MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 8

1. (**8.3.16**) Find the power series solution for $y'' - 2y' + y = 0$ near $x = 0$.

Solution:

Assume (since $x = 0$ is an ordinary point)

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}.$$

Substituting the above into the d.e., we obtain

$$\sum_{n=2} n(n-1)a_n x^{n-2} - 2\sum_{n=1} n a_n x^{n-1} + \sum_{n=0} a_n x^n = 0.$$

Letting $k = n-2$ in the first summation, $k = n-1$ in the second summation, and $n = k$ in the third summation we arrive at

$$\sum_{k=0}(k+2)(k+1)a_{k+2}x^k - 2\sum_{k=0}(k+1)a_{n+1}x^k + \sum_{k=0} a_k x^k = 0$$

$$\Longrightarrow \qquad \sum_{n=0}[(k+2)(k+1)a_{k+2} - 2(k+1)a_{k+1} + a_k]x^k = 0.$$

The recursion formula is

$$a_{k+2} = \frac{2a_{k+1}}{k+2} - \frac{a_k}{(k+2)(k+1)}, \qquad n \geq 0.$$

Calculating the first few coefficients:

$$a_2 = \frac{2a_1}{-}\frac{a_0}{2} = a_1 - \frac{1}{2}a_0, \qquad a_0,\ a_1 \text{arbitrary}$$

$$a_3 = \frac{2a_2}{3} - \frac{a_1}{3(2)} = \frac{2}{3}\left(a_1 - \frac{1}{2}a_0\right) - \frac{1}{6}a_1 = \frac{1}{2}a_1 + \frac{1}{3}a_0$$

$$a_4 = \frac{2a_3}{4} - \frac{a_2}{4(3)} = \frac{1}{2}\left(\frac{1}{2}a_1 + \frac{1}{3}a_0\right) - \frac{1}{12}\left(a_1 - \frac{1}{2}a_0\right) = \frac{1}{6}a_1 + \frac{5}{24}a_0.$$

So the solution is

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + a_1 x + \left(a_1 - \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{2}a_1 + \frac{1}{3}a_0\right)x^3 + \cdots \\ &= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right) + a_1\left(x + x^2 + \frac{1}{2}x^3 + \cdots\right). \end{aligned}$$

2. (8.3.22) Find a power series solution to $y'' - xy = 0$ about $x = 0$.

Solution:

We have

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

The differential equation becomes

$$\sum_{n=2} n(n-1) a_n x^{n-2} - \sum_{n=0} a_n x^{n+1} = 0.$$

Let $k = n - 2$ and $k = n + 1$, resp. in the first and second summations. This gives

$$\sum_{k=0} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1} a_{k-1} x^k = 0$$

$$2a_2 + \sum [(k+2)(k+1) a_{k+2} - a_{k-1}] x^k = 0.$$

So $a_2 = 0$, a_1 and a_0 arbitrary. The recurrence relation is then

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}, \qquad k \geq 1.$$

Calculating the first few coefficients we find that

$$a_{3n-1} = 0, \qquad n \geq 1$$

$$a_{3n} = \frac{a_0}{(3n)(3n-1)(3n-3)(3n-4)(3n-6)\cdots 3 \cdot 2}, \qquad n \geq 1$$

$$a_{3n+1} = \frac{a_1}{(3n+1)(3n)(3n-2)(3n-1)\cdots 4 \cdot 3}, \qquad n \geq 1$$

Then the solution is

$$y = a_0 + a_0 \sum_{n=1} \frac{1}{(3n)(3n-1)\cdots 2} x^{3n} + a_1 x + a_1 \sum_{n=1} \frac{1}{(3n+1)(3n)\cdots 3} x^{3n+1}$$

or $$y = a_0 \left(1 + \frac{1}{3(2)} x^3 + \frac{1}{6(5)(3)(2)} x^6 + \cdots\right) + a_1 \left(x + \frac{1}{4(3)} x^4 + \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots\right).$$

3. **(8.3.28)** Solve $y'' + (x-2)y' - y = 0, \qquad y(0) = -1, \;\; y'(0) = 0.$

Solution:

Assume

$$y = \sum_{n=0} a_n x^n, \qquad y' = \sum_{n=1} n a_n x^{n-1}, \qquad y'' = \sum_{n=2} n(n-1) a_n x^{n-2}.$$

Substitute into the d.e.

$$\sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=1} n a_n x^n - 2\sum_{n=1} n a_n x^{n-1} - \sum_{n=0} a_n x^n = 0.$$

Let $k = n-2$, $k = n$, $k = n-1$, and $k = n$, resp. in the summations we obtain

$$\sum_{k=0}(k+2)(k+1)a_{k+2}x^k + \sum_{k=1} k a_k x^k - 2\sum_{k=0} a_{k+1}(k+1)x^k - \sum_{k=0} a_k x^k = 0$$

so that

$$2a_2 - 2a_1 - a_0 + \sum_{k=1}[(k+2)(k+1)a_{k+2} - 2(k+1)a_{k+1} + (k-1)a_k]x^k = 0.$$

Hence $a_2 = a_1 + \frac{1}{2}a_0$ where a_0 and a_1 are arbitrary. The recursion formula is

$$a_{k+2} = \frac{2a_{k+1}}{k+2} - \frac{(k-1)a_k}{(k+2)(k+1)}, \qquad k = 1, 2, \ldots$$

And the first few coefficients are

$$\begin{aligned} a_3 &= \frac{2}{3}a_1 + \frac{1}{3}a_0 \\ a_4 &= \frac{1}{4}a_1 + \frac{1}{8}a_0. \end{aligned}$$

Therefore the general solution is

$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right) + a_1\left(x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots\right).$$

Since $y(0) = -1$ and $y'(0) = 0$, $a_0 = -1$ and $a_1 = 0$. Then the solution to the I.V.P. is

$$y = -1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \cdots.$$

4. **(8.4.6)** Find the minimum value for the radius of convergence of the power series solution to

$$(1+x^3)y'' - xy' + 3x^2y = 0, \quad \text{about} \quad x_0 = 1.$$

Solution:

Since $p(x) = \dfrac{-x}{1+x^3}$ and $q(x) = \dfrac{3x^2}{1+x^3}$, and

$$1 + x^3 = (1+x)(x^2 - x + 1) = 0$$

the singular points are: $x = -1$, $\dfrac{1 \pm i\sqrt{3}}{2}$. Then the distance between $x_0 = 1$ and $x = -1$ is 2, and the distance between x_0 and $x = \dfrac{1 \pm i\sqrt{3}}{2}$ is

$$d = \sqrt{\left(\frac{1}{2}\right)^2 + \frac{3}{4}} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 < 2.$$

Therefore the minimum value for the radius of convergence of the solution is 1.

5. (8.4.8) Find the solution for $y' - 2xy = 0$ near $x_0 = -1$.

Solution:

We want a solution of the form $y = \sum\limits_{n=0}^{\infty} a_n(x+1)^n$. To simplify our work let $u = x + 1$ so that $x = u - 1$ and $y(x) = y(u-1)$. Also

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d^2y}{du^2}\frac{du}{dx} = \frac{d^2y}{du^2}.$$

The d.e. becomes $y'(u) - 2(u-1)y(u) = 0$ near $u = 0$. Assume a solution of the form $y(u) = \sum_{n=0} a_n u^n$, then $y'(u) = \sum_{n=1}^{\infty} n a_n u^{n-1}$. Then

$$\begin{aligned}
& \sum_{n=1} n a_n u^{n-1} - 2(u-1)\sum_{n=0} a_n u^n = 0 \\
\Longrightarrow \quad & \underbrace{\sum_{n=1} n a_n u^{n-1}}_{k=n-1} \underbrace{-2\sum_{n=0} a_n u^{n+1}}_{k=n+1} \underbrace{+2\sum_{n=0} a_n u^n}_{k=n} = 0 \\
\Longrightarrow \quad & \sum_{k=0}(k+1)a_{k+1}u^k - 2\sum_{k=1} a_{k-1}u^k + 2\sum_{k=0} a_k u^k = 0 \\
\Longrightarrow \quad & a_1 + 2a_0 + \sum_{k=1}[(k+1)a_{k+1} - 2a_{k-1} + 2a_k]u^k = 0.
\end{aligned}$$

Thererfore $a_1 = -2a_0$ and

$$a_{k+1} = \frac{2a_{k-1} - 2a_k}{k+1} = \frac{2}{k+1}(a_{k-1} - a_k), \qquad k \geq 1.$$

Calculating some of the coefficients,

$$\begin{aligned}
a_2 &= a_0 - (-2a_0) = 3a_0, \\
a_3 &= \frac{2}{3}(a_1 - a_0) = \frac{2}{3}(-2a_0 - 3a_0) = -\frac{10}{3}a_0.
\end{aligned}$$

The general solution is

$$\begin{aligned}
y(u) &= \sum_{n=0} a_n u^n = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots \\
&= a_0 - 2a_0 u + 3a_0 u^2 - \frac{10}{3}u^3 + \cdots \\
\therefore \quad y(x) &= a_0(1 - 2(x+1) + 3(x+1)^2 - \tfrac{10}{3}(x+1)^3 + \cdots).
\end{aligned}$$

6. **(8.4.16)** Solve $y'' + ty' + e^t y = 0, \qquad y(0) = 0, \quad y'(0) = -1.$

Solution:

Assume

$$y = \sum_{n=0} a_n t^n, \qquad y' = \sum_{n=1} n a_n t^{n-1}, \qquad y'' = \sum_{n=2} n(n-1) a_n t^{n-2}.$$

Then

$$\sum_{n=2} n(n-1) a_n t^{n-2} + \sum_{n=1} n a_n t^n + \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \sum_{n=0} a_n t^n = 0.$$

Writing out the first few terms

$$\begin{aligned} &(2a_2 + 6a_3 t + 12 a_4 t^2 + \cdots) + (a_1 t + 2a_t^2 + 3a_3 t^3 + \cdots) \\ &\qquad + a_0 + a_0 t + a_0 \frac{t^2}{2!} + a_0 \frac{t^3}{3!} + a_1 t + a_1 t^2 + a_1 \frac{t^3}{2!} \\ &\qquad\qquad + a_2 t^2 + a_2 t^3 + a_2 \frac{t^4}{2!} + \cdots = 0 \end{aligned}$$

Equating the coefficients of the like terms:

$$\begin{aligned} \text{consts.:} &\quad 2a_2 + a_0 = 0 \\ t: &\quad 6a_3 + a_1 + a_0 + a_1 = 0 \\ t^2: &\quad 12a_4 + 2a_2 + \frac{a_0}{2} + a_1 + a_2 = 0 \\ t^3: &\quad 20a_5 + 3a_3 + \frac{a_0}{6} + \frac{a_1}{2} + a_2 + a_3 = 0 \\ \text{etc.} & \end{aligned}$$

Since $a_0 = 0$ and $a_1 = -1$, we have

$$a_2 = 0, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{12}, \quad a_5 = -\frac{1}{24},$$

and the solution is

$$\begin{aligned} y &= a_0 + a_1 t + a_2 t^2 + \ldots \\ &= -t + \frac{1}{3} t^3 + \frac{1}{12} t^4 - \frac{1}{24} t^5 + \ldots . \end{aligned}$$