MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 9

1. (**10.2.10**) Find the values of λ (eigenvalues) for which the given problem has a nontrivial solution. Also find the corresponding nontrivial solutions (eigenfunctions):

$$y'' + \lambda y = 0, \quad 0 < x < \pi, \quad y'(0) = 0, \;\; y(\pi) = 0.$$

Solution:

Case 1: Assume $\lambda = 0$.
We have $y = c_1 x + c_2$. Using the conditions $y(\pi) = 0$ and $y'(0) = 0$ we get $c_1 = 0$ and $c_2 = 0$. So there is only the trivial solution.
Case 2: Assume $\lambda > 0$.
The auxiliary equation is $r^2 + \lambda = 0$ so that $r = \pm i\sqrt{\lambda}$. Then

$$y = c_1 cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$y'(0) = 0 \quad \Longrightarrow \quad c_2\sqrt{\lambda} = 0 \quad \therefore \quad c_2 = 0$$

$$y(\pi) = 0 \quad \Longrightarrow \quad c_1\cos\sqrt{\lambda}\pi = 0.$$

If we want $c_1 \neq 0$, $\cos\sqrt{\lambda}\pi = 0$,i.e.,

$$\sqrt{\lambda} = \frac{(2n+1)}{2}$$

$$\lambda = \frac{(2n+1)^2}{4}, \quad n = 0, 1, \ldots \quad \text{(eigenvalues)}$$

$$\therefore \quad y_n = c_n \cos\left(\frac{2n+1}{2}\right) x \quad \text{(eigenfunctions).}$$

Case 3: Assume $\lambda < 0$.
So $r^2 + \lambda = 0$ given $r^2 = -\lambda = K^2 > 0$ and $r = \pm K$. Then $y = c_1 e^{Kx} + c_2 e^{-Kx}$. Using the conditions $y'(0) = 0$ and $y(\pi) = 0$ we arrive at $c_1 = c_2 = 0$. So we have only the trivial solution in this case.

2. (**10.2.16**) Solve the heat flow problem:

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,\ t > 0$$
$$u(0,t) = u(\pi,t) = 0, \qquad t > 0$$
$$u(x,0) = \sin 3x + 5\sin 7x - 2\sin 13x.$$

Solution:

We use the results we arrived at in class: Since $u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = f(x)$,

$$\sum_{n=1}^{\infty} c_n \sin nx = \sin 3x + 5\sin 7x - 2\sin 13x$$

so that $c_3 = 1$, $c_7 = 5$, $c_{13} = -2$ and all other $c_i = 0$. Then

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-3(\frac{n\pi}{L})^2 t} \sin\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \sin nx \\ &= e^{-27t}\sin 3x + 5e^{-147t}\sin 7x - 2e^{-507t}\sin 13x. \end{aligned}$$

3. (**10.2.22**) Solve the heat flow problem:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 x}{\partial x^2}, \qquad 0 < x \leq \pi,\ t > 0$$
$$u(0,t) = u(\pi,t) = 0, \qquad t \geq 0$$
$$u(x,0) = \sin x - \sin 2x + \sin 3x = f(x)$$
$$u_t(x,0) = 6\sin 3x - 7\sin 5x = g(x).$$

Solution:

Using the results from class:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = \sin x - \sin 2x + \sin 3x$$
$$u_t(x,0) = \sum_{n=1}^{\infty} 3nb_n \sin nx = 6\sin 3x - 7\sin 5x.$$

Therefore

$$a_1 = 1, \qquad a_2 = -1, \qquad a_3 = 1$$
$$9b_3 = 6 \quad \Longrightarrow \quad b_3 = \frac{2}{3}$$
$$15b_5 = -7 \quad \Longrightarrow \quad b_5 = -\frac{7}{15}$$

The solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} [a_n \cos 3nt + b_n \sin 3nt] \sin nx \\ &= \sum_{n=1}^{\infty} a_n \cos 3nt \sin nx + \sum_{n=1}^{\infty} b_n \sin 3nt \sin nx \\ &= \cos 3t \sin x - \cos 6t \sin 2x + \cos 9t \sin 3x \\ &\qquad + \frac{2}{3}\sin 9t \sin 3x - \frac{7}{15}\sin 15t \sin 5x. \end{aligned}$$

4. (**10.2.28**) Show that $u_{tt} + u_t + u = \alpha^2 u_{xx}$ with $u(x,t) = X(x)T(t)$ yields

$$X''(x) - \lambda X(x) = 0,$$
$$T''(t) + T'(t) + (1 - \lambda\alpha^2)T(t) = 0,$$

where λ is a constant.

Solution:

Let $u(x,t) = X(x)T(t)$. Then

$$u_t = X(x)T'(t), \qquad u_{tt} = X(x)T''(t), \qquad u_{xx} = X''(x)T(t),$$

and substituting into the P.D.E. gives

$$X(x)T''(t) + X(x)T'(t) + X(x)T(t) = \alpha^2 X''(x)T(t)$$
$$X(x)(T''(t) + T'(t) + T(t)) = \alpha^2 X''(x)T(t).$$

Separating the variables,

$$\frac{T''(t) + T'(t) + T(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

i.e.,

$$\frac{T''(t) + T'(t) + T(t)}{\alpha^2 T(t)} = \lambda \qquad \text{and} \qquad \frac{X''(x)}{X(x)} = \lambda.$$

So we have the desired equations:

$$X''(x) - \lambda X(x) = 0$$
$$T''(t) + T'(t) + (1 - \lambda\alpha^2)T(t) = 0.$$

5. (**10.3.10**) Find the Fourier series of $f(x) = |x|,\ -\pi < x < \pi$.

Solution:

The Fourier series of f on $[-\pi, \pi]$ is given by (with $T = \pi$)

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \\
&\text{(since } |x| \sin nx \text{ is odd)} \\
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \\
&= \frac{2}{\pi} \left[\frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx \right|_0^{\pi} = \frac{2}{\pi} \left[\frac{1}{n^2} \cos n\pi - \frac{1}{n^2} \right] \\
&= \frac{2}{\pi n^2} [(-1)^n - 1], \qquad n = 1, 2, \ldots .
\end{aligned}$$

Therefore

$$\begin{aligned}
f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx \\
\text{or} \qquad &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{-2}{n^2} \cos nx \\
\text{or} \qquad &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1} \frac{1}{(2k-1)^2} \cos(2k-1)x .
\end{aligned}$$

6. (**10.3.14**) Compute the Fourier series for

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ x + \pi, & -\pi < x < 0. \end{cases}$$

Solution:

We compute the coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\,dx + \int_0^{\pi} x\,dx\right] \\
&= \frac{1}{\pi}\left[\frac{x^2}{2} + \pi x\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi}\right] = \pi. \\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right] \\
&= \frac{1}{\pi}\left[\frac{1}{n}(x+\pi)\sin nx + \frac{1}{n^2}\cos nx\Big|_{-\pi}^{0} + \left(\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\Big|_0^{\pi}\right.\right. \\
&= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\cos(-n\pi) + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = 0 \\
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right] \\
&= \frac{1}{\pi}\left[-\frac{x+\pi}{n}\cos nx + \frac{1}{n^2}\sin nx\Big|_{-\pi}^{0} + \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right.\right. \\
&= \frac{1}{\pi}\left[-\frac{\pi}{n} - \frac{\pi}{n}\cos n\pi\right] = -\frac{1}{n}[1 + (-1)^n] \\
&= \begin{cases} 0 & \text{if } n \text{ odd} \\ -\frac{2}{n} & \text{if } n \text{ even.} \end{cases}
\end{aligned}$$

Then the Fourier series for $f(x)$ is

$$f(x) \sim \frac{\pi}{2} + \sum_{n\ \text{even}}\left(-\frac{2}{n}\right)\sin nx$$

$$\text{or} \qquad = \frac{\pi}{2} - 2\sum_{k=1}^{\infty}\frac{\sin 2kx}{2k} = \frac{\pi}{2} - \sum_{k=1}^{\infty}\frac{1}{k}\sin 2kx.$$