MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 5

1. (**7.2.6**) Use the definition to find the Laplace transform of $\cos bt$, b a constant.

Solution:

We find $\mathcal{L}\{\cos bt\} = \displaystyle\int_0^\infty e^{-st} \cos bt\, dt$. First we will evaluate the integral using integration by parts,

$$\begin{aligned} I = \int e^{-st} \cos bt\, dt &= \frac{1}{b} e^{-st} \sin bt + \frac{s}{b} \int e^{-st} \sin bt\, dt \\ &= \frac{1}{b} e^{-st} \sin bt + \frac{s}{b} \left\{ -\frac{1}{b} e^{-st} \cos bt - \frac{s}{b} \int e^{-st} \cos bt\, dt \right\}. \end{aligned}$$

Collecting the integrals we obtain

$$\begin{aligned} I + \frac{s^2}{b^2} I &= \frac{1}{b} e^{-st} \sin bt - \frac{s}{b^2} e^{-st} \cos bt + C \\ \left(\frac{b^2 + s^2}{b^2}\right) I &= \frac{1}{b} e^{-st} \sin bt - \frac{s}{b^2} e^{-st} \cos bt + C \\ I &= \frac{b^2}{b^2 + s^2} \frac{1}{b^2} e^{-st} (b \sin bt - s \cos bt) + C \\ &= \frac{1}{b^2 + s^2} e^{-st} (b \sin bt - s \cos bt) + C. \end{aligned}$$

Then using the above result,

$$\begin{aligned} \mathcal{L}\{\cos bt\} &= \lim_{r \to \infty} \int_0^r e^{-st} \cos bt\, dt \\ &= \lim_{r \to \infty} \left[\frac{1}{b^2 + s^2} \left\{ e^{-sr} (b \sin br - s \cos br) - (-s) \right\} \right] \\ &= \frac{1}{b^2 + s^2} \left[\lim_{r \to \infty} \left(e^{-sr} (b \sin br - s \cos br) \right) + s^2 \right] \\ &= \frac{s}{s^2 + b^2}, \qquad s > 0. \end{aligned}$$

Here we used the fact that $\lim_{r \to \infty} e^{-sr}(b \sin br - s \cos br) = 0$ since $b \sin br - s \cos br$ is a bounded function, $e^{-sr}(b \sin br - s \cos br) \to 0$ as $r \to \infty$ and $s > 0$.

2. (7.2.10) Use the definition to find the Laplace transform of

$$f(t) = \begin{cases} 1-t, & 0 < t < 1, \\ 0, & 1 < t. \end{cases}$$

Solution:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^1 e^{-st}(1-t)\, dt \\
&= -\frac{1}{s}(1-t)e^{-st}\Big|_0^1 - \frac{1}{s}\int_0^1 e^{-st}\, dt \qquad \text{(using int'n by parts)} \\
&= -\frac{1}{s}(1-t)e^{-st} + \frac{1}{s^2}e^{-st}\Big|_0^1 \\
&= \frac{1}{s^2}e^{-s} + \frac{1}{s} - \frac{1}{s^2}.
\end{aligned}$$

3. (7.2.14) Determine the Laplace transform: $\mathcal{L}\left\{5 - e^{2t} + 6t^2\right\}$.

Solution:

$$\begin{aligned}
\mathcal{L}\left\{5 - e^{2t} + 6t^2\right\} &= 5\mathcal{L}\{1\} - \mathcal{L}\left\{e^{2t}\right\} + 6\mathcal{L}\left\{t^2\right\} \\
&= 5\left(\frac{1}{s}\right) - \frac{1}{s-2} + 6\left(\frac{2}{s^3}\right) \\
&= \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}.
\end{aligned}$$

4. (**7.3.10**) Find $\mathcal{L}\left\{te^{2t}\cos 5t\right\}$.

Solution:

$$
\begin{aligned}
\mathcal{L}\left\{t\,e^{2t}\cos 5t\right\} &= -\frac{dF}{ds} \qquad \text{where} \quad F(s) = \mathcal{L}\left\{e^{2t}\cos 5t\right\} = \frac{s-2}{(s-2)^2+25} \\
&= -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right) \\
&= -\left[\frac{(s-2)^2+25-(s-2)(2(s-2))}{((s-2)^2+25)^2}\right] \\
&= \frac{s^2-4s-21}{((s-2)^2+25)^2} \\
\text{or} \qquad &= \frac{(s-7)(s+3)}{((s-2)^2+25)^2}.
\end{aligned}
$$

5. (**7.3.12**) Find $\mathcal{L}\{\sin 3t\cos 3t\}$.

Solution:

We use the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$. So we have

$$
\begin{aligned}
\mathcal{L}\{\sin 3t\cos 3t\} &= \frac{1}{2}\mathcal{L}\{\sin 6t\} \\
&= \frac{1}{2}\,\frac{6}{s^2+36} = \frac{3}{s^2+36}.
\end{aligned}
$$

6. (**7.3.22**) Starting with the transform $\mathcal{L}\{1\} = 1/s$, use the formula for the derivatives of the Laplace transform to show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$.

Solution:

Since $\mathcal{L}\{1\} = 1/s$, we have

$$\begin{aligned}
\frac{d}{ds}\left(\frac{1}{s}\right) &= -\frac{1}{s^2} \\
\frac{d^2}{ds^2}\left(\frac{1}{s}\right) &= \frac{2}{s^3} \\
\frac{d^3}{ds^3}\left(\frac{1}{s}\right) &= \frac{2(-3)}{s^4} \\
&\vdots\ \vdots \\
\frac{d^n}{ds^n}\left(\frac{1}{s}\right) &= (-1)^n \frac{n!}{s^{n+1}}.
\end{aligned}$$

Since $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$, we obtain

$$\begin{aligned}
\mathcal{L}\{t^n\} &= (-1)\frac{d^n}{ds^n}\mathcal{L}\{1\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) \\
&= (-1)^n(-1)^n \frac{n!}{s^{n+1}} \\
&= (-1)^{2n} \frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.
\end{aligned}$$

7. (**7.4.10**) Determine the inverse Laplace transform of $\dfrac{s-1}{2s^2+s+6}$.

Solution:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s-1}{2s^2+s+6}\right\} &= \mathcal{L}^{-1}\left\{\frac{s-1}{2(s+\frac{1}{4})^2+\frac{47}{8}}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+\frac{1}{4}-\frac{5}{4}}{2[(s+\frac{1}{4})^2+\frac{47}{16}]}\right\} \\
&= \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{2+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{47}{16}}\right\} - \mathcal{L}^{-1}\left\{\frac{\frac{5}{4}(\frac{\sqrt{47}}{4})}{\frac{\sqrt{47}}{4}[(s+\frac{1}{4})^2+\frac{47}{16}]}\right\}\right] \\
&= \frac{1}{2}\left[e^{-\frac{1}{4}t}\cos\frac{\sqrt{47}}{4}t - \frac{5}{\sqrt{47}}e^{-\frac{1}{4}t}\sin\frac{\sqrt{47}}{4}t\right].
\end{aligned}$$

8. (**7.4.24**) Determine $\mathcal{L}^{-1}\{F(s)\}$ where $F(s) = \dfrac{7s^2 - 41s + 84}{(s-1)(s^2 - 4s + 13)}$.

Solution:

We use partial fractions decomposition:

$$\begin{aligned}
\frac{7s^2 - 41s + 84}{(s-1)(s^2-4s+13)} &= \frac{A}{s-1} + \frac{Bs + C}{s^2 - 4s + 13} \\
7s^2 - 41s + 84 &= A(s^2 - 4s + 13) + (s-1)(Bs + C) \\
&= (A + B)s^2 + (-4A - B + C)s + 13A - C.
\end{aligned}$$

Letting $s = 1$, we get $50 = 10A$, i.e., $A = 5$. Equating the coefficients of the s^2 terms, and then the constant terms, we obtain

$$\begin{aligned}
A + B = 7 \quad &\Longrightarrow \quad B = 2 \\
13A - C = 84 \quad &\Longrightarrow \quad C = -19.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{5}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2s - 19}{s^2 - 4s + 13}\right\} \\
&= 5e^t + \mathcal{L}^{-1}\left\{\frac{2(s-2) - 15}{(s-2)^2 + 9}\right\} \\
&= 5e^t + 2\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2 + 9}\right\} - 5\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2 + 9}\right\} \\
&= 5e^t + 2e^{2t}\cos 3t - 5e^{2t}\sin 3t.
\end{aligned}$$

9. (**7.4.28**) Determine $\mathcal{L}^{-1}\{F(s)\}$ where

$$s^2F(s) + sF(s) - 6F(s) = \frac{s^4+4}{s^3+s^2}.$$

Solution:

We have

$$(s^2+s-6)F(s) = \frac{s^4+4}{s^2(s+1)}.$$

Using partial fractions decomposition,

$$\begin{aligned} F(s) &= \frac{s^4+4}{s^2(s+1)(s+3)(s-2)} \\ &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+3} + \frac{E}{s-2} \end{aligned}$$

$$\begin{aligned} s^4+4 = {} & As(s+1)(s+3)(s-2) + B(s+1)(s+3)(s-2) \\ & + Cs^2(s+3)(s-2) + Ds^2(s+1)(s-2) + Es^2(s+1)(s+3). \end{aligned}$$

Substituting the zeros of the factors: $s,\ s+1,\ s+3,\ s-2$, resp. we get

$$\begin{aligned} s=0: &\quad 4 = -6B &\implies& \quad B = -\frac{2}{3} \\ s=-1: &\quad 5 = -6C &\implies& \quad C = -\frac{5}{6} \\ s=-3: &\quad 85 = 90D &\implies& \quad D = \frac{17}{18} \\ s=2: &\quad 20 = 60E &\implies& \quad E = \frac{1}{3}. \end{aligned}$$

To find A equate the coefficients of the s^4 terms: $1 = A + C + D + E \implies A = 1 + \frac{5}{6} - \frac{17}{18} - \frac{1}{3} = \frac{5}{9}$. So then

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{5/9}{s} - \frac{2/3}{s^2} - \frac{5/6}{s+1} + \frac{17/18}{s+3} + \frac{1/3}{s-2}\right\} \\ &= \frac{5}{9} - \frac{2}{3}t - \frac{5}{6}e^{-t} + \frac{17}{18}e^{-3t} + \frac{1}{3}e^{2t}. \end{aligned}$$

10. (**7.4.34**) Compute $\mathcal{L}^{-1}\left\{F(s)\right\}$ where $F(s) = \ln\left(\dfrac{s-4}{s-3}\right)$.

Solution:

We have

$$\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\} = -tf(t) = -t\mathcal{L}^{-1}\left\{F(s)\right\}. \tag{1}$$

Since $F(s) = \ln\left(\dfrac{s-4}{s-3}\right) = \ln(s-4) - \ln(s-3)$,

$$F'(s) = \frac{1}{s-4} - \frac{1}{s-3}.$$

Then

$$\mathcal{L}^{-1}\left\{F'(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4} - \frac{1}{s-3}\right\} = e^{4t} - e^{3t}. \tag{2}$$

Using (1) and (2) we have

$$e^{4t} - e^{3t} = -t\mathcal{L}^{-1}\left\{F(s)\right\}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{F(s)\right\} = -\frac{1}{t}\left(e^{4t} - e^{3t}\right).$$