MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 7

1. (**7.7.22**) Solve the integro-differential equation:

$$y'(t) - 2\int_0^t e^{t-v} y(v)\,dv = t, \quad y(0) = 2.$$

Solution:

Taking Laplace transform of the equation

$$\begin{aligned}
sY(s) - y(0) - 2\left(\frac{1}{s-1}\right)Y(s) &= \frac{1}{s^2} \\
\left(s - \frac{2}{s-1}\right)Y(s) &= \frac{1}{s^2} + 2 \\
\frac{(s-2)(s+1)}{s-1}\,Y(s) &= \frac{1+2s^2}{s^2} \\
Y(s) &= \frac{(1+2s^2)(s-1)}{s^2(s-2)(s+1)} = \frac{2s^3 - 2s^2 + s - 1}{s^2(s-2)(s+1)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s+1}.
\end{aligned}$$

So

$$2s^3 - 2s^2 + s - 1 = As(s-2)(s+1) + B(s-2)(s+1) + Cs^2(s+1) + Ds^2(s-2).$$

Carrying on with the partial fraction decomposition:

$$\begin{array}{llll}
s = 0: & -1 = -2B & \Longrightarrow & B = \dfrac{1}{2} \\
s = 2: & 9 = 12C & \Longrightarrow & C = \dfrac{3}{4} \\
s = -1: & -6 = -3D & \Longrightarrow & D = 2 \\
s^3: & 2 = A + C + D & \Longrightarrow & A = -\dfrac{3}{4}.
\end{array}$$

We then have

$$Y(s) = \frac{-3/4}{s} + \frac{1/2}{s^2} + \frac{3/4}{s-2} + \frac{2}{s+1}.$$

Then the solution is

$$y(t) = -\frac{3}{4} + \frac{1}{2}t + \frac{3}{4}e^{2t} + 2e^{-t}.$$

2. (7.9.4) Solve the system of linear equations:

$$(1) \qquad x' - 3x + 2y = \sin t, \qquad x(0) = 0$$

$$(2) \qquad 4x - y' - y = \cos t, \qquad y(0) = 0.$$

Solution:

Taking Laplace transform:

$$\begin{aligned} sX(s) - x(0) - 3X(s) + 2Y(s) &= \frac{1}{s^2+1} \\ 4X(s) - sY(s) + y(0) - Y(s) &= \frac{s}{s^2+1}. \end{aligned}$$

This leads to:

$$(3) \qquad (s-3)X(s) + 2Y(s) = \frac{1}{s^2+1}$$

$$(4) \qquad 4X(s) - (s+1)Y(s) = \frac{s}{s^2+1}.$$

Multiplying (3) by 4 and (4) by $s-3$ and subtracting the equations we arrive at:

$$\begin{aligned} (s^2 - 2s + 5)Y(s) &= \frac{-s^2+3s+4}{s^2+1} \\ Y(s) &= \frac{-s^2+3s+4}{(s^2+1)(s^2-2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+5}. \end{aligned}$$

We have

$$-s^2 + 3s + 4 = (As+B)(s^2-2s+5) + (Cs+D)(s^2+1).$$

Equating the coefficients of the similar terms:

$$\begin{aligned} s^3: &\qquad 0 = A + C \\ s^2: &\qquad -1 = -2A + B + D \\ s: &\qquad 3 = 5A - 2B + C \\ \text{const.} &\qquad 4 = 5B + D. \end{aligned}$$

Solving the system we get:

$$A = \frac{11}{10}, \qquad B = \frac{7}{10}, \qquad C = -\frac{11}{10}, \qquad D = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{\frac{11}{10}s + \frac{7}{10}}{s^2+1} + \frac{-\frac{11}{10}s + \frac{1}{2}}{(s-1)^2+4} \\ &= \frac{\frac{11}{10}s}{s^2+1} + \frac{\frac{7}{10}}{s^2+1} - \frac{11}{10}\frac{s-1}{(s-1)^2+4} - \frac{3}{5}\frac{2}{2[(s-1)^2+4]}. \end{aligned}$$

Therefore

$$y(t) = \frac{11}{10}\cos t + \frac{7}{10}\sin t - \frac{11}{10}e^t\cos 2t - \frac{3}{10}e^t\sin 2t.$$

To find $x(t)$ use the second equation in the original system:

$$\begin{aligned} x(t) &= \frac{1}{4}(\cos t + y' + y) \\ &= \frac{7}{10}\cos t - \frac{1}{10}\sin t - \frac{7}{10}e^t\cos 2t + \frac{2}{5}e^t\sin 2t. \end{aligned}$$

So the solution to the system is

$$\begin{aligned} x(t) &= \frac{7}{10}\cos t - \frac{1}{10}\sin t - \frac{7}{10}e^t\cos 2t + \frac{2}{5}e^t\sin 2t \\ y(t) &= \frac{11}{10}\cos t + \frac{7}{10}\sin t - \frac{11}{10}e^t\cos 2t - \frac{3}{10}e^t\sin 2t. \end{aligned}$$

3. (**7.9.14**) Solve the system:

$$x'' = y + u(t-1), \qquad x(0) = 1, \quad x'(0) = 0$$

(1) $$y'' = x + 1 - u(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution:

Taking Laplace transform we arrive at:

$$\begin{aligned} s^2X(s) - \quad Y(s) &= s + \frac{e^{-s}}{s} \\ - \quad X(s) + s^2Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s}. \end{aligned}$$

Multiply the second equation by s^2 and add the two equations and solve for $Y(s)$:

$$\begin{aligned} (s^4-1)Y(s) &= 2s + \left(\frac{1}{s} - s\right) e^{-s} \\ Y(s) &= \frac{2s}{s^4-1} + \frac{1-s^2}{s(s^4-1)} e^{-s} \\ &= \frac{2s}{(s^2+1)(s-1)(s+1)} - \frac{1}{s(s^2+1)} e^{-s}. \end{aligned}$$

Applying partial fraction decomposition on the RHS we arrive at

$$Y(s) = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} - \frac{s}{s^2+1} - \frac{1}{s}e^{-s} + \frac{s}{s^2+1}e^{-s}.$$

Hence

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \cos t - u(t-1) + \cos(t-1)u(t-1).$$

Using equation (1)

$$\begin{aligned} x(t) &= y'' - 1 + u(t-1) \\ &= \frac{1}{2}e^t + \frac{1}{2}e^{-t} + \cos t - \cos(t-1)u(t-1) - 1 + u(t-1). \end{aligned}$$

The solution to the system is

$$\begin{aligned} x(t) &= \frac{1}{2}(e^t + e^{-t}) + \cos t - 1 + u(t-1)(1 - \cos(t-1)) \\ y(t) &= \frac{1}{2}(e^t + e^{-t}) - \cos t - u(t-1)(1 - \cos(t-1)). \end{aligned}$$

4. (**7.Rev.12**) Find $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^2-4s+6}\right\}$.

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2-4s+6}\right\} &= \mathcal{L}^{-1}\left\{\frac{2(s-2)+4-1}{(s-2)^2+2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2(s-2)}{(s-2)^2+2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+2}\right\} \\
&= 2e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{3e^{2t}}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} \\
&= 2e^{2t}\cos\sqrt{2}t + \frac{3}{\sqrt{2}}e^{2t}\sin\sqrt{2}t.
\end{aligned}$$

5. (**7.Rev.28**) Solve the integro-differential equation

$$y'(t) - 2\int_0^t y(v)\sin(t-v)\,dv = 1, \qquad y(0) = -1.$$

Solution:

Taking Laplace transform of the d.e. and solving for Y(s)

$$\begin{aligned}
sY(s) - y(0) - 2\left(Y(s)\,\frac{1}{s^2+1}\right) &= \frac{1}{s}\\
\left(s - \frac{2}{s^2+1}\right)Y(s) &= \frac{1}{s} - 1\\
\frac{s^3+s-2}{s^2+1}Y(s) &= \frac{1}{s} - 1\\
\therefore \qquad Y(s) &= \frac{1-s}{s}\,\frac{s^2+1}{s^3+s-2}\\
&= \frac{1-s}{s}\,\frac{s^2+1}{(s-1)(s^2+s+2)} = \frac{-(s^2+1)}{s(s^2+s+2)}.
\end{aligned}$$

Partial fraction decomposition on the RHS gives

$$\begin{aligned}
Y(s) &= \frac{-1/2}{s} + \frac{(-1/2)s + 1/2}{s^2+s+2}\\
&= -\frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{(s+\frac{1}{2}) - \frac{3}{2}}{(s+\frac{1}{2})^2 + \frac{7}{4}}\right]\\
&= -\frac{1}{2}\frac{1}{s} - \frac{1}{2}\left[\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{7}{4}} - \frac{3(\sqrt{7}/2)}{\sqrt{7}[(s+\frac{1}{2})^2+\frac{7}{4}]}\right].
\end{aligned}$$

The solution is

$$y(t) = -\frac{1}{2} - \frac{1}{2}\left(e^{-\frac{1}{2}t}\cos\frac{\sqrt{7}}{2}t - \frac{3}{\sqrt{7}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{7}}{2}t\right).$$

6. (**8.2.26**) Express the series $\sum_{n=1}^{\infty} \frac{a_n}{n+3} x^{n+3}$ in terms of x^k.

Solution:

Let $k = n + 3$. So $n = k - 3$. Substituting into the given series we obtain

$$\sum_{k=4}^{\infty} \frac{a_{k-3}}{k} x^k.$$

7. (**8.2.28**) Show that

$$2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} n b_n x^{n-1} = b_1 + \sum_{n=1}^{\infty} [2a_{n-1} + (n+1)b_{n+1}]x^n.$$

Solution:

Let $k = n + 1$ in the first summation and $k = n - 1$ in the second summation. Then

$$\begin{aligned} \text{LHS} &= 2\sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=0}^{\infty} (k+1) b_{k+1} x^k \\ &= 2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} (n+1) b_{n+1} x^n \\ &= b_1 + \sum_{n=1}^{\infty} [2a_{n-1} + b_{n+1}(n+1)]x^n = \text{RHS}. \end{aligned}$$

8. (**8.2.32**) Find the Taylor series for $f(x) = \ln(1+x)$ about $x_0 = 0$.

Solution:

The Taylor series for $f(x)$ is given by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$. Calculate the coefficients.

$$
\begin{aligned}
f(x) &= \ln(1+x) & f(0) &= 0 \\
f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\
f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -1 \\
f'''(x) &= \frac{2}{(1+x)^3} & f'''(0) &= 2 \\
&\vdots & &\vdots
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x) &= f(0) + \frac{f'(0)}{x} + \frac{f''(0)}{2!}x^2 + \cdots \\
&= x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \cdots \\
&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n.
\end{aligned}
$$