MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 4

1. (**2.Rev.9**) Solve the equation

$$(x^2 + y^2)dx + 3xydy = 0.$$

Solution:

This equation is homogeneous as well as Bernoulli.

(1) *Using the method for homogeneous equations.*

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{3xy}.$$

Let $y = vx$. So $y' = v'x + v$. This leads to

$$v'x + v = -\frac{x^2 + v^2x^2}{3vx^2}.$$

Simplifying we arrive at

$$\frac{dv}{dx} = \frac{1}{x}\left(-\frac{1}{3v} - \frac{4}{3}v\right) = -\frac{1 + 4v^2}{3vx}.$$

Separating the variables,

$$\frac{v\,dv}{1 + 4v^2} = -\frac{1}{3x}\,dx.$$

Integrating,

$$\frac{1}{8}\ln(1 + 4v^2) = -\frac{1}{3}\ln|x| + \ln C$$

$$\left[1 + 4\left(\frac{y}{x}\right)^2\right]^{1/8} = \frac{C}{|x|^{1/3}},$$

or

$$x^2 + 4y^2 = Kx^{-2/3}.$$

Solution:

(2) *Using the Bernoulli equation method.*

$$\frac{dy}{dx} + \frac{1}{3x}y = -\frac{x}{3}y^{-1}.$$

Let $v = y^{1-n} = y^2$. So $dv/dx = 2y dy/dx$ and $dy/dx = (1/2y)dv/dx$. Substituting into the equation we obtain

$$\begin{aligned} \frac{1}{2y}\frac{dv}{dx} + \frac{1}{3x}y &= -\frac{x}{3}y^{-1} \\ \therefore \qquad \frac{dv}{dx} + \frac{2}{3x}y^2 &= -\frac{2}{3}x \\ \therefore \qquad \frac{dv}{dx} + \frac{2}{3x}v &= -\frac{2}{3}x. \end{aligned}$$

This is a linear equation in v. We look for an integrating factor,

$$\mu(x) = e^{\int \frac{2}{3x}dx} = e^{\frac{2}{3}\ln x} = x^{2/3}.$$

Hence

$$\begin{aligned} d(x^{2/3}v) &= -\frac{2}{3}x^{5/3}\,dx \\ x^{2/3}v &= -\frac{1}{4}x^{8/3} + C \\ v &= -\frac{1}{4}x^2 + Cx^{-2/3}, \\ \text{i.e.,} \qquad y^2 &= -\frac{1}{4}x^2 + Cx^{-2/3}, \end{aligned}$$

or

$$4y^2 + x^2 = Cx^{-2/3}.$$

2. (**4.Review.20**) Solve the equation

$$\frac{dy}{d\theta} + \frac{y}{\theta} = -4\theta y^{-2}.$$

Solution:

Let $v = y^3$, so that

$$\frac{dv}{d\theta} = 3y^2 \frac{dy}{d\theta} \qquad \text{and} \qquad \frac{dy}{d\theta} = \frac{1}{3} y^{-2} \frac{dv}{d\theta}.$$

Substituting into the equation we obtain

$$\frac{dv}{d\theta} + \frac{3}{\theta} v = -12\theta. \qquad \text{(linear)}$$

An integrating factor is

$$\mu(\theta) = e^{\int \frac{3}{\theta} d\theta} = e^{3 \ln \theta} = \theta^3.$$

Multiplying the equation by θ^3 we arrive at

$$\begin{aligned} \theta^3 \frac{dv}{d\theta} + 3\theta^2 v &= -12\theta^4, \\ d\theta^3 v &= -12\theta^4 \, d\theta. \end{aligned}$$

Integrating,

$$\begin{aligned} \theta^3 v &= -\frac{12}{5}\theta^5 + C \\ v &= -\frac{12}{5}\theta^2 + C\theta^{-3} \\ y^3 &= -\frac{12}{5}\theta^2 + C\theta^{-3}, \end{aligned}$$

or

$$5y^3 + 12\theta^2 = K\theta^{-3}.$$

3. (**2.Rev.22**) Solve the equation

$$(2x - 2y - 8)\, dx + (x - 3y - 6)\, dy = 0.$$

Solution:

Rewriting the equation we obtain

$$\frac{dy}{dx} = -\frac{2x - 2y - 8}{x - 3y - 6} = \frac{-2x + 2y + 8}{x - 3y - 6}.$$

Noting that $-2(-3) = 6 \neq (2)(1)$, we use the translation of axes $x = u + h$ and $y = v + k$. This gives

$$\begin{aligned} -2x + 2y + 8 &= -2(u + h) + 2(v + k) + 8 = -2u + 2v - 2h + 2k + 8 \\ x - 3y - 6 &= u + h - 3(v + k) - 6 \qquad = u - 3v + h - 3k - 6. \end{aligned}$$

Set $-2h + 2k + 8 = 0$ and $h - 3k - 6 = 0$ and solve for h and k. We obtain $h = 3$ and $k = -1$ so that $x = u + 3$ and $y = v - 1$. The d.e. becomes

$$\frac{dv}{du} = \frac{-2u + 2v}{u - 3v} = \frac{-2 + 2(\frac{v}{u})}{1 - 3(\frac{v}{u})}.$$

Let $z = v/u$ so that $dv/du = z + u dz/du$. Then

$$\begin{aligned} z + u\frac{dz}{du} &= \frac{-2 + 2z}{1 - 3z} \\ u\frac{dz}{du} &= \frac{-2 + 2z}{1 - 3z} - z = \frac{-2 + z + 3z^2}{1 - 3z}. \end{aligned}$$

Separating the variables,

$$\frac{1 - 3z}{3z^2 + z - 2}\, dz = \frac{1}{u}\, du. \tag{1}$$

We integrate the LHS: (use the hint given in class)

$$\begin{aligned} \int \frac{1 - 3z}{(3z - 2)(z + 1)}\, dz &= \int \frac{-3/5}{3z - 2}\, dz + \int \frac{-4/5}{z + 1}\, dz \\ &= -\frac{3}{5}\frac{1}{3} \ln|3z - 2| - \frac{4}{5} \ln|z + 1| + C_1. \end{aligned}$$

From Eqn. (1) we get

$$
\begin{aligned}
-\frac{1}{5}\ln|3z-2| - \frac{4}{5}\ln|z+1| &= \ln|u| + \ln C \\
\ln|3z-2|^{-1/5} + \ln|z+1|^{-4/5} &= \ln C|u| \\
\ln\left|(3z-2)^{-1/5}(z+1)^{-4/5}\right| &= \ln C|u| \\
(3z-2)^{-1/5}(z+1)^{-4/5} &= Ku \\
\left[3\left(\frac{v}{u}\right)-2\right]^{-1/5}\left(\frac{v}{u}+1\right)^{-4/5} &= Ku \\
\left[3\left(\frac{y+1}{x-3}\right)-2\right]^{-1/5}\left[\frac{y+1}{x-3}+1\right]^{-4/5} &= K(x-3).
\end{aligned}
$$

4. (**4.8.20**) Solve the equation

$$y'' - 2y' - 3y = 3x^2 - 5.$$

Solution:

We first solve the associated homogeneous equation $y''-2y'-3y = 0$. Solving the auxiliary equation $r^2 - 2r - 3 = (r-3)(r+1) = 0$ we obtain $r = 3,\ -1$. Hence the complementary function is

$$y_c = c_1 e^{3x} + c_2 e^{-x}.$$

We now look for a particular solution to the nonhomogeneous equation. We try a solution of the form $y_p = Ax^2 + Bx + C$ (note that there is no duplication between this choice and the solution y_c). Now $y_p' = 2Ax + B$ and $y_p'' = 2A$. Substitute into the D.E.:

$$\begin{aligned} 2A - 4Ax - 2B - 3Ax^2 - 3Bx - 3C &= 3x^2 - 5 \\ -3Ax^2 + (-4A - 3B)x + 2A - 2B - 3C &= 3x^2 - 5. \end{aligned}$$

Hence equating the corresponding coefficients

$$\begin{aligned} -3A = 3 \qquad &\Longrightarrow \qquad A = -1 \\ -4A - 3B = 0 \qquad &\Longrightarrow \qquad B = \frac{4}{3} \\ 2A - 2B - 3C = -5 \qquad &\Longrightarrow \qquad C = \frac{1}{9}. \end{aligned}$$

So a particular solution is $y_p = -x^2 + \frac{4}{3}x + \frac{1}{9}$. Then a general solution is

$$y = c_1 e^{3x} + c_2 e^{-x} - x^2 + \frac{4}{3}x + \frac{1}{9}.$$

5. (**4.8.36**) Solve the initial value problem

$$y'' - y = e^{\theta} - e^{-\theta} + 2, \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution:

The auxiliary equation for the associated homogeneous equation is $r^2 - 1 = 0$ so $r = \pm 1$. Hence $y_c = c_1 e^{\theta} + c_2 e^{-\theta}$. If we try a particular solution of the form $y_p = Ae^{\theta} + Be^{-\theta} + C$ we note that the first 2 terms duplicate the y_c. So y_p should be

$$\begin{aligned} & y_p = A\theta e^{\theta} + B\theta e^{-\theta} + C \\ \Longrightarrow \quad & y_p' = Ae^{\theta} + A\theta e^{\theta} + Be^{-\theta} - B\theta e^{-\theta} \\ \Longrightarrow \quad & y_p'' = 2Ae^{\theta} + A\theta e^{\theta} - 2Be^{-\theta} + B\theta e^{-\theta}. \end{aligned}$$

The differential equation becomes

$$\begin{aligned} & 2Ae^{\theta} + A\theta e^{\theta} - 2Be^{-\theta} + B\theta e^{-\theta} - A\theta e^{\theta} - B\theta e^{-\theta} - C = e^{\theta} - e^{-\theta} + 2 \\ \Longrightarrow \quad & 2Ae^{\theta} - 2Be^{-\theta} - C = e^{\theta} - e^{-\theta} + 2. \end{aligned}$$

Equating the coefficients we get $A = \frac{1}{2}$, $B = \frac{1}{2}$ and $C = -2$. So $y_p = \frac{1}{2}\theta e^{\theta} + \frac{1}{2}\theta e^{-\theta} - 2$ and a general solution is

$$y = c_1 e^{\theta} + c_2 e^{-\theta} + \frac{1}{2}\theta e^{\theta} + \frac{1}{2}\theta e^{-\theta} - 2,$$

and

$$y' = c_1 e^{\theta} - c_2 e^{-\theta} + \frac{1}{2}e^{\theta} + \frac{1}{2}\theta e^{\theta} + \frac{1}{2}e^{-\theta} - \frac{1}{2}\theta e^{-\theta}.$$

Using the initial conditions,

$$\begin{aligned} y(0) = 0 \quad & \Longrightarrow \quad 0 = c_1 + c_2 - 2 \\ y'(0) = 0 \quad & \Longrightarrow \quad 0 = c_1 - c_2 + \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

Solving the system we arrive at $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$. Then the solution is

$$y = \frac{1}{2}e^{\theta} + \frac{3}{2}e^{-\theta} + \frac{1}{2}\theta e^{\theta} + \frac{1}{2}\theta e^{-\theta} - 2.$$

6. (**4.9.6**) Solve the differential equation (using variation of parameters)

$$y'' + 2y' + y = e^{-x}.$$

Solution:

Solving the auxiliary equation $r^2 + 2r + 1 = (r+1)^2 = 0$ we get $r = -1, -1$. So $y_c = c_1 e^{-x} + c_2 x e^{-x}$. We look for a particular solution of the form $y_p = v_1 e^{-x} + v_2 x e^{-x}$. This leads to the system

$$v_1' e^{-x} + v_2' x e^{-x} = 0 \tag{1}$$

$$-v_1' e^{-x} + v_2'(e^{-x} - xe^{-x}) = e^{-x}. \tag{2}$$

From Eqn. (1) $\quad v_1' = -v_2' x$. Substitute into Eqn. (2):

$$\begin{aligned} xe^{-x} v_2' + e^{-x} v_2' - xe^{-x} v_2' &= e^{-x} \\ v_2' &= 1 \\ v_2 &= x. \end{aligned}$$

Hence $v_1' = -x$ and $v_1 = -x^2/2$. Then

$$y_p = -\frac{x^2}{2} e^{-x} + x^2 e^{-x} = \frac{x^2}{2} e^{-x},$$

and a general solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2}{2} e^{-x}.$$

7. (**4.9.24**) Given that $y_1 = x^2$ and $y_2 = x^3$ are linearly independent solutions to the homogeneous equation, find a general solution to:

$$x^2y'' - 4xy' + 6y = x^3 + 1, \qquad x > 0.$$

Solution:

Note that the homogeneous equation is a Caucy-Euler equation so even if the solutions are not given we can solve it. So

$$y_c = c_1x^2 + c_2x^3$$

and

$$y_p = v_1x^2 + v_2x^3.$$

We remember to put the equation in standard form

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x + x^{-2}.$$

We then solve the system

$$v_1'x^2 + v_2'x^3 = 0 \tag{1}$$

$$v_1'(2x) + v_2'(3x^2) = x + x^{-2}. \tag{2}$$

From (1) we get $v_1' = -v_2'x$. Substituting it into (2) gives

$$\begin{aligned} -2x(xv_2') + 3x^2v_2' &= x + x^{-2} \\ x^2v_2' &= x + x^{-2} \\ v_2' &= x^{-1} + x^{-4} \\ v_2 &= \ln x - \frac{x^{-3}}{3}. \end{aligned}$$

Then

$$\begin{aligned} v_1' &= -x(x^{-1} + x^{-4}) = -1 - x^{-3} \\ v_1 &= -x + \frac{x^{-2}}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} y_p &= x^2\left(-x+\frac{x^{-2}}{2}\right)+x^3\left(\ln x-\frac{x^{-3}}{3}\right) \\ &= -x^3+\frac{1}{2}+x^3\ln x-\frac{1}{3} \\ &= x^3\ln x-x^3+\frac{1}{6}. \end{aligned}$$

The general solution is then

$$y = c_1x^2+c_2x^3+x^3\ln x-x^3+\frac{1}{6}.$$