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MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 3

1. (**4.4.8**) The equation

$$xy''' + (1-x)y'' + xy' - y = 0$$

has $f(x) = x$ as a solution. Use the substitution $y(x) = v(x)f(x)$ to reduce this third order equation to a homogeneous linear second order equation in the variable $w = v'$.

Solution:

Let $y = vx$. Then

$$\begin{aligned} y' &= v'x + v \\ y'' &= v''x + 2v' \\ y''' &= v'''x + 3v''. \end{aligned}$$

Substitute into the differential equation. We obtain

$$x(v'''x + 3v'') + (1-x)(v''x + 2v') + x(v'x + v) - vx = 0$$

which simplifies to

$$x^2v''' + (4x - x^2)v'' + (x^2 - 2x + 2)v' = 0.$$

Let $w = v'$ so that $w' = v''$ and $w'' = v'''$. Then the equation becomes

$$x^2w'' + x(4-x)w' + (x^2 - 2x + 2)w = 0,$$

or

$$w'' + \left(\frac{4-x}{x}\right) w' + \left(\frac{x^2 - 2x + 2}{x^2}\right) w = 0,$$

which is a homogeneous second order linear equation.

2. (**4.5.20**) Solve the I.V.P.

$$y'' - 4y' + 4y = 0; \qquad y(1) = 1, \quad y'(1) = 1.$$

Solution:

The auxiliary equation for the d.e. is $r^2 - 4r + 4 = 0$. The roots are $r = 2,\ 2$. Hence a general solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}.$$

Differentiating we get

$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}.$$

We now use the initial conditions:

$$\begin{aligned} y(1) = 1 \quad &\Longrightarrow \quad & c_1 e^2 + c_2 e^2 &= 1 \\ y'(1) = 1 \quad &\Longrightarrow \quad & 2c_1 e^2 + c_2 e^2 + 2c_2 e^2 &= 1. \end{aligned}$$

We arrive at the following system of 2 equations

$$\begin{aligned} c_1 + c_2 &= e^{-2} \\ 2c_1 + 3c_2 &= e^{-2}. \end{aligned}$$

Multiplying the first equation by 2 and subtracting the 2nd equation give us

$$c_2 = -e^{-2} \qquad \text{and} \qquad c_1 = 2e^{-2}.$$

Hence we obtain the solution

$$y(x) = 2e^{-2} e^{2x} - e^{-2} x e^{2x},$$

or

$$y(x) = 2e^{2x-2} - x e^{2x-2},$$

or

$$y(x) = e^{2x-2}(2 - x).$$

3. (**4.5.34**) Find the general solution to the Cauchy-Euler equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y=0, \qquad x>0.$$

Solution:

Let $x=e^t$. We have $dx/dt=e^t=x$. Then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt}=x\frac{dy}{dx}\\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(x\frac{dy}{dx}\right)\\
&= \frac{dx}{dt}\frac{dy}{dx}+x\frac{d}{dt}\left(\frac{dy}{dx}\right)\\
&= \frac{dy}{dt}+x\frac{d^2y}{dx^2}\frac{dx}{dt}\\
&= \frac{dy}{dt}+x^2\frac{d^2y}{dx^2}.
\end{aligned}$$

So

$$x\frac{dy}{dx}=\frac{dy}{dt} \qquad \text{and} \qquad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}.$$

So the differential equation transforms to

$$y''(t)+y'(t)-6y(t)=0.$$

The auxiliary equation is $r^2+r-6=0$, so that $(r+3)(r-2)=0$ and $r=-3,\ r=2$. Then

$$y(t)=c_1e^{-3t}+c_2e^{2t}.$$

Using $x=e^t$ so that $\ln x=t$, we have the general solution

$$y(x)=c_1x^{-3}+c_2x^2.$$

4. (**4.5.40**) Solve the I.V.P.

$$x^2 y''(x) + 7xy'(x) + 5y(x) = 0; \qquad y(1) = -1, \quad y'(1) = 13.$$

Solution:

Using the method used in #34 we arrive at

$$y''(t) + 6y'(t) + 5y(t) = 0.$$

The auxiliary equation is

$$r^2 + 6r + 5 = 0$$

which gives $(r+5)(r+1) = 0$, and $r = -5, \ -1$. The general solution is

$$y(x) = c_1 x^{-5} + c_2 x^{-1},$$

whose derivative is

$$y'(x) = -5c_1 x^{-6} - c_2 x^{-2}.$$

Using the initial conditions

$$\begin{aligned} y(1) = -1 \quad &\Longrightarrow \quad c_1 + c_2 = -1 \\ y'(1) = 13 \quad &\Longrightarrow \quad -5c_1 - c_2 = 13. \end{aligned}$$

Solving the system we arrive at $c_1 = -3$ and $c_2 = 2$. Hence the solution is

$$y(x) = -3x^{-5} + 2x^{-1}.$$

5. (**4.6.14**) Find a general solution

$$y'' - 2y' + 26y = 0.$$

Solution:

The auxiliary equation is $r^2 - 2r + 26 = 0$. Using the quadratic formula to solve:

$$r = \frac{2 \pm \sqrt{4 - 4(26)}}{2} = 1 \pm 5i.$$

So the general solution is

$$y = c_1 e^x \cos 5x + c_2 e^x \sin 5x.$$

6. (**4.6.20**) Find a general solution to

$$y''' - y'' + 2y = 0.$$

Solution:

The auxiliary equation is $r^3 - r^2 + 2 = 0$. We note that $r = -1$ is a root. Dividing $r^3 - r^2 + 2$ by $r + 1$ using long division we find the other factor $r^2 - 2r + 2$ whose roots are $r = 1 \pm i$. So the general solution is

$$y = c_1 e^{-x} + e^x(c_1 \cos x + c_2 \sin x).$$

7. (**4.6.22**) Solve the initial value problem

$$y'' + 2y' + 17y = 0; \qquad y(0) = 1, \quad y'(0) = -1.$$

Solution:

We find the roots for the auxiliary equation $r^2 + 2r + 17 = 0$ as $r = -1 \pm i4$. So the general solution is

$$y = c_1 e^{-x} \cos 4x + c_2 e^{-x} \sin 4x.$$

Differentiating, we arrive at

$$\begin{aligned} y' &= -c_1 e^{-x} \cos 4x - 4c_1 e^{-x} \sin 4x - c_2 e^{-x} \sin 4x + 4c_2 e^{-x} \cos 4x \\ &= (-c_1 + 4c_2)e^{-x} \cos 4x + (-4c_1 - c_2)e^{-x} \sin 4x. \end{aligned}$$

Using the initial values,

$$\begin{aligned} y(0) = 1 \qquad &\Longrightarrow \qquad c_1 = 1 \\ y'(0) = -1 \qquad &\Longrightarrow \qquad -c_1 + 4c_2 = -1. \end{aligned}$$

So $c_1 = 1$ and $c_2 = 0$. The solution is

$$y = e^{-x} \cos 4x.$$

8. (**4.7.12(b)**) Let $L[y] := y'' + 2y' + 4y$, $y_1(x) = (1/4)\sin 2x$, and $y_2(x) = x/4 - 1/8$. Verify that $L[y_1](x) = \cos 2x$ and $L[y_2](x) = x$. Then use the superposition principle to find solution to the differential equation

$$L[y](x) = 2x - 3\cos 2x.$$

Solution:

We first verify that $L[y_1](x) = \cos 2x$ and $L[y_2](x) = x$. Since

$$\begin{aligned} y_1 &= \frac{1}{4}\sin 2x, & y_1' &= \frac{1}{2}\cos 2x, & y_1'' &= -\sin 2x, \\ y_2 &= \frac{x}{4} - \frac{1}{8}, & y_2' &= \frac{1}{4}, & y_2'' &= 0, \end{aligned}$$

we have

$$\begin{aligned} L[y_1](x) &= -\sin 2x + 2\left(\frac{1}{2}\cos 2x\right) + 4\left(\frac{1}{4}\sin 2x\right) \\ &= -\sin 2x + \cos 2x + \sin 2x = \cos 2x, \\ L[y_2](x) &= 0 + 2\left(\frac{1}{4}\right) + 4\left(\frac{x}{4} - \frac{1}{8}\right) \\ &= \frac{1}{2} + x - \frac{1}{2} = x. \end{aligned}$$

To solve $L[y] = 2x - 3\cos 2x$ we use the superposition principle, i.e.,

$$y(x) = 2\left(\frac{x}{4} - \frac{1}{8}\right) - 3\left(\frac{1}{4}\sin 2x\right).$$

So the solution is

$$y = \frac{x}{2} - \frac{1}{4} - \frac{3}{4}\sin 2x.$$