Math 214 — Solutions to Assignment #9

<u>14.2</u>

8. (a) Sketch the plane curve $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$.

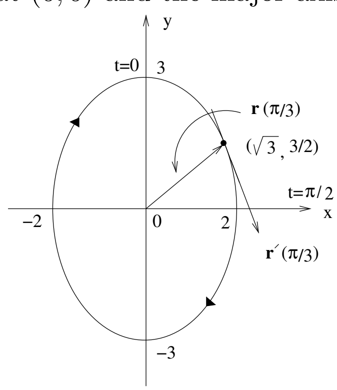
(b) Find $\mathbf{r}'(t)$.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for $t = \pi/3$.

Solution. (a) $x = 2\sin t$, $y = 3\cos t \implies \sin t = x/2$, and $\cos t = y/3$. So

$$\sin^2 t + \cos^2 t = \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The curve is an ellipse with center at $(0,0)$ and the major axis along the y-axis.



(b) $\mathbf{r}'(t) = \langle 2\cos t, -3\sin t\rangle$.

(c) See the graph above.

26. Find parametric equations for the tangent line to the curve: $x = \ln t$, $y = 2\sqrt{t}$, $z = t^2$ at $(0,2,1)$.

Solution. $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$. $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle$. At $(0,2,1)$, $t = 1$. So $\mathbf{r}'(1) = \langle 1,1,2\rangle$ is a direction vector for the tangent line whose parametric equations are

$$x = t, \quad y = 2 + t, \quad z = 1 + 2t.$$

30. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2\sin \pi t, \cos \pi t\rangle$ at the points where $t = 0$ and $t = 0.5$.

Solution. We first find $r'(t) = \langle \pi\cos\pi t, 2\pi\cos\pi t, -\pi\sin\pi t\rangle$. We can use this for direction vectors for the 2 tangent lines.

Let $t = 0$. $r'(0) = \langle \pi, 2\pi, 0\rangle$. The point on the curve is $(0, 0, 1)$. The tangent line is

$$x = \pi t, \quad y = 2\pi t, \quad z = 1.$$

Let $t = 0.5$. $r'(1/2) = \langle 0, 0, -\pi\rangle$. The point on the curve is $(1, 2, 0)$ and the tangent line is

$$x = 1, \quad y = 2, \quad z = -\pi s.$$

At the point of intersection of these tangent lines: $x: \ \pi t = 1 \implies t = 1/\pi$ and $z: \ -\pi s = 1 \implies s = -1/\pi$. So the point is $(1, 2, 1)$.

40. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution.

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = -\cos t - \sin t + t^2 + \mathbf{C} = \langle -\cos t, -\sin t, t^2\rangle + \langle a, b, c\rangle.$$

Since $\mathbf{r}(0) = \langle 1, 1, 2\rangle$, $t = 0$. Then

$$\langle 1, 1, 2\rangle = \langle -\cos 0, -\sin 0, 0^2\rangle + \langle a, b, c\rangle = \langle -1, 0, 0\rangle + \langle a, b, c\rangle$$

$$\therefore \quad \langle a, b, c\rangle = \langle 2, 1, 2\rangle$$

$$\therefore \quad \mathbf{r}(t) = \langle -\cos t, -\sin t, t^2\rangle + \langle 2, 1, 2\rangle = (2 - \cos t)\mathbf{i} + (1 - \sin t)\mathbf{j} + (2 + t^2)\mathbf{k}.$$

14.3

2. Find the length of the curve $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$, $0 \le t \le \pi$.

Solution. $\mathbf{r}'(t) = \langle 2t, \cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t\rangle = \langle 2t, t\sin t, t\cos t\rangle$. So

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$$

since $t \in [0, \pi]$. Therefore, the length of the curve is

$$L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}\,t\,dt = \sqrt{5}\,\frac{t^2}{2}\Big|_0^\pi = \frac{\sqrt{5}}{2}\pi^2.$$

14. Let $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$, $t > 0$.

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use Formula 9 to find the curvature.

Solution. (a) By the previous problem (#2), $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{5}\,t$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2t, t\sin t, t\cos t\rangle}{\sqrt{5}t} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}\cos t \right\rangle.$$

Then

$$\mathbf{T}'(t) = \left\langle 0, \frac{1}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}\sin t \right\rangle \implies |\mathbf{T}'(t)| = \sqrt{\frac{1}{5}(\cos^2 t + \sin^2 t)} = \frac{1}{\sqrt{5}},$$

$$\therefore \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 0, \frac{1}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}\sin t\rangle}{1/\sqrt{5}} = \langle 0, \cos t, -\sin t\rangle.$$

(b) The curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1/\sqrt{5}}{\sqrt{5}\,t} = \frac{1}{5t}.$$

18. Use Theorem 10 to find the curvature of $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + (1+t^2)\mathbf{k}$.

Solution. $\mathbf{r}(t) = \langle t, t, 1+t^2\rangle$, $\mathbf{r}'(t) = \langle 1, 1, 2t\rangle$, $\mathbf{r}''(t) = \langle 0, 0, 2\rangle$.

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} = \langle 2, -2, 0\rangle$$

$$\therefore \quad |\mathbf{r}' \times \mathbf{r}''| = \sqrt{4+4} = 2\sqrt{2},$$

$$\text{and} \quad |\mathbf{r}'| = \sqrt{1+1+4t^2} = \sqrt{2+4t^2} = \sqrt{2}\sqrt{1+2t^2}$$

$$\therefore \quad \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{2}}{(\sqrt{2})^3(1+2t^2)^{\frac{3}{2}}} = \frac{1}{(1+2t^2)^{\frac{3}{2}}}.$$

26. At what point does the curve $y = \ln x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

Solution. Use $\kappa(x) = |f''(x)|/(1+(f'(x))^2)^{3/2}$.

$$y = \ln x \implies y' = \frac{1}{x} \implies y'' = -\frac{1}{x^2} \quad (x > 0)$$

$$\therefore \quad \kappa(x) = \frac{|-\frac{1}{x^2}|}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{1/x^2}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{1/x^2}{(x^2+1)^{\frac{3}{2}}/x^3} = \frac{x}{(1+x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \kappa'(x) = \frac{(1+x^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(1+x^2)^{\frac{1}{2}}(2x)}{(1+x^2)^3} = \frac{(1+x^2)^{\frac{1}{2}}[1+x^2-3x^2]}{(1+x^2)^3} = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}.$$

The critical point is $x = 1/\sqrt{2}$ (remember the domain of f is $x > 0$). Then on $(0, 1/\sqrt{2})$, $\kappa'(x) > 0$ so κ is increasing; and on $(1/\sqrt{2}, \infty)$, $\kappa'(x) < 0$ so κ is decreasing. Hence the curvature is a maximum at $x = 1/\sqrt{2}$. The maximum curvature occurs at $(1/\sqrt{2}, \ln(1/\sqrt{2}))$. Also,

$$\begin{aligned}\lim_{x\to\infty} \kappa(x) &= \lim_{x\to\infty} \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}} = \lim_{x\to\infty} \frac{x^2(\frac{1}{x^2}-2)}{(x^2(\frac{1}{x^2}+1))^{\frac{5}{2}}} \\ &= \lim_{x\to\infty} \frac{x^2(\frac{1}{x^2}-2)}{x^5(\frac{1}{x^2}+1)^{\frac{5}{2}}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-2}{x^3(\frac{1}{x^2}+1)^{\frac{5}{2}}} = \lim_{x\to\infty} \frac{-2}{x^3} = 0.\end{aligned}$$

42. Find equations of the normal plane and osculating plane of the curve $x = t, \quad y = t^2, \quad z = t^3$ at the point $(1,1,1)$.

Solution. At $(1,1,1)$, $t = 1$. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. The normal plane is determined by the vectors $\mathbf{B}$ and $\mathbf{N}$ so a normal vector is the unit tangent vector $\mathbf{T}$ (or $\mathbf{r}'$. Now

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\langle 1,2,3\rangle}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle.$$

Using $\langle 1,2,3\rangle$ and the point $(1,1,1)$, an equation of the normal plane is

$$x - 1 + 2(y-1) + 3(z-1) = 0 \implies x + 2y + 3z = 6.$$

The osculating plane is determined by the vectors $\mathbf{N}$ and $\mathbf{T}$. So we can use for a normal vector $\mathbf{n} = \mathbf{B} = \mathbf{T} \times \mathbf{N}$. Now

$$\begin{aligned}\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4t^2+9t^4}}\,\langle 1, 2t, 3t^2\rangle \\ \implies \mathbf{T}'(t) &= \frac{1}{2}(1+4t^2+9t^4)^{-\frac{3}{2}}(8t+36t^3)\,\langle 1,2t,3t^2\rangle + \frac{1}{\sqrt{1+4t^2+9t^4}}\,\langle 0,2,6t\rangle, \\ \implies \mathbf{T}'(1) &= \frac{1}{2}\frac{8+36}{(\sqrt{1+4+9})^3}\,\langle 1,2,3\rangle + \frac{1}{\sqrt{1+4+9}}\,\langle 0,2,6\rangle = \frac{1}{7\sqrt{14}}\,\langle 11,8,-9\rangle, \\ \therefore \quad \mathbf{N}(1) &= \frac{\frac{1}{7\sqrt{14}}\,\langle 11,8,-9\rangle}{\sqrt{121+64+81}} = \frac{\langle 11,8,-9\rangle}{\sqrt{266}}\end{aligned}$$

For a normal vector use

$$\mathbf{n} = \langle 1,2,3\rangle \times \langle 11,8,-9\rangle = \langle -42, 42, -14\rangle = 14\langle 3,-3,1\rangle.$$

Then the osculating line has equation

$$3(x-1) - 3(y-1) + (z-1) = 0 \implies 3x - 3y + z = 1.$$

15.1

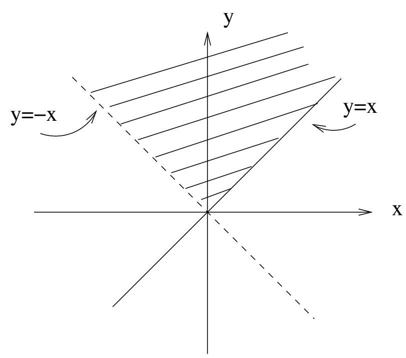
For #16 and 18 find and sketch the domain of the function.

16. $f(x,y) = \sqrt{y-x}\,\ln(y+x)$.

Solution. The domain of f is

$$D = \{(x,y)\,|\,y \geq x \text{ and } y > -x\} = \{(x,y)\,|\,-y < x \leq y,\ y > 0\}.$$

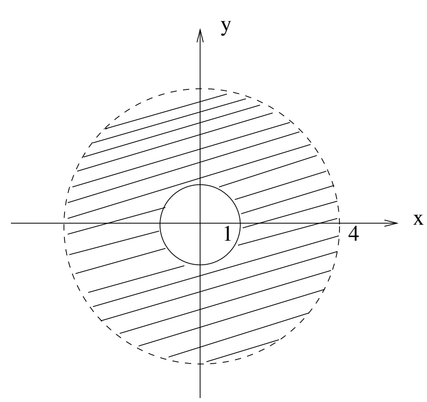
The graph of D is



18. $f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$.

Solution. For the domain of f we need $x^2+y^2-1 \geq 0$, i.e., $x^2+y^2 \geq 1$ and $4-x^2-y^2 > 0$, i.e., $x^2+y^2 < 4$. So

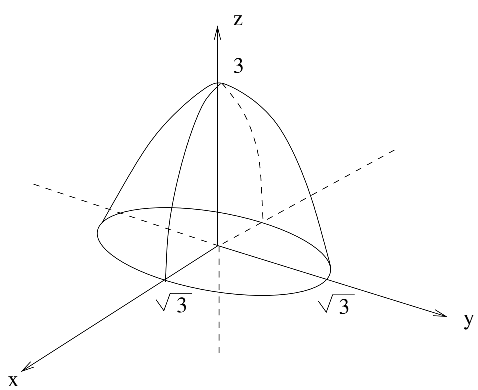
$$D = \{(x,y)\,|\,1 \leq x^2+y^2 < 4\}$$



26. Sketch the graph of the function $f(x, y) = 3 - x^2 - y^2$.

Solution. Let $z = 3 - x^2 - y^2$. We look at various traces of of f.

$$\begin{aligned}
z = 0 &: \quad x^2 + y^2 = 3 \\
z = k &: \quad x^2 + y^2 = 3 - k \quad \text{(a family of circles, } k \leq 3\text{)} \\
x = 0 &: \quad z - 3 = -y^2 \\
x = k &: \quad z - 3 + k^2 = -y^2 \quad \text{(a family of parabolas, opens down)} \\
y = 0 &: \quad z - 3 = -x^2 \\
y = k &: \quad z - 3 + k^2 = -x^2 \quad \text{(a family of parabolas, opens down)}
\end{aligned}$$



38. Draw a contour map of the function $f(x, y) = x^2 - y^2$ showing several level curves.

Solution. The level curves are $x^2 - y^2 = k$ s.t.

$$\begin{aligned}
k = 0 &: \quad x^2 - y^2 = 0 \implies y^2 = x^2 \implies y = \pm x \\
k > 0 &: \quad \frac{x^2}{k} - \frac{y^2}{k} = 1 \quad \text{(a family of hyperbolas, } x\text{-int: } x = \pm\sqrt{k}\text{)} \\
k < 0 &: \quad \frac{x^2}{k} - \frac{y^2}{k} = 1 \quad \text{(a family of hyperbolas, } y\text{-int: } y = \pm\sqrt{k}\text{)}
\end{aligned}$$