xi_1 < \phi_2$ and $\phi_2 < \xi_2 < \phi_1 + 2\pi$. And,

$$\begin{aligned} \left| \int_{\alpha}^{\beta} u_r \right| &\leq \max_{\theta \in [-\pi, \pi]} v_r(\theta) \left| \int_{\xi_2-2\pi}^{\xi_1} f \right| \\ &\leq \|f\|. \end{aligned}$$

It follows that $\|u_r\| \leq \|f\|$.

(b) Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. We have

$$\begin{aligned} \int_{\alpha}^{\beta} [u_r(\theta) - f(\theta)] d\theta &= \int_{\alpha}^{\beta} \left[\int_{-\pi}^{\pi} \Phi_r(\phi - \theta) f(\phi) d\phi - f(\theta) \int_{-\pi}^{\pi} \Phi_r(\phi) d\phi \right] d\theta \\ &= \int_{-\pi}^{\pi} \Phi_r(\phi) \int_{\alpha}^{\beta} [f(\theta + \phi) - f(\theta)] d\theta d\phi. \end{aligned} \quad (8)$$

The reversal of integrals in (8) is justified as above. We now have

$$\begin{aligned} \|u_r - f\| &\leq \sup_{0 \leq \beta - \alpha \leq 2\pi} \left| \int_{-\pi}^{\pi} \Phi_r(\phi) \int_{\alpha}^{\beta} [f(\theta + \phi) - f(\theta)] d\theta d\phi \right| \\ &\leq P[g](r) \quad \text{where } g(\phi) = \|f(\phi + \cdot) - f(\cdot)\| \\ &\rightarrow 0 \quad \text{as } r \rightarrow 1 \quad \text{since } f \text{ is continuous in the Alexiewicz norm.} \end{aligned}$$

(c) Let f be positive on $(0, 1)$ and vanish elsewhere. Then u_r is positive for $0 \leq r < 1$. We then have

$$\begin{aligned} \|u_r - f\| &\geq \int_{-\pi}^0 u_r(\phi) d\phi \\ &= \int_0^1 f(\theta) P[\chi_{[-\pi, 0]}](re^{i\theta}) d\theta. \end{aligned}$$

Now, as $r \rightarrow 1$

$$P[\chi_{[-\pi, 0]}](re^{i\theta}) \rightarrow \begin{cases} 0, & 0 < \theta < \pi \\ 1/2, & \theta = -\pi, 0, \pi \\ 1, & -\pi < \theta < 0. \end{cases}$$

But, the convergence is not uniform. Let a decay rate be given by $A: (0, 1) \rightarrow (0, 1/2)$, where $A(r)$ decreases to 0 as r increases to 1. It is easy to show, for example, using a cubic spline, that A has a C^1 majorant with limit 0 as $r \rightarrow 1$. So, we can assume $A \in C^1([0, 1])$. By keeping θ close enough to 0 we can keep $P[\chi_{[-\pi, 0]}](re^{i\theta})$ bounded away from 0 for all r . To see this, write $\rho := (1+r)/(1-r)$. Then

$$\begin{aligned} \|u_r - f\| &\geq \int_0^{1-r} f(\theta) P[\chi_{[-\pi, 0]}](re^{i\theta}) d\theta \\ &= \frac{1}{\pi} \int_0^{1-r} f(\theta) \left\{ \frac{\pi}{2} - \arctan \left[\rho \tan \left(\frac{\theta}{2} \right) \right] + \arctan \left[\frac{1}{\rho} \tan \left(\frac{\theta}{2} \right) \right] \right\} d\theta \\ &\geq \int_0^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{1}{\pi} \arctan \left[\rho \tan \left(\frac{\theta}{2} \right) \right] \right\} d\theta \\ &\geq \int_0^{1-r} f(\theta) \left\{ \frac{1}{2} - \frac{\rho \theta}{2\pi \cos(\theta/2)} \right\} d\theta \\ &\geq \left(\frac{1}{2} - \frac{1}{\pi \cos(1/2)} \right) \int_0^{1-r} f(\theta) d\theta. \end{aligned}$$

We can now let

$$f(\theta) := \begin{cases} -\left(\frac{1}{2} - \frac{1}{\pi \cos(1/2)}\right)^{-1} A'(1-\theta), & 0 < \theta < 1 \\ 0, & \text{otherwise.} \end{cases}$$

And,

$$\|u_r - f\| \geq -\int_0^{1-r} A'(1-\theta) d\theta = A(r). \quad \blacksquare$$

Remarks 7 1. We have equality in (a) when f is of one sign.

2. Part (a) and dilation show that if $0 \leq r \leq s < 1$ then $\|u_r\| = \|P[u_s]_r\| \leq \|u_s\|$ (cf. [1, Corollary 6.6]).
3. The triangle inequality and (b) show that $\|u_r\| \rightarrow \|f\|$ as $r \rightarrow 1$.
4. In (c), the decay of $\|u_r - f\|$ can be arbitrarily rapid. Take f to be constant!
5. The same proof shows that we can choose $f \in L^1$ to make $\|u_r - f\|_1$ tend to 0 arbitrarily slowly. Jensen's inequality then shows the same holds true for $\|u_r - f\|_p$ for some $f \in L^p$, for each $1 \leq p < \infty$. \blacksquare

The Poisson integral induces an isometry from \mathcal{HK} into (but onto) $h^{\mathcal{HK}}$.

Theorem 8 *The mapping $P: \mathcal{HK} \rightarrow h^{\mathcal{HK}}$, $f \mapsto P[f]$, is an isometry into, but not onto, $h^{\mathcal{HK}}$.*

Proof: Let $f \in \mathcal{HK}$ and $u = P[f]$. From Remarks 7.2 and 7.3,

$$\|u\|_{\mathcal{HK}} = \sup_{0 \leq r < 1} \|u_r\| = \lim_{r \rightarrow 1} \|u_r\| = \|f\|.$$

Hence, P is an isometry.

However, P is not onto $h^{\mathcal{HK}}$. Let F be continuous on $[-\pi, \pi]$ such that $F(-\pi) = 0$, F is 2π -periodic and F is not in ACG^* , i.e., F is not an indefinite Henstock/Kurzweil integral. See [3] for the definition of ACG^* . The function

$$v_r(\theta) := F(\pi)\Phi_r(\pi - \theta) - \int_{-\pi}^{\pi} \Phi_r'(\phi - \theta)F(\phi) d\phi \quad (9)$$

is harmonic on D (using dominated convergence). Let $\alpha \in \mathbb{R}$ and $0 < \beta - \alpha \leq 2\pi$. Then

$$\begin{aligned} \int_{\alpha}^{\beta} v_r(\theta) d\theta &= F(\pi) \int_{\alpha}^{\beta} \Phi_r(\pi - \theta) d\theta + \int_{-\pi}^{\pi} F(\phi) \int_{\alpha}^{\beta} \Phi_r'(\phi - \theta) d\theta d\phi \\ &= F(\pi)P[\chi_{[\alpha, \beta]}](-r) + P[F](re^{i\beta}) - P[F](re^{i\alpha}). \end{aligned}$$

So, $\|v_r\| \leq 1 + 2 \max |F|$ and $v \in h^{\mathcal{HK}}$. If there was $f \in \mathcal{HK}$ such that $v = P[f]$ then write $G(\theta) := \int_{-\pi}^{\theta} f$. Since $G \in ACG^*$, we have

$$v(re^{i\theta}) = G(\pi)\Phi_r(\pi - \theta) - \int_{-\pi}^{\pi} \Phi_r'(\phi - \theta)G(\phi) d\phi. \quad (10)$$

Comparing (9) and (10), letting $r \rightarrow 0$ shows $G(\pi) = F(\pi)$. Write $H := F - G$. Expand $\Phi_r'(\theta) = (-1/\pi) \sum_{n=1}^{\infty} nr^n \sin(n\theta)$. The series converges uniformly and absolutely on compact subsets of D . Then for all $re^{i\theta} \in D$,

$$\begin{aligned} 0 &= \int_{-\pi}^{\pi} H(\phi) \sum_{n=1}^{\infty} nr^n \sin[n(\theta - \phi)] d\phi \\ &= \sum_{n=1}^{\infty} nr^n \int_{-\pi}^{\pi} H(\phi) \sin[n(\theta - \phi)] d\phi. \end{aligned}$$

For all $n \geq 1$ and all $\theta \in \mathbb{R}$ we have $\int_{-\pi}^{\pi} H(\phi) \sin[n(\theta - \phi)] d\phi = 0$. Since H is continuous it is constant. But then F differs from G by a constant. This contradicts the assumption that $F \notin ACG^*$. Thus, no such F exists and P is not onto $h^{\mathcal{HK}}$. ■

3 Bounded variation

Define the functions of normalised bounded variation by $\mathcal{NBV} := \{g: \mathbb{R} \rightarrow \mathbb{R} \mid g \text{ is } 2\pi\text{-periodic, } Vg < \infty, g(-\pi) = 0, g \text{ is right continuous}\}$. Then \mathcal{NBV} is the dual of \mathcal{HK} [6]. The norm on \mathcal{NBV} is the variation.

Theorem 9 *Let $g \in \mathcal{BV}$. For $re^{i\theta} \in D$ let $v = P[g]$.*

- (a) *If $g \in \mathcal{NBV}$ then $v_r \rightarrow g$ weak* in \mathcal{NBV} as $r \rightarrow 1$.*
- (b) *For all $0 \leq r < 1$, $\|v_r\|_{\infty} \leq \inf |g| + Vg$.*

- (c) If $g \in \mathcal{NBV}$ then $\|v_r\|_\infty \leq Vg$ for all $0 \leq r < 1$.
- (d) $Vv_r \leq Vg$ for all $0 \leq r < 1$
- (e) There is $g \in \mathcal{NBV}$ such that $V[v_r - g] \not\rightarrow 0$ as $r \rightarrow 1$. And, there is $g \in \mathcal{BV}$ such that $g(\theta) = [g(\theta+) + g(\theta-)]/2$ for all $\theta \in [-\pi, \pi]$ and yet $V(v_r - g) \not\rightarrow 0$ as $r \rightarrow 1$.
- (f) Let $h^{\mathcal{BV}} := \{u : D \rightarrow \mathbb{R} \mid \Delta u = 0, \|v\|_{\mathcal{BV}} < \infty\}$, where $\|v\|_{\mathcal{BV}} := \sup_{0 \leq r < 1} Vv_r$. The mapping $P : \mathcal{NBV} \rightarrow h^{\mathcal{BV}}$, $g \mapsto P[g]$, is an isometric isomorphism between the Banach spaces \mathcal{NBV} and $h^{\mathcal{BV}}$.

Proof: (a) Let $f \in \mathcal{HK}$. Write $u = P[f]$. Then, using (5) and of Theorem 6(b),

$$\left| \int_{-\pi}^{\pi} f(v_r - g) \right| = \left| \int_{-\pi}^{\pi} (u_r - f)g \right| \quad (11)$$

$$\leq \|u_r - f\| Vg \quad (12)$$

$$\rightarrow 0 \quad \text{as } r \rightarrow 1. \quad (13)$$

The interchange of orders of integration in (11) is valid by [3, pp. 58, Theorem 57].

(b), (c) These follow immediately from (5).

(d) Let $\{(s_n, t_n)\}$ be a sequence of disjoint intervals in $(-\pi, \pi)$. Then

$$\begin{aligned} \sum |v_r(s_n) - v_r(t_n)| &= \sum \left| \int_{-\pi}^{\pi} \Phi_r(\phi) [g(\phi + s_n) - g(\phi + t_n)] d\phi \right| \\ &\leq P[1](r) Vg \\ &= Vg. \end{aligned}$$

(e) Let $-\pi < a < b < \pi$ and $g = \chi_{[a,b]}$. Then $g \in \mathcal{NBV}$ and

$$\begin{aligned} |v_r(b) - g(b) - v_r(-\pi) + g(-\pi)| &= |v_r(b) - v_r(-\pi)| \\ &\rightarrow 1/2 \quad \text{as } r \rightarrow 1. \end{aligned}$$

So, $V(v_r - g) \not\rightarrow 0$.

Note that if we replace $g(\theta)$ by $[g(\theta+) + g(\theta-)]/2$ then $v_r(\theta) \rightarrow g(\theta)$ for all $\theta \in [-\pi, \pi]$. But, $V(v_r - g) \rightarrow 4$ as $r \rightarrow 1$.

(f) By (d), $\|v\|_{\mathcal{BV}} \leq Vg$. Since $v_r \rightarrow g$ weak* in \mathcal{NBV} (a), we have (cf. [1, Theorem 6.13]),

$$Vg \leq \liminf_{r \rightarrow 1} Vv_r \leq \liminf_{r \rightarrow 1} \|v\|_{\mathcal{BV}} = \|v\|_{\mathcal{BV}}.$$

Hence, P is an isometry.

To show P is onto $h^{\mathcal{BV}}$, let $v \in h^{\mathcal{BV}}$. Since \mathcal{HK} is separable, every norm-bounded sequence contains a weak* convergent subsequence. But $\{v_r\}$ is norm-bounded in \mathcal{NBV} so there is a subsequence $\{v_{r_j}\}$ and $g \in \mathcal{NBV}$ such that for all $f \in \mathcal{HK}$ we have $\int_{-\pi}^{\pi} f v_{r_j} \rightarrow \int_{-\pi}^{\pi} f g$ as $r_j \rightarrow 1$. To show $v = P[g]$, fix $re^{i\theta} \in D$. Then

$$v(re^{i\theta}) = \int_{-\pi}^{\pi} \Phi_r(\phi - \theta) v_{r_j}(\phi) d\phi. \quad (14)$$

Now, v is continuous on D , $\Phi_r(\cdot - \theta) \in \mathcal{HK}$ and v_{r_j} is of uniform bounded variation. Using weak* convergence, taking the limit $r_j \rightarrow 1$ in (14) yields $v(re^{i\theta}) = P[g](re^{i\theta})$. Thus, \mathcal{NBV} and $h^{\mathcal{BV}}$ are isomorphic. Since \mathcal{NBV} is a Banach space, $h^{\mathcal{BV}}$ is as well. ■

4 The Dirichlet problem

Under an Alexiewicz norm boundary condition, we can prove uniqueness for the Dirichlet problem.

Theorem 10 *Let $f \in \mathcal{HK}$. The Dirichlet problem*

$$u \in C^2(D) \quad (15)$$

$$\Delta u = 0 \quad \text{in } D \quad (16)$$

$$\|u_r - f\| \rightarrow 0 \quad \text{as } r \rightarrow 1 \quad (17)$$

has the unique solution $u = P[f]$.

Proof: First note that from Theorem 6(b) and [2, Proposition 1], $u = P[f]$ is certainly a solution of (15), (16) and (17).

Suppose there were two solutions u and v . Write $w = u - v$. Then w satisfies (15) and (16). And, $\|w_r\| \leq \|u_r - f\| + \|v_r - f\|$, which has limit 0 as $r \rightarrow 1$. Since w is harmonic in D it has the trigonometric expansion

$$w(re^{i\theta}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)], \quad (18)$$

the series converging uniformly and absolutely on compact subsets of D . Fix $0 \leq r < 1$. We have $\|w_r\| \geq |\int_{-\pi}^{\pi} w_r| = \pi|a_0|$. Letting $r \rightarrow 1$ shows $a_0 = 0$. And, for $n \geq 1$, we have $\|w_r \cos(n \cdot)\| \geq |\int_{-\pi}^{\pi} w_r(\theta) \cos(n\theta) d\theta| = \pi r^n |a_n|$. As well,

$$\begin{aligned} \|w_r \cos(n \cdot)\| &\leq \|w_r\| \left\{ \inf_{|\theta| \leq \pi} |\cos(n\theta)| + V[\theta \mapsto \cos(n\theta)] \right\} \\ &= 4n\|w_r\|. \end{aligned}$$

Therefore, $4n\|w_r\| \geq r^n |a_n|$. Letting $r \rightarrow 1$ shows $a_n = 0$. Similarly, $b_n = 0$. It follows that $w = 0$ and we have uniqueness. ■

In [4], Shapiro gave a uniqueness theorem that combined a pointwise limit with an L^p condition. There is an analogue for the Alexiewicz norm.

Theorem 11 *Suppose $\Delta u = 0$ in D and there exists $f \in \mathcal{HK}$ such that*

$$u_r(\theta) \rightarrow f(\theta) \quad \text{for each } \theta \in [-\pi, \pi] \quad (19)$$

$$\|u_r\| = o(1/(1-r)) \quad \text{as } r \rightarrow 1. \quad (20)$$

Then $u = P[f]$.

Proof: As in Theorem 10, suppose w is a solution of the corresponding homogeneous problem ($f = 0$). Let $\alpha, \beta \in \mathbb{R}$ with $0 < \beta - \alpha \leq 2\pi$. Following the proof of Theorem 3 in [4] and using (5),

$$\begin{aligned} |w(r^2 e^{i\theta})| &= |P[w_r](re^{i\theta})| \\ &\leq \frac{\|w_r\| g(r)}{2\pi(1-r)}, \end{aligned}$$

where $g(r) := (1 + 6r + r^2)/(1+r)$. But, $g(r) \leq g(1) = 4$. Hence, $\|w_{r^2}\|_{\infty} = o(1/(1-r)^2)$ and so $\|w_r\| = o(1/(1-r)^2)$ as $r \rightarrow 1$. It follows from [4, Theorem 1] that $w = 0$. ■

As pointed out in [4], neither (19) nor (20) can be relaxed. If $u_r \rightarrow f$ except for one value $\theta_0 \in [-\pi, \pi)$ then we can add a multiple of $\Phi_r(\theta - \theta_0)$ to $u(re^{i\theta})$. If in place of (20) we had $\|u_r - f\| = O(1/(1-r))$ then we could add a multiple of Φ'_r to u_r .

Examples 12 (a) Let $v(z) = (1+z)/(1-z)$ and $w(z) = v(z)e^{-v(z)}$. Define

$$\begin{aligned} u(re^{i\theta}) &= \operatorname{Re}(w(re^{i\theta})) \\ &= \frac{(1-r^2) \cos\left(\frac{2r \sin \theta}{1-2r \cos \theta + r^2}\right) + 2r \sin \theta \sin\left(\frac{2r \sin \theta}{1-2r \cos \theta + r^2}\right)}{\exp(2\pi\Phi_r(\theta))(1-2r \cos \theta + r^2)}. \end{aligned}$$

Let

$$f(\theta) := \lim_{r \rightarrow 1} u_r(\theta) = \begin{cases} \left(\frac{1+\cos \theta}{\sin \theta}\right) \sin\left(\frac{1+\cos \theta}{\sin \theta}\right), & 0 < |\theta| < \pi \\ 0, & |\theta| = 0, \pi. \end{cases}$$

Note that $f \notin L^p$ for any $1 \leq p \leq \infty$. The set function μ defined by $\mu(A) = \int_A f$ is not a signed Borel measure. Thus, u is not the Lebesgue/Poisson integral of any L^p function or measure. Since $f(\theta) \sim (2/\theta) \sin(2/\theta)$ as $\theta \rightarrow 0$ we have $f \in \mathcal{HK}$. And,

$$\begin{aligned} |(1-r)u_r(\theta)| &\leq (1-r)e^{-1} + \frac{2r e^{-1}}{1+r} \\ &\leq 1/2. \end{aligned}$$

By dominated convergence, $\|(1-r)u_r\| \rightarrow 0$ as $r \rightarrow 1$. And, by Theorem 11, $u = P[f]$. There is a similar result for the imaginary part of w .

(b) Let $w(z) = [1/(1-z)]e^{1/(1-z)}$ and define

$$\begin{aligned} u(re^{i\theta}) &= \operatorname{Re}(w(re^{i\theta})) \\ &= \frac{(1-r \cos \theta) \cos\left(\frac{r \sin \theta}{1-2r \cos \theta + r^2}\right) - r \sin \theta \sin\left(\frac{r \sin \theta}{1-2r \cos \theta + r^2}\right)}{\exp\left(\frac{1-r \cos \theta}{1-2r \cos \theta + r^2}\right) (1-2r \cos \theta + r^2)}. \end{aligned}$$

Let

$$f(\theta) := \lim_{r \rightarrow 1} u_r(\theta) = \begin{cases} \frac{(1-\cos \theta) \cos\left(\frac{\sin \theta}{2(1-\cos \theta)}\right) - \sin \theta \sin\left(\frac{\sin \theta}{2(1-\cos \theta)}\right)}{2\sqrt{e}(1-\cos \theta)}, & 0 < |\theta| \leq \pi \\ \infty, & \theta = 0. \end{cases}$$

Although $f \in \mathcal{HK}$, Theorem 11 does not apply since f is not a real-valued function. Indeed, $(1-r)u_r(0) = \exp(1/(1-r)) \rightarrow \infty$ as $r \rightarrow 1$. From Theorem 1, u is not the Poisson integral of any function in \mathcal{HK} (nor L^p function nor measure). In particular, $u \neq P[f]$.

In these examples, the origin is the only point of nonabsolute summability of f . For each $0 \leq \lambda < 2\pi$, an example is given in [2] of the Poisson integral of a function in \mathcal{HK} whose set of points of nonabsolute summability in $(-\pi, \pi)$ has measure λ .

References

- [1] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, New York, Springer-Verlag, 2001.
- [2] M. Benedicks and W.F. Pfeffer, *The Dirichlet problem with Denjoy–Perron integrable boundary condition*, *Canad. Math. Bull.* **28** (1985) 113–119.
- [3] V.G. Čelidze and A.G. Džvaršeišvili, *The theory of the Denjoy integral and some applications* (trans. P.S. Bullen), Singapore, World Scientific, 1989.
- [4] V.L. Shapiro, *The uniqueness of functions harmonic in the interior of the unit disc*, *Proc. London Math. Soc.* (3) **13** (1963) 639–652.
- [5] C. Swartz, *An introduction to functional analysis*, New York, Marcel Dekker, 1992.
- [6] C. Swartz, *Introduction to gauge integrals*, Singapore, World Scientific, 2001.
- [7] E. Talvila, *Henstock/Kurzweil Fourier transforms*, *Illinois J. Math.* (to appear).
- [8] F. Wolf, *The Poisson integral. A study in the uniqueness of harmonic functions*, *Acta. Math.* **74** (1941) 65–100.