11.1

10. Let \( x = t^2, \ y = t^3 \).

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as \( t \) increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

Solution.

\[
\begin{array}{c|cccccc}
 t & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
x & 9 & 4 & 1 & 0 & 1 & 4 & 9 \\
y & -27 & -8 & -1 & 0 & 1 & 8 & 27
\end{array}
\]

(b) \( t = \pm \sqrt{x} \implies y = \pm x^{3/2}, \ x \geq 0, \ y \in \mathbb{R}. \) Or \( x = y^{2/3}, \ x \geq 0, \ y \in \mathbb{R}. \)

For #12 and 16

(a) Eliminate the parameter to find a Cartesian equation of the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

12. \( x = 4 \cos \theta, \ y = 5 \sin \theta, \ -\pi/2 \leq \theta \leq \pi/2. \)
Solution.

(a) Since $\cos^2 \theta + \sin^2 \theta = 1$, we get $(x/4)^2 + (y/5)^2 = 1$, i.e., \( \frac{x^2}{16} + \frac{y^2}{25} = 1 \) which is an ellipse with the $x$-intercepts $x = \pm 4$, the $y$-intercepts $y = \pm 5$. But since $-\pi/2 \leq \theta \leq \pi/2$, we have $0 \leq \cos \theta \leq 1$ so the graph consists of only the portion on the right side of the $y$-axis.

(b)

\[ x = \ln t, \quad \sqrt{t}, \quad t \geq 1. \]

Solution.

(a) $x = \ln t \implies t = e^x \implies y = \sqrt{e^x} = e^{x/2}, \quad x \geq 0.$

(b)

22. Describe the motion of a particle with position $(x, y)$ as $t$ varies in the given interval:

\[ x = \cos^2 t, \quad t = \cos t, \quad 0 \leq t \leq 4\pi. \]

Solution. \( x = y^2 \) is a parabola opening to the right with the vertex $(0, 0)$. The particle starts at the point $(1, 1)$ ($t = 0$). It then moves along the parabola to
the point \((0, 0)\) \((t = \pi/2)\) down to \((1, -1)\) \((t = \pi)\), then back to \((0, 0)\) \((t = 3\pi/2)\) and to \((1, 1)\) \((t = 2\pi)\). The same motion is repeated from \(2\pi\) to \(4\pi\).

11.2

8. Find an equation of the tangent to the curve \(x = \tan \theta, y = \sec \theta\) at the point \((1, \sqrt{2})\) by 2 methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

Solution. (a)

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec \theta \tan \theta}{\sec^2 \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta,
\]

at \((1, \sqrt{2})\), \(1 = \tan \theta, \sqrt{2} = \sec \theta \implies \theta = \frac{\pi}{4}\) (or \(\pi/4 + 2n\pi\)),

(since \(\tan \theta > 0\) in quadrant I and III while \(\sec \theta > 0\) in quad. I and IV). Hence the slope of the tangent at \((1, \sqrt{2})\) is

\[
y' \left( \frac{\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},
\]

and the equation is

\[
y - \sqrt{2} = \frac{\sqrt{2}}{2}(x - 1).
\]

(b) Since \(\tan^2 \theta + 1 = \sec^2 \theta \implies x^2 + 1 = y^2\), differentiating implicitly we obtain

\[
2x = 2yy' \implies \frac{dy}{dx} = \frac{x}{y} \implies y'(1, \sqrt{2}) = \frac{1}{\sqrt{2}}
\]

which gives the same equation as in part (a).

16. Find \(dy/dx\) and \(d^2y/dx^2\) if \(x = \cos 2t, y = \cos t, 0 < t < \pi\). For which values of \(t\) is the curve concave upward?
Solution.

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2 \sin 2t} = \frac{\sin t}{4 \sin t \cos t} = \frac{1}{4 \cos t} = \frac{1}{4} \sec t, \]

\[ \frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dx/dt}{dx/dt} = \frac{\frac{1}{3} \sec t \tan t}{-2 \sin 2t} = -\frac{1}{16} \frac{\sin t}{\cos^2 t \sin t \cos t} = \frac{1}{16 \cos^3 t} \text{ or } -\frac{1}{16 \sec^3 t}. \]

The curve is concave upward if \( y''(x) > 0 \) on \( 0 < t < \pi \), i.e.,

\[-\frac{1}{16 \sec^3 t} > 0 \iff \sec t < 0 \iff \cos t < 0 \iff t \in (\pi/2, \pi). \]

18. Find the points on the curve \( x = 2t^3 + 3t^2 - 12t, \ y = 2t^3 + 3t^2 + 1 \) where the tangent is horizontal or vertical.

**Solution.** We find the derivative

\[ \frac{dy}{dx} = \frac{6t^2 + 6t}{6t^2 + 6t - 12} = \frac{6(t+1)}{6(t+2)(t-1)} = \frac{t(t+1)}{(t+2)(t-1)}. \]

Horizontal tangents occur when \( y' = 0 \), i.e., \( t = 0, -1 \). So the points are

\[ t = 0 \implies x = 0, \ y = 1, \ i.e., (0, 1), \]
\[ t = -1 \implies x = -2 + 3 + 12 = 13, \ y = -2 + 3 + 1 = 2, \ i.e., (13, 2). \]

Vertical tangents occur at \( t = -2, 1 \) (\( y' \) does not exist there). Then the points are:

\[ t = -2 \implies x = -16 + 12 + 24 = 20, \ y = -16 + 12 + 1 = -3, \ i.e., (20, -3), \]
\[ t = 1 \implies x = 2 + 3 - 12 = -7, \ y = 2 + 3 + 1 = 6, \ i.e., (-7, 6). \]

34. Find the area of the region enclosed by the astroid \( x = a \cos^3 \theta, \ y = a \sin^3 \theta. \)

**Solution.** The graph of the astroid is
Using symmetry

\[ A = 4 \int_{0}^{a} y \, dx = 4 \int_{0}^{\frac{\pi}{4}} a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) \, d\theta \]

\[ = -12a^2 \int_{0}^{\frac{\pi}{4}} \sin^4 \theta \cos^2 \theta \, d\theta = -12a^2 \int_{0}^{\frac{1}{4}} (1 - \cos 2\theta)^2 \cdot \frac{1}{2} (1 + \cos 2\theta) \, d\theta \]

\[ = -\frac{3}{2}a^2 \int_{0}^{\frac{\pi}{4}} (1 - 2 \cos 2\theta + \cos^2 2\theta)(1 + \cos 2\theta) \, d\theta \]

\[ = \frac{3}{2}a^2 \left( \frac{1}{2} \theta - \frac{1}{2} \sin 2\theta - \frac{1}{8} \sin 4\theta \right) \bigg|_{0}^{\frac{\pi}{4}} + \int_{0}^{\frac{\pi}{4}} \left( \frac{\cos^2 2\theta}{1 - \sin^2 2\theta} \right) \cos 2\theta \, d\theta \]

\[ = \frac{3}{2}a^2 \left( \frac{1}{2} \cdot \frac{\pi}{4} - 0 - 0 \right) + \frac{1}{2} \int_{0}^{0} (1 - u^2) \, du \quad (u = \sin 2\theta) \]

\[ = \frac{3}{2}a^2 \cdot \frac{\pi}{4} = \frac{3}{8}a^2 \pi. \]

44. Find the length of the curve \( x = e^t + e^{-t}, \ y = 5 - 2t, \ 0 \leq t \leq 3. \)

**Solution.** \( \frac{dx}{dt} = e^t - e^{-t} \) and \( \frac{dy}{dt} = -2. \) So

\[ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = (e^t - e^{-t})^2 + 4 = e^{2t} - 2 + e^{-2t} + 4 \]

\[ = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2 \]

\[ \therefore \ L = \int_{0}^{3} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{3} (e^t + e^{-t}) \, dt \]

\[ = e^t - e^{-t} \bigg|_{0}^{3} = e^3 - e^{-3} - 1 + 1 = e^3 - e^{-3}. \]

60. Find the area of the surface obtained by rotating the curve \( x = 3t - t^3, \ y = 3t^2, \)
\( 0 \leq t \leq 1 \) about the \( x \)-axis.

**Solution.**

\[ S = 2\pi \int_{a}^{b} y \, ds = 2\pi \int_{0}^{1} 3t^2 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

\[ = 2\pi \int_{0}^{1} 3t^2 \sqrt{3^2 + (-6t)^2} \, dt \]

\[ = 2\pi \int_{0}^{1} 3t^2 \sqrt{9 + 36t^2} \, dt \]

\[ = 2\pi \left[ \frac{t}{3} \left( 3t^2 + 9 \right)^{\frac{3}{2}} \right]_{0}^{1} \]

\[ = 2\pi \left( \frac{1}{3} (3 + 9)^{\frac{3}{2}} - \frac{0}{3} \right) \]

\[ = 2\pi \left( \frac{1}{3} \cdot 12^{\frac{3}{2}} \right) \]

\[ = 2\pi \cdot 4 \cdot 3 \]

\[ = 24\pi. \]
where
\[
\left(\frac{dx}{dt}\right)^2 = (3 - 3t^2)^2 = 9 - 18t^2 + 9t^4
\]
\[
\left(\frac{dy}{dt}\right)^2 = (6t)^2 = 36t^2
\]
\[
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9 - 18t^2 + 9t^4 + 36t^2 = 9 + 18t^2 + 9t^4 = (3 + 3t^2)^2
\]

\[S = 2 \pi \int_0^1 3t^2(3 + 3t^2) dt = 2 \pi \int_0^1 9t^2(1 + t^2) dt\]
\[= 18 \pi \int_0^1 (t^2 + t^4) dt = 18 \pi \left( t^3 + t^5 \right)^1_0 = 18 \pi \left( \frac{1}{3} + \frac{1}{5} \right) = \frac{48 \pi}{5}. \]

66. Find the surface area generated by rotating the given curve about the y-axis.

\[x = e^t - t, \quad y = 4t^{1/2}, \quad 0 \leq t \leq 1\]

Solution. \[S = 2 \pi \int_0^1 x \, ds \text{ where } ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt.\]
\[
\left(\frac{dx}{dt}\right)^2 = (e^t - 1)^2 = e^{2t} - 2e^t + 1, \quad \left(\frac{dy}{dt}\right)^2 = (2e^{t/2})^2 = 4e^t
\]

\[\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2
\]

\[\therefore \quad S = 2 \pi \int_0^1 (e^t - t)(e^t + 1) \, dt
\]
\[= 2 \pi \int_0^1 (e^{2t} + e^t - te^t - t) \, dt
\]
\[= 2 \pi \left( \frac{1}{2}e^{2t} + e^t - (te^t - e^t) - \frac{t^2}{2} \right)^1_0
\]
\[= 2 \pi \left( \frac{1}{2}e^2 + e - e + e - \frac{1}{2} - \frac{1}{2} - 1 - 1 \right) = \pi(e^2 + 2e - 6). \]

(Here to do \[\int te^t \, dt, \text{ integration by parts was used.}\)