MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 2

1. (**2.6.20**) Solve $dy/dx = \sin(x-y)$.

Solution:

Let $u = x - y$. Then

$$\frac{du}{dx} = 1 - \frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = 1 - \frac{du}{dx}.$$

So the d.e. becomes

$$\frac{du}{dx} = 1 - \sin u$$

Separation of variables yields

$$\frac{du}{1-\sin u} = dx.$$

Then

$$\begin{aligned}\int \frac{du}{1-\sin u} &= \int \frac{(1+\sin u)}{1-\sin^2 u}\, du = \int \frac{1+\sin u}{\cos^2 u}\, du \\ &= \int \left(\frac{1}{\cos^2 u} + \frac{\sin u}{\cos^2 u}\right) du \\ &= \int (\sec^2 u + \tan u \sec u)\, du \\ &= \tan u + \sec u + C.\end{aligned}$$

Using $u = x - y$ we obtain the general solution

$$\tan(x-y) + \sec(x-y) = x + C.$$

2. (**2.6.26**) Solve $\dfrac{dy}{dx} + y = e^x y^{-2}$.

Solution:

This is a Bernoulli equation with $n = -2$. Let $v = y^{1-n} = y^3$. Then we have

$$\frac{dv}{dx} = 3y^2\frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{1}{3}y^{-2}\frac{dv}{dx}.$$

Substituting into the d.e.

$$\begin{aligned} \frac{1}{3}y^{-2}\frac{dv}{dx} + y &= e^x y^{-2} \\ \Longrightarrow \qquad \frac{dv}{dx} + 3y^3 &= 3e^x \\ \Longrightarrow \qquad \frac{dv}{dx} + 3v &= 3e^x \end{aligned}$$

This is a linear equation in v. So an integrating factor is $\mu(x) = e^{3x}$. Multiplying the above equation by e^{3x} we obtain

$$\begin{aligned} e^{3x}\frac{dv}{dx} + 3e^{3x}v &= 3e^{4x} \\ d(e^{3x}v) &= 3e^{4x}dx \\ e^{3x}v &= \frac{3}{4}e^{4x} + C \\ v &= \frac{3}{4}e^x + Ce^{-3x} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y^3 &= \frac{3}{4}e^x + Ce^{-3x} \\ \text{or} \qquad y &= \left(\frac{3}{4}e^x + Ce^{-3x}\right)^{\frac{1}{3}}. \end{aligned}$$

3. (2.6.30) Solve $(x+y-1)dx+(y-x-5)dy=0$.

Solution:

We rewrite the equation

$$\begin{aligned}\frac{dy}{dx} &= -\frac{x+y-1}{y-x-5} \\ &= \frac{x+y-1}{x-y+5}.\end{aligned}$$

Since $a_1b_2=-1\neq a_2b_1=1$, use the translation $x=u+h$ and $y=v+k$. Then

$$\begin{aligned}x+y-1 &= u+h+v+k-1 = u+v+h+k-1 \\ x-y+5 &= u+h-v-k+5 = u-v+h-k+5\end{aligned}$$

We want $h+k-1=0$ and $h-k+5=0$. It is easy to see that $h=-2$ and $k=3$ satisfy the equations. So we let $x=u-2$ and $y=v+3$. Then the d.e. becomes

$$\begin{aligned}\frac{dv}{dx} &= \frac{u+v}{u-v} \\ &= \frac{1+\frac{v}{u}}{1-\frac{v}{u}}\end{aligned} \tag{1}$$

Let $z=v/u$, i.e., $v=uz$. Differentiating w.r.t. u we get $\frac{dv}{du}=1+u\frac{dz}{du}$. Then (1) becomes

$$\begin{aligned}1+u\frac{dz}{du} &= \frac{1+z}{1-z} \\ u\frac{dz}{du} &= \frac{1+z}{1-z}-1 \\ &= \frac{1+z-1+2z}{1-z} = \frac{2z}{1-z}\end{aligned}$$

Separating the variables and integrating,

$$\int \frac{1-z}{2z}\,dz = \int \frac{1}{u}\,du$$

$$\frac{1}{2}(\ln|z| - z) = \ln|u| + C_1$$

$$\ln|\frac{v}{u}| - \frac{v}{u} = 2\ln|u| + C_2$$

$$\ln\left|\frac{y-3}{x+2}\right| - \frac{y-3}{x+2} = 2\ln|x+2| + C_2$$

or $$(x+2)\ln\left|\frac{y-3}{x+2}\right| - y + 3 = 2(x+2)\ln|x+2| + C_2(x+2)$$

4. (4.2.14) Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ are solutions to the homogeneous equation

$$y'' - y' - 2y = 0,$$

find solutions to this equation that satisfy the following initial conditions:

(a) $y(0) = -1, \quad y'(0) = 4.$

(b) $y(0) = 3/2, \quad y'(0) = 0.$

Solution:

(a) Given $y_1 = e^{2x}$ and $y_2 = e^{-x}$, $y = c_1e^{2x} + c_2e^{-x}$ is a solution, and $y' = 2c_1e^{2x} - c_2e^{-x}$.

$$\begin{aligned} y(0) = -1 \quad &\Longrightarrow \quad -1 = c_1 + c_2 \\ y'(0) = 4 \quad &\Longrightarrow \quad 4 = 2c_1 - c_2 \end{aligned}$$

Solving the system we obtain $c_1 = 1$ and $c_2 = -2$. So the solution is

$$y = e^{2x} - 2e^{-x}.$$

(b) Using y and y' of part (a) we have

$$\begin{aligned} y(0) = \tfrac{3}{2} \quad &\Longrightarrow \quad \tfrac{3}{2} = c_1 + c_2 \\ y'(0) = 0 \quad &\Longrightarrow \quad 0 = 2c_1 - c_2 \end{aligned}$$

So $c_1 = \frac{1}{2}$ and $c_2 = 1$. The solution is then

$$y = \frac{1}{2}e^{2x} - e^{-x}.$$

5. (4.2.30) Boundary Value Problems. Given that every solution to

$$y'' + y = 0 \tag{14}$$

is of the form

$$y(x) = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants, show that

(a) there is a unique solution to (14) that satisfies the B.C. $y(0) = 2$ and $y(\pi/2) = 0$;

(b) there is no solution to (14) that satisfies $y(0) = 2$ and $y(\pi) = 0$;

(c) there are infinitely many solutions to (14) that satisfy $y(0) = 2$ and $y(\pi) = -2$.

Solution:

(a) Since $y = c_1 \cos x + c_2 \sin x$,

$$\begin{aligned} y(0) = 2 &\quad\Longrightarrow\quad 2 = c_1 \\ y(\tfrac{\pi}{2}) = 0 &\quad\Longrightarrow\quad 0 = c_2 \end{aligned}$$

So $y = 2\cos x$ is the unique solution.

(b)

$$\begin{aligned} y(0) = 2 &\quad\Longrightarrow\quad 2 = c_1 \\ y(\pi) = 0 &\quad\Longrightarrow\quad 0 = -c_1 \end{aligned}$$

This is a contradiction (2 different values of c_1). So the equation has no solution.

(c)

$$\begin{aligned} y(0) = 2 &\quad\Longrightarrow\quad 2 = c_1 \\ y(\pi) = -2 &\quad\Longrightarrow\quad -2 = -c_1 \end{aligned}$$

So the solution is $y = 2\cos x + c_2 \sin x$, i.e., there is an infinite number of solutions.

6. (4.3.10) (a) Verify that the functions y_1 and y_2 are linearly independent solutions, (b) find a general solution, and (c) find the solution to the I.V.P.

$$y'' - 5y' = 0; \quad y_1(x) = 2, \quad y_2(x) = e^{5x};$$
$$y(0) = 2, \quad y'(0) = 5.$$

Solution:

(a) It is easy to verify that $y_1 = 2$ and $y_2 = e^{5x}$. Just show that they satisfy the equation. To show that they are linearly independent, find the Wronskian

$$W[2, e^{5x}] = \begin{vmatrix} 2 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} = 10e^{5x} \neq 0, \quad \forall\, x$$

So they are linearly independent solutions of the d.e.

(b) The general solution is

$$y(x) = c_1 \cdot 2 + c_2 e^{5x} \qquad \text{or} \qquad y = c_3 + c_2 e^{5x}.$$

(c) From part (b) $y' = 5c_2 e^{5x}$. So

$$y(0) = 2 \quad \Longrightarrow \quad 2c_1 + c_2 = 2$$
$$y'(0) = 5 \quad \Longrightarrow \quad 5c_2 = 5.$$

Therefore, $c_2 = 1$ and $c_1 = \frac{1}{2}$. The solution is

$$y = 1 + e^{5x}.$$

7. (**4.3.22**) Show that if y_1 and y_2 are linearly independent solutions of $y'' + py' + qy = 0$ on (a, b), then they cannot both have an extremum at the same point x_0 in (a, b).

Solution:

If y_1 and y_2 are linearly independent solutions of the equation, then

$$\begin{aligned} W[y_1(x_0), y_2(x_0)] &= \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \\ &= y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \quad \text{for some } x_0 \in (a, b). \end{aligned}$$

If they have an extremum at the same point x_0, then $y_1'(x_0) = 0$ and $y_2'(x_0) = 0$. But this gives $W = 0$. This is a contradiction. So the two linearly independent solutions cannot have an extremum at the same point x_0.

8. **(4.4.6)** Find a second linearly independent solution, given a nontrivial solution f for

$$xy'' + (1-2x)y' + (x-1)y = 0, \quad x > 0; \quad f(x) = e^x.$$

Solution:

We look for a solution of the form $y = ve^x$. Then

$$\begin{aligned} y' &= v'e^x + ve^x \\ y'' &= v''e^x + 2v'e^x + ve^x. \end{aligned}$$

Substitute them into the differential equation to obtain

$$x(v''e^x + 2v'e^x + ve^x) + (1-2x)(v'e^x + ve^x) + (x-1)ve^x = 0.$$

Multiplying out and dividing by e^x,

$$xv'' + 2xv' + xv + v' + v - 2xv' - 2xv + xv - v = 0,$$

which simplifies to $xv'' + v' = 0$. Let $g = v'$. Then we can solve for v by the following procedure:

$$\begin{aligned} g' + \frac{1}{x}g &= 0 \\ \frac{dg}{g} &= -\frac{1}{x}dx \\ \ln g &= -\ln x \\ g &= \frac{1}{x} \\ v' &= \frac{1}{x} \\ \therefore \qquad v &= \ln x \end{aligned}$$

So $y = e^x \ln x$ is a 2nd linearly independent solution.

9. (4.4.14(c)) Consider Legendre's equation

$$(1-x^2)y'' - 2xy + \lambda(\lambda+1)y = 0, \qquad -1 < x < 1,$$

where λ is a parameter. Use the reduction of order formula to obtain an integral representation of a second linearly independent solution if $\lambda = 3$ and corresponding solution $f(x) = 5x^3 - 3x$ are given.

Solution:

With $\lambda = 3$ the equation becomes

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

The standard form of the equation is then

$$y'' - \frac{2x}{1-x^2}\,y' + \frac{12}{1-x^2}\,y = 0$$

The reduction of order formula is

$$y(x) = f(x)\int \frac{e^{-\int p(x)dx}}{f(x)^2}\,dx.$$

So $p(x) = -\frac{2x}{1-x^2}$ and

$$\begin{aligned} e^{-\int p(x)dx} &= e^{\int \frac{2x}{1-x^2}dx} = e^{-\ln(1-x^2)} \\ &= (1-x^2)^{-1} = \frac{1}{1-x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= (5x^3-3x)\int \frac{\frac{1}{1-x^2}}{(5x^3-3x)^2}\,dx \\ &= (5x^3-3x)\int \frac{1}{(1-x^2)(5x^3-3x)^2}\,dx \end{aligned}$$

So an integral representation of a second linearly independent solution is

$$y(x) = (5x^3-3x)\int \frac{1}{(1-x^2)(5x^3-3x)}\,dx.$$