MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 10

1. (**10.3.18**) Determine the function to which the Fourier series for $f(x) = |x|$, $-\pi < x < \pi$ converges.

Solution:

We found in the previous assignment that the Fourier series for f is

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1} \frac{1}{(2k-1)^2}\cos(2k-1)x.$$

Since f is a continuous function on $[-\pi, \pi]$, by the theorem on convergence, the Fourier series of f converges to $f(x)$ including the endpoints.

2. (**10.4.6**) Compute the Fourier sine series for $f(x) = \cos x$, $0 < x < \pi$.

Solution:

The Fourier sine series for f is

$$\sum_{n=1}^{\infty} b_n \sin nx \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx, \qquad n \geq 1.$$

Using the trigonometric identity $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$, ans assuming $n \neq 1$ $(n \geq 2)$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x]\, dx \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x - \frac{1}{n-1}\cos(n-1)x\right|_0^{\pi} \\
&= -\frac{1}{\pi}\left[\frac{1}{n+1}\underbrace{\cos(n+1)\pi}_{(-1)^{n+1}} + \frac{1}{n-1}\underbrace{\cos(n-1)\pi}_{(-1)^{n-1}} - \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{\pi}\left[\frac{1}{n+1}((-1)^{n+1}+1) + \frac{1}{n-1}((-1)^{n-1}+1)\right]
\end{aligned}$$

(note that if n is odd, $b_n = 0$). Now assume $n = 1$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x \sin x\, dx = \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} \sin 2x\, dx \\
&= -\frac{1}{2\pi}\cos 2x\Big|_0^{\pi} = -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = 0.
\end{aligned}$$

The Fourier sine series for f on $[0, \pi]$ is given by

$$\frac{1}{\pi} \sum_{n=2}^{\infty} \left[\frac{(-1)^{n+1} + 1}{n+1} + \frac{(-1)^{n+1} + 1}{n-1} \right] \sin nx.$$

3. (**10.4.14**) Compute the Fourier cosine series for $f(x) = e^{-x},\ 0 < x < 1$.

Solution:

The Fourier cosine series for f on $[0, 1]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{where} \quad a_n = 2\int_0^1 e^{-x}\cos n\pi x\, dx, \quad n = 0, 1, \ldots.$$

Using integration by parts we derive

$$\int e^{au}\cos nu\, du = \frac{1}{a^2+n^2} e^{au}(a\cos nu + n\sin nu) + C.$$

So

$$\begin{aligned}
a_0 &= 2\int_0^1 e^{-x}\, dx = -2e^{-x}\Big|_0^1 = 2(1-e^{-1}),\\
a_n &= 2\int_0^1 e^{-x}\cos n\pi x\, dx = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{1}{\pi}u}\cos nu\, du\\
&= \frac{2}{\pi}\left[\frac{1}{(-\frac{1}{\pi})^2+n^2} e^{-\frac{1}{\pi}u}\left(-\frac{1}{\pi}\cos nu + n\sin nu\right)\Big|_0^{\pi}\right.\\
&= \frac{2}{\pi}\left[\frac{\pi^2}{1+(n\pi)^2}\left\{e^{-1}\left(-\frac{1}{\pi}\cos n\pi + n\sin n\pi\right) - \left(-\frac{1}{\pi}\cos 0\right)\right\}\right]\\
&= \frac{2\pi}{1+(n\pi)^2}\left\{\frac{e^{-1}}{\pi}(-1)^{n+1} + \frac{1}{\pi}\right\} = \frac{(-1)^{n+1}e^{-1}+1}{1+(n\pi)^2}, \qquad n = 1, 2, \ldots.
\end{aligned}$$

The Fourier cosine series for f on $[0, 1]$ is

$$1 - e^{-1} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-1}+1}{1+(n\pi)^2}\cos n\pi x.$$

4. (10.4.18) Find the solution to the heat flow problem:

$$u_t = 5u_{xx}, \quad 0 < x < \pi,\ t > 0$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0$$
$$u(x,0) = x(\pi - x), \quad 0 < x < \pi.$$

Solution:

This is the heat flow problem with $\beta = 5$ and $L = \pi$, so

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin nx$$
$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = x(\pi - x).$$

Then

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx \\ &= \frac{2}{\pi} \left[\int_0^{\pi} \pi x \sin nx \, dx - \int_0^{\pi} x^2 \sin nx \, dx \right] \end{aligned}$$

where

$$\begin{aligned} \int_0^{\pi} x \sin nx \, dx &= \frac{1}{n^2} (\sin u - u \cos u \Big|_0^{n\pi} \quad (u = nx) \\ &= \frac{1}{n^2} (-n\pi \cos n\pi) = \frac{\pi}{n} (-1)^{n+1}, \end{aligned}$$

and

$$\begin{aligned} \int_0^{\pi} x^2 \sin nx \, dx &= \frac{1}{n^3} \int_0^{n\pi} u^2 \sin u \, du \\ &= \frac{1}{n^3} [-u^2 \cos u + n(\cos u + u \sin u) \Big|_0^{n\pi} \\ &= \frac{1}{n^3} [-(n\pi)^2 \cos n\pi + n(\cos n\pi) - n] \\ &= \frac{1}{n^3} [(n\pi)^2 (-1)^{n+1} + n(-1)^n - n] = \frac{\pi^2 (-1)^{n+1}}{n} + \frac{1}{n^2}((-1)^n - 1) \end{aligned}$$

Therefore

$$c_n = \frac{2}{\pi} \left[\pi \, \frac{\pi}{n} (-1)^{n+1} - \frac{\pi^2}{n} (-1)^{n+1} - \frac{1}{n^2} ((-1)^n - 1) \right] = \frac{2}{\pi n^2} [(-1)^{n+1} + 1].$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^{n+1}+1]e^{-5n^2 t}\sin nx$$

$$\text{or} \qquad = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}e^{-5(2k-1)^2 t}\sin(2k-1)x.$$

5. (10.5.2) Find a solution to the initial-boundary value problem:

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0$$

$$u(x,t) = x^2, \quad 0 < x < \pi.$$

Solution:

This is the heat flow problem with $\beta = 1$ and $L = \pi$. So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin nx$$

where $\sum_{n=1}^{\infty} c_n \sin nx = x^2$. We calculate c_n

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx \\ &= \frac{2}{\pi} \left[\frac{\pi^2}{n} (-1)^{n+1} + \frac{1}{n^2} ((-1)^n - 1) \right] \quad \text{(using 10.4.18 results)} \\ &= \frac{2\pi}{n} (-1)^{n+1} + \frac{2}{\pi n^2} ((-1)^n - 1). \end{aligned}$$

6. (**10.5.8**) Solve the heat flow problem:

$$u_t = u_{xx}, \quad 0 < x < \pi, \ t > 0$$
$$u(0,t) = 0, \quad u(\pi,t) = 3\pi, \quad t > 0$$
$$u(x,0) = 0, \quad 0 < x < \pi.$$

Solution:

The solution is of the form $u(x,t) = v(x) + w(x,t)$ where $v(x)$ is a steady-state solution and $w(x,t)$ is a transient solution where as $t \to \infty$, w, w_t, w_{xx} all tend to 0. This is a nonhomogeneous problem we discussed in class with $A = 0$, $B = 3\pi$, $L = \pi$ and $\beta = 1$. Then

$$v(x) = A + \frac{(B-A)}{L}x = 3x,$$

(since $v''(x) = 0$ and $v(x) = c_1 + c_2 x$, $v(0) = 0$, $v(\pi) = 3\pi$). To find $w(x,t)$ we solve the homogeneous problem with $B = 0$. Then by separation of variables we arrive at

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin nx \qquad \text{with } \beta = 1 \text{ and } L = \pi.$$

Hence $u(x,t) = 3x + \sum\limits_{n=1}^{\infty} c_n e^{-n^2 t} \sin nx$ where $3x + \sum\limits_{n=1} c_n \sin nx = 0$, i.e., $\sum_{n=1} c_n \sin nx = -3x$. Then

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi} (-3x) \sin nx\, dx \\ &= -\frac{6}{\pi}\int_0^{\pi} x \sin nx\, dx \\ &= -\frac{6}{\pi}\left[\frac{1}{n^2}(\sin u - u\cos u\right|_0^{n\pi} \\ &= -\frac{6}{\pi n^2}(-n\pi \cos n\pi) = \frac{6}{\pi n^2} n\pi(-1)^n = \frac{6}{n}(-1)^n, \quad n \geq 1. \end{aligned}$$

So the solution is

$$u(x,t) = 3x + \sum_{n=1}^{\infty} \frac{6(-1)^n}{n} e^{-n^2 t} \sin nx.$$