MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 6

1. (**7.5.8**) Solve the IVP: $y'' + 4y = 4t^2 - 4t + 10, \quad y(0) = 0, \;\; y'(0) = 3.$

Solution:

Taking Laplace transform of the equation

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= 4\left(\frac{2}{s^3}\right) - 4\left(\frac{1}{s^2}\right) + \frac{10}{s} \\ (s^2+4)Y(s) &= \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} + 3 \\ Y(s) &= \frac{8 - 4s + 10s^2 + 3s^3}{s^3(s^2+4)}. \end{aligned}$$

Partial fraction decomposition is

$$\frac{3s^3 + 10s^2 - 4s + 8}{s^3(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+4}$$

$$3s^3 + 10s^2 - 4s + 8 = As^2(s^2+4) + Bs(s^2+4) + C(s^2+4) + (Ds+E)s^3.$$

We then have

$$\begin{aligned} s = 0: &\qquad 8 = 4C \\ s^4: &\qquad 0 = A + D \\ s^3: &\qquad 3 = B + E \\ s^2: &\qquad 10 = 4A + C \\ s: &\qquad -4 = 4B. \end{aligned}$$

Solving the system of equations we obtain

$$A = 2, \qquad B = -1, \qquad C = 2, \qquad D = -2, \qquad E = 4.$$

Hence

$$Y(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} + \frac{-2s+4}{s^2+4} = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} - \frac{2s}{s^2+4} + \frac{2\cdot 2}{s^2+4}.$$

So the solution is

$$y(t) = 2 - t + t^2 - 2\cos 2t + 2\sin 2t.$$

2. (**7.5.12**) Solve the IVP: $w'' - 2w' + w = 6t - 2,\ \ w(-1) = 3,\ \ w'(-1) = 7.$

Solution:

Set $y(t) = w(t-1),\ y'(t) = w'(t-1),\ \ y''(t) = w''(t-1)$. Replace t by $t-1$ in the differential equation. Then

$$w''(t-1) - 2w'(t-1) + w(t-1) = t(t-1) - 2$$

$$y''(t) - 2y'(t) + y(t) = 6t - 8, \quad y(0) = 3, \quad y'(0) = 7.$$

Taking Laplace transform of the equation we arrive at

$$(s^2 - 2s + 1)Y(s) = \frac{6}{s^2} - \frac{8}{s} + 3s + 1 = \frac{6 - 8s + 3s^3 + s^2}{s^2}$$

$$Y(s) = \frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2}.$$

Partial fraction decomposition leads to:

$$\frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$3s^3 + s^2 - 8s + 6 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$$

$$s = 0: \qquad 6 = B$$

$$s = 1: \qquad 2 = D$$

$$s: \qquad -8 = A - 2B \quad \Longrightarrow \quad A = 4$$

$$s^2: \qquad 1 = -2A + B - C + D \quad \Longrightarrow \quad C = -1.$$

Hence

$$Y(s) = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2},$$

and

$$y(t) = 4 + 6t - e^t + 2e^t t$$

$$\Longrightarrow \qquad w(t) = y(t+1) = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}$$

$$= 10 + 6t + e^{t+1} + 2te^{t+1}.$$

3. (**7.5.36**) Solve the IVP: $ty'' - ty' + y = 2, \quad y(0) = 2, \quad y'(0) = -1.$

Solution:

First note that

$$\begin{aligned}
\mathcal{L}\{ty''\} &= -\frac{d}{ds}\mathcal{L}\{y''(t)\} = -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) \\
&= -\frac{d}{ds}(s^2Y(s) - 2s + 1) = -2sY(s) - s^2Y'(s) + 2 \\
\mathcal{L}\{ty'\} &= -\frac{d}{ds}\mathcal{L}\{y'(t)\} = -\frac{d}{ds}(sY(s) - y(0)) \\
&= -Y(s) - sY'(s).
\end{aligned}$$

Taking Laplace of the differential equation

$$-2sY(s) - s^2Y'(s) + 2 + Y(s) + sY'(s) + Y(s) = \frac{2}{s}$$

$$(-s^2 + s)Y'(s) + (2 - 2s)Y(s) = \frac{2}{s} - 2$$

$$Y'(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}. \qquad \text{(a linear equation)} \tag{1}$$

An integrating factor $\mu(s) = e^{\int \frac{2}{s} ds} = e^{2\ln s} = s^2$. Multiplying the equation (1) by s^2 gives

$$d(s^2Y(s)) = 2ds$$

$$s^2Y(s) = 2s + C$$

$$Y(s) = \frac{2}{s} + Cs^{-2}.$$

Since $\lim_{s\to\infty} Y(s) = \lim_{s\to\infty}\left(\frac{2}{s} + \frac{C}{s^2}\right) = 0$ for any C,

$$Y(s) = \frac{2}{s} + \frac{C}{s^2}$$

$$y(t) = 2 + Ct \quad \Longrightarrow \quad y' = C$$

$$y'(0) = -1 \quad \Longrightarrow \quad y'(0) = C = -1.$$

Therefore the solution to the IVP is

$$y(t) = 2 - t.$$

4. (**7.6.18**) Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}\right\}$.

Solution:

Using partial fraction decomposition:

$$\begin{aligned}\frac{3s^2-s+2}{(s-1)(s^2+1)} &= \frac{4}{s-1}+\frac{Bs+C}{s^2+1}\\ 3s^2-s+2 &= A(s^2+1)+(Bs+C)(s-1).\end{aligned}$$

We then have

$$\begin{array}{llll} s=1: & 4=2A & \Longrightarrow & A=2 \\ s^2: & 3=A+B & \Longrightarrow & B=1 \\ s: & -1=-B+C & \Longrightarrow & C=0. \end{array}$$

Hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{e^{-s}(3s^2-s+2)\right\}(s-1)(s^2+1) &= \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{2}{s-1}+\frac{s}{s^2+1}\right)\right\}\\ &= \mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s-1}\right\}+\mathcal{L}^{-1}\left\{\frac{se^{-s}}{s^2+1}\right\}\\ &= 2u(t-1)e^{t-1}+u(t-1)\cos(t-1).\end{aligned}$$

5. (**7.6.20**) Solve

$$I''(t) + 4I(t) = g(t), \qquad I(0) = 1, \quad I'(0) = 3$$

$$g(t) = \begin{cases} 3\sin t, & 0 \le t \le 2\pi \\ 0, & t > 2\pi. \end{cases}$$

Solution:

We write

$$\begin{aligned} g(t) &= 3\sin t - 3(\sin t)u(t - 2\pi) \\ &= 3\sin t - 3\sin(t - 2\pi)u(t - 2\pi) \\ \Longrightarrow \qquad \mathcal{L}\{g(t)\} &= \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1}. \end{aligned}$$

Let $\mathcal{L}\{I(t)\} = J(s)$. Then the differential equation becomes

$$\begin{aligned} s^2J(s) - sI(0) - I'(0) + 4J(s) &= \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1} \\ (s^2+4)J(s) &= \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1} + s + 3 \\ \therefore \qquad J(s) &= \frac{3}{(s^2+1)(s^2+4)} - e^{-2\pi s}\frac{3}{(s^2+1)(s^2+4)} \\ &\quad + \frac{s}{s^2+4} + \frac{3}{s^2+4}. \end{aligned}$$

Using partial fractions decomposition we arrive at

$$J(s) = \frac{1}{s^2+1} - \frac{1}{s^2+4} - e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) + \frac{s}{s^2+4} + \frac{3(2)}{2(s^2+4)}.$$

Hence the solution is

$$\begin{aligned} I(t) &= \sin t - \frac{1}{2}\sin 2t - u(t-2\pi)\left(\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)\right) + \cos 2t + \frac{3}{2}\sin 2t \\ &= \sin t + \sin 2t + \cos 2t - u(t-2\pi)\left(\sin t - \frac{1}{2}\sin 2t\right). \end{aligned}$$

6. (**7.6.36**) Solve the IVP: $y'' + 5y' + 6y = t\,u(t-2), \quad y(0) = 0, \quad y'(0) = 1.$

Solution:

Taking Laplace transforms and solving for $Y(s)$ we obtain

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 6Y(s) &= \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s} \\
(s^2 + 5s + 6)Y(s) &= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + 1 \\
Y(s) &= e^{-2s}\left(\frac{1+2s}{s^2(s^2+5s+6)}\right) + \frac{1}{s^2+5s+6}.
\end{aligned}
$$

Partial fractions decompositions on $\frac{2s+1}{s^2(s+3)(s+2)}$ and $\frac{1}{(s+3)(s+2)}$ lead to

$$
Y(s) = e^{-2s}\left(\frac{7}{36}\,\frac{1}{s} + \frac{1}{6}\,\frac{1}{s^2} + \frac{5}{9}\,\frac{1}{s+3} - \frac{3}{4}\frac{1}{s+2}\right) - \frac{1}{s+3} + \frac{1}{s+2}.
$$

So the solution to the IVP is

$$
\begin{aligned}
y(t) &= u(t-2)\left[\frac{7}{36} + \frac{1}{6}(t-2) + \frac{5}{9}e^{-3(t-2)} - \frac{3}{4}e^{-2(t-2)}\right] - e^{-3t} + e^{-2t} \\
&= u(t-2)\left[-\frac{5}{36} + \frac{1}{6}t + \frac{5}{9}e^{-3(t-2)} - \frac{3}{4}e^{-2(t-2)}\right] - e^{-3t} + e^{-2t}.
\end{aligned}
$$

7. (**7.6.40**) Solve the IVP:

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 2, \quad y'(0) = -1$$

$$g(t) = \begin{cases} e^{-t}, & 0 \le t < 3 \\ 1, & 3 < t. \end{cases}$$

Solution:

Rewrite $g(t)$ using the unit step function:

$$\begin{aligned} g(t) &= e^{-t} - e^{-t}u(t-3) + u(t-3) \\ &= e^{-t} - e^{-3}e^{-(t-3)}u(t-3) + u(t-3), \end{aligned}$$

so that

$$\mathcal{L}\left\{g(t)\right\} = \frac{1}{s+1} - e^{-3}\frac{e^{-3s}}{s+1} + \frac{e^{-3s}}{s}.$$

Taking Laplace transform of the d.e., solving for $Y(s)$ and performing p.f.d.:

$$\begin{aligned} (s^2+3s+2)Y(s) &= \frac{1}{s+1} - e^{-3}\frac{e^{-3s}}{s+1} + \frac{e^{-3s}}{s} + 2s + 5 \\ Y(s) &= \frac{2s+3}{(s+1)^2} - e^{-3}e^{-3s}\left(\frac{1}{(s+1)^2(s+2)}\right) + e^{-3s}\left(\frac{1}{s(s+1)(s+2)}\right) \\ &= \frac{2}{s+1} + \frac{1}{(s+1)^2} - e^{-3}e^{-3s}\left(-\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}\right) \\ &\quad + e^{-3s}\left(\frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}\right). \end{aligned}$$

Then the solution is

$$\begin{aligned} y(t) &= 2e^{-t} + te^{-t} + e^{-3}u(t-3)[e^{-(t-3)} - (t-3)e^{-(t-3)} - e^{-2(t-3)}] \\ &\quad + u(t-3)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-3)} - e^{-(t-3)}\right]. \end{aligned}$$

8. (**7.7.18**) Solve the following integral equation

$$y(t) + \int_0^t (t - v)y(v)\,dv = t^2.$$

Solution:

Taking Laplace transform of the differential equation (noting that the integral is $t * y(t)$), we arrive at

$$\begin{aligned} Y(s) + \mathcal{L}\{t * y\} &= \frac{2}{s^3} \\ Y(s) + \frac{1}{s^2}\,Y(s) &= \frac{2}{s^3} \\ \left(1 + \frac{1}{s^2}\right) Y(s) &= \frac{2}{s^3}. \end{aligned}$$

So

$$\begin{aligned} Y(s) &= \frac{2}{s^3}\,\frac{s^2}{s^2+1} = \frac{2}{s(s^2+1)} \\ &= \frac{2}{s} - \frac{2s}{s^2+1}. \qquad \text{(by inspection or p.f.d.)} \end{aligned}$$

Then the solution is

$$y(t) = 2 - 2\cos t.$$