LUSIN'S THEOREM AND BOCHNER INTEGRATION

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Abstract. It is shown that approximating functions used to define the Bochner integral can be formed using geometrically nice sets, such as balls, from a differentiation basis. This follows from the ubiquity of Lebesgue points, which is a consequence of Lusin's theorem, for which a simple proof is included in the discussion.

1. Introduction

An attractive feature of the Riemann and Henstock–Kurzweil integrals is that they can be defined in terms of Riemann sums using geometrically nice sets. In [7], we showed how Lebesgue points and points of approximate continuity could be used to approximate the Lebesgue integral with such sums. We show here that a similar result holds for the Bochner integral, where the domain of the integrand is a measure space $(X, \mathcal{M}, \mu)$ on which is defined a differentiation basis. Although somewhat more general settings are possible, we will assume that X is a finite dimensional normed vector space and the differentiation basis is obtained using the Besicovitch or Morse covering theorem (see Section 3). The geometrically nice sets we will use for the approximating functions will be balls or the more general starlike sets described in Section 3. We will assume that μ is a complete **Radon measure**; i.e., μ is a regular measure on $\mathcal{M}$, which includes the Borel sets, and compact sets have finite measure. We will let $\mathbb{N}$ denote the natural numbers and $\mathbb{R}$ the real numbers.

In what follows, an integrand f will take its values in a Banach space $(Y, \|\cdot\|)$ and will be μ-**essentially separably valued**. This means that off of some μ-null set, the range of f is in the closure of a countable dense subset of Y, whence there is a countable base for the relative topology of that part of the range. This will allow us to use Lusin's theorem. An elementary proof of that theorem, simple even for real-valued functions on $\mathbb{R}$, is given in the next section.

Our approximating sums will be constructed using a gauge function δ mapping a subset Ω of X into $(0, 1]$. In the theory of Henstock and McShane

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integration, the appearance of a gauge function is somewhat mysterious. We show in proving Theorem 5.1 how the properties of Lebesgue points (defined below in Section 3) can be used to determine an appropriate gauge.

We note that in [6], Gordon has studied the Bochner integral in terms of gauge functions and partitions on $\mathbb{R}$.

2. Lusin's Theorem

Recall that a function f from X into a topological space $(Y, \mathcal{T})$ is **Borel measurable** if the inverse image of each open set in Y is in $\mathcal{M}$. In our case, where Y is a Banach space with norm $\|\cdot\|$ and f is μ-essentially separably valued on X, it follows from Theorem III.6.10 of [4] that f is Borel measurable on X if and only if it is **μ-measurable**. This means that there is a sequence of simple functions f_n with $\lim_n \|f_n - f\| = 0$ μ-a.e. on X.

In any case, Lusin's theorem holds for the restriction of f to a set $\Omega \subseteq X$ with $\mu(\Omega) < +\infty$. (An extension to all of X follows from the σ-finiteness and regularity of μ.) Here is an elementary proof of that theorem stated for the general case but appropriate even for the simplest setting.

2.1. **Theorem** (Lusin). *Let Y be a Hausdorff space with a countable base $\{V_n\}$ for the topology, and let f be a Borel measurable function from a Radon measure space of finite measure $(\Omega, \mathcal{A}, \mu)$ into Y. Given $\varepsilon > 0$, there is a compact set K with $\mu(\Omega \setminus K) < \varepsilon$ such that f restricted to K is continuous.*

Proof. Fix compact sets $K_n \subseteq f^{-1}[V_n]$ and $K'_n \subseteq \Omega \setminus f^{-1}[V_n]$ for each n so that $\mu(\Omega \setminus K) < \varepsilon$ when $K := \bigcap_n (K_n \cup K'_n)$. Given $x \in K$ and an n with $f(x) \in V_n$, $x \in O := \Omega \setminus K'_n$ and $f[O \cap K] \subseteq V_n$. □

2.2. **Remark.** *The authors have learned of a similar principle used in Oxtaby's text* [10] *to show directly that a measurable $f : \mathbb{R} \to \mathbb{R}$ is continuous when restricted to a large measurable subset of* $\mathbb{R}$.

3. Covering Theorems

Our integration result is based on calculations using a covering theorem. Here we use either Besicovitch's theorem [1] for a covering by balls, or the theorem of Morse [9] involving more general sets. In [7], we have established strengthened versions of these theorems that hold for our finite dimensional normed vector space X. These covering theorems are also valid, however, for a space that is locally isometric to X since one only needs to bound the cardinality of a finite collection of sets with small diameter forming a "τ-satellite configuration" as defined in [7]. We leave it to the interested reader to consider our results in this more general setting, as well as settings using Vitali's covering theorem.

We will denote the closed (and compact) ball in X with center a and radius $r > 0$ by $B(a, r) := \{x \in X : \|x - a\| \leq r\}$. The sets used by Morse involve a parameter $\lambda \geq 1$. When $\lambda = 1$, the sets are closed balls.

Given $\lambda \geq 1$ and $a \in X$, we say that a set $S(a) \subseteq X$ is a **λ-Morse set** or just a **Morse set** associated with a and λ if there is an $r > 0$ such that $B(a,r) \subseteq S(a) \subseteq B(a,\lambda r)$ and $S(a)$ is **starlike** with respect to $B(a,r)$. This means that for each $y \in B(a,r)$ and each $x \in S(a)$, the line segment $\alpha y + (1-\alpha)x$, $0 \leq \alpha \leq 1$, is contained in $S(a)$. We call a the **tag** for $S(a)$. Note that the closure $\mathrm{cl}\,(S(a))$ of a λ-Morse set $S(a)$ is again a λ-Morse set. We say that a λ-Morse set $S(a)$ is **δ-fine** with respect to a gauge function δ defined at a if $\lambda r \leq \delta(a)$.

A collection $\mathcal{S}$ of subsets of X consisting of at least one Morse set associated with each point a in a set $\Omega \subseteq X$ is called a **Morse cover** of Ω provided the same parameter $\lambda \geq 1$ is used for sets in the cover and there is a finite upper bound to the diameters of the sets in the cover. We will also call such a cover a **λ-Morse cover** of Ω. A Morse cover of Ω is called **fine** if each point of Ω is the tag of sets in the cover having arbitrarily small diameter. Given a Radon measure μ, a Morse cover of a measurable set $\Omega \subseteq X$ is called a **μ-a.e. cover** of Ω if **i)** it is fine, **ii)** each set in the cover is μ-measurable, and **iii)** for any $\varepsilon > 0$ and for any gauge function $\delta : \Omega \to (0,1]$, there is a finite or infinite sequence of disjoint, δ-fine sets $S_n \in \mathcal{S}$ such that $\mu(\Omega \setminus \cup_n S_n) = 0$ and $\mu(\cup_n S_n \setminus \Omega) < \varepsilon$. In [7] we have established the following result.

3.1. **Proposition.** *Let μ be a Radon measure on X. Let $\Omega \subseteq X$ be measurable, and suppose that $\mathcal{S}$ is a fine Morse cover of Ω consisting of μ-measurable sets. Then $\mathcal{S}$ is a μ-a.e. cover of Ω if $\mathcal{S}$ consists of closed sets or if for each set $S \in \mathcal{S}$, $\mu(\Omega \cap (\mathrm{cl}(S) \setminus S)) = 0$.*

We have also shown that when dealing with Morse sets that are not closed, the conditions in Proposition 3.1 are easily fulfilled when the Morse cover $\mathcal{S}$ is **scaled**. This means that for each $S(a) \in \mathcal{S}$ and each $p \in (0,1]$, the set $S^{(p)}(a)$ is also in $\mathcal{S}$ where $S^{(p)}(a) = \{a + px : a + x \in S(a)\}$.

4. Approximate Continuity and Lebesgue Points

For this section, we fix a μ-a.e. λ-Morse cover $\mathcal{S}$ of a measurable set $\Omega \subseteq X$. We work with a μ-measurable $f : X \to Y$. If f is only defined on a measurable subset of X containing Ω, we extend it to the rest of X with the value 0. The following notions depend on the choice of $\mathcal{S}$.

4.1. **Definition.** *Given a μ-measurable $f : X \to Y$, a point $a \in \Omega$ is a* ***point of approximate continuity*** *for f if for all positive ε and η there is an $R > 0$ such that if $S(a)$ is a set in $\mathcal{S}$ with tag a and $S(a) \subseteq B(a,R)$, then for $E(a,\eta) := \{y \in S(a) : \|f(a) - f(y)\| > \eta\}$ we have $\mu(E(a,\eta)) \leq \varepsilon\, \mu(S(a))$. A point $a \in \Omega$ is a* ***Lebesgue point*** *of f with respect to some $y_0 \in Y$ if for any $\varepsilon > 0$ there is an $R > 0$ such that if $S(a)$ is a set in $\mathcal{S}$ with tag a and $S(a) \subseteq B(a,R)$, then*

$$\int_{S(a)} \|f(y) - y_0\|\ \mu(dy) \leq \varepsilon\, \mu(S(a)).$$

It follows from the fact that $\mathcal{S}$ is a differentiation basis that if g is a μ-integrable, nonnegative, real-valued function on X, then μ-almost all points of Ω are Lebesgue points and points of approximate continuity for g. (See, for example, [2].) In particular, if A is a measurable subset of Ω, then almost all points of A are **points of density**, that is, points of approximate continuity with respect to the characteristic function χ_A of A. Therefore, we have the following consequence of Lusin's theorem.

4.2. **Proposition.** *If $f : X \to Y$ is μ-measurable, then μ-almost all points of Ω are points of approximate continuity for f.*

Proof. If $\mu(\Omega) < +\infty$, then by Lusin's theorem (Theorem 2.1), there is an increasing sequence of compact sets $K_n \subseteq \Omega$ such that for each n, $f|K_n$ is continuous and $\mu(\Omega \setminus \cup_n K_n) = 0$. For this case, the result follows from the fact that for each n, μ-almost every point of K_n is a point of density of K_n. The general case follows since Ω has σ-finite measure. □

We also have the following relationship between points of approximate continuity and Lebesgue points.

4.3. **Proposition.** *If $f : X \to Y$ is μ-measurable and $a \in \Omega$ is a point of approximate continuity for f and also a Lebesgue point for $\|f\|$, then a is a Lebesgue point for f with respect to $f(a)$.*

Proof. Let $k \geq 0$ be the constant with respect to which a is a Lebesgue point for $\|f\|$. Let $c = \max(k, \|f(a)\|)$, and fix $\varepsilon > 0$. Choose $R > 0$ so that if S is a set in $\mathcal{S}$ with tag a and $S \subseteq B(a, R)$, then

$$\int_S |\ \|f(y)\| - k\ |\ \ \mu(dy) < \varepsilon \cdot \mu(S),$$

and for $E := \{y \in S : \|f(y) - f(a)\| > \varepsilon\}$, we have $\mu(E) \leq \frac{\varepsilon}{2c+1} \cdot \mu(S)$. Now

$$\begin{aligned}&\int_{E\cap\{\|f\|>c\}} (\|f(y)\| - c)\ \ \mu(dy)\\ &\leq \int_{E\cap\{\|f\|>c\}} (\|f(y)\| - k)\ \ \mu(dy) < \varepsilon \cdot \mu(S),\end{aligned}$$

whence

$$\int_{E\cap\{\|f\|>c\}} \|f(y)\|\ \ \mu(dy) \leq 2\varepsilon \cdot \mu(S).$$

Therefore,

$$\begin{aligned}&\int_S \|f(y) - f(a)\|\ \ \mu(dy)\\ &\leq \int_{S\setminus E} \varepsilon\ \mu(dy) + \int_E 2c\ \mu(dy) + \int_{E\cap\{\|f\|>c\}} \|f(y)\|\ \ \mu(dy)\\ &\leq 4\varepsilon \cdot \mu(S).\end{aligned}$$

Since the choice of $\varepsilon > 0$ is arbitrary, the result follows. $\square$

5. Integration

Recall that if $\mu(\Omega) < +\infty$, then a μ-measurable function $f : \Omega \to Y$ is **Bochner integrable** (with respect to μ) if there is a sequence of simple functions $f_n : \Omega \to Y$ with $\lim_n \int_\Omega \|f_n - f\| \, d\mu = 0$. In this case, the sequence of integrals $\int_\Omega f_n \, d\mu$ is Cauchy in Y, and the limit is the Bochner integral of f. Also, f is Bochner integrable if and only if the norm $\|f\|$ is integrable. (See, for example, Chapter II of [3].) We will consider the analogous approximations and integral for the case that $\|f\|$ is integrable on our σ-finite measure space $(X, \mathcal{M}, \mu)$, with the understanding that a simple function is defined on all of X but must vanish off of a set of finite measure. In Theorem 12 of [7], we have established a necessary and sufficient condition for the integrability of a real-valued function, such as $\|f\|$, in terms of approximations by Riemann sums using sets from a μ-a.e. Morse cover. Now, assuming that $\|f\|$ is μ-integrable on X, we show that the Bochner integral of f can also be approximated by the integral of simple functions using sets from a μ-a.e. Morse cover of X.

5.1. **Theorem.** *Assume that $\|f\|$ is μ-integrable on X. Fix $\lambda \geq 1$, and let $\mathcal{S}$ be a fine λ-Morse cover of X such that, $\mathcal{S}$ is a μ-a.e., λ-Morse cover of the ball $B(0, m)$ for each $m \in \mathbb{N}$. Fix $\varepsilon > 0$, and choose m so that for $\Omega := B(0, m)$, $\int_{X \setminus \Omega} \|f\| \; d\mu < \varepsilon/4$. Then there is a gauge function $\delta : \Omega \to (0, 1]$ such that for any finite or countably infinite disjoint sequence $\langle S_i(x_i) \rangle$ of δ-fine sets from $\mathcal{S}$ covering all but a set of measure 0 of Ω we have a finite subset for which the sum $\sum_{n=1}^N f(x_n) \chi_{S_n}$ approximates f in the sense that*

$$\int_X \left\| f(y) - \sum\nolimits_{n=1}^N f(x_n) \chi_{S_n}(y) \right\| \; \mu(dy) < \varepsilon.$$

Proof: Fix $\gamma > 0$ so that for each $A \subseteq \Omega$ with $\mu(A) < \gamma$, $\int_A \|f\| \; d\mu < \varepsilon/4$. Let L be the set of points $x \in \Omega$ that are Lebesgue points of f with respect to $f(x)$. Since $\mu(\Omega \setminus L) = 0$, there is an open set G containing $\Omega \setminus L$ with $\mu(G) < \gamma$. For each $x \in \Omega \setminus L$, we choose $\delta(x)$ with $0 < \delta(x) \leq 1$ so that $B(x, \delta(x)) \subseteq G$. If $x \in L$, we choose $\delta(x)$ with $0 < \delta(x) \leq 1$ so that if $S(x)$ is a set in $\mathcal{S}$ with tag x and $S(x) \subseteq B(x, \delta(x))$, then

$$\int_{S(x)} \|f(y) - f(x)\| \; \mu(dy) \leq \frac{\varepsilon}{4\left[1 + \mu\left(B\left(0, m+1\right)\right)\right]} \cdot \mu(S(x)).$$

With this choice for the gauge δ, we now let $\langle S_i(x_i) \rangle$ be any finite or countably infinite disjoint sequence of δ-fine sets from $\mathcal{S}$ covering all but a set of measure 0 of Ω. Choose a finite subset of this sequence covering all but a set of measure γ of Ω, and discard those sets $S_i(x_i)$ with tag x_i not in L. Let $\langle S_n(x_n) : 1 \leq n \leq N \rangle$ be an enumeration of the remaining sets, and let $A = \cup_{n=1}^{N} S_n(x_n)$. Then $\int_{X\setminus A} \|f\| \ d\mu < 3\varepsilon/4$. Moreover, since each set $S_n(x_n)$ is contained in $B\,(0, m+1)$,

$$\begin{aligned}
&\int_A \left\| f(y) - \sum\nolimits_{n=1}^{N} f(x_n)\chi_{S_n}(y) \right\| \ \mu(dy) \\
&\leq \sum\nolimits_{n=1}^{N} \int_{S_n} \|f(y) - f(x_n)\| \ \mu(dy) \\
&\leq \sum\nolimits_{n=1}^{N} \frac{\varepsilon}{4\,[1+\mu\,(B\,(0,m+1))]} \cdot \mu(S_n(x_n)) \\
&\leq \frac{\varepsilon}{4\,[1+\mu\,(B\,(0,m+1))]} \cdot \mu(A) \leq \frac{\varepsilon}{4}. \ \square
\end{aligned}$$

5.2**. Corollary.** *By an appropriate choice of a sequence of* ε*'s, one may find a sequence of such simple functions* $f_k = \sum_n f(x_n^k)\chi_{S_n^k}(y)$ *converging to* f *both in measure and* μ*-almost everywhere on* X*.*

5.3**. Remark.** *Note that our simple functions are not formed using a partition of* X*, as is usual for the Henstock–Kurzweil integral; they are, however, defined on all of* X*.*

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