MATHEMATICS 201 (R1)

SOLUTIONS TO ASSIGNMENT 1

1. (**2.2.16**) Solve $(x + xy^2)dx + e^{x^2} y dy = 0$.

Solution:

Note that the equation can be written as

$$e^{x^2} y dy = -x(1 + y^2)dx.$$

Separating the variables we have

$$\frac{y}{1+y^2}\, dy = -xe^{-x^2}\, dx.$$

Integrating,

$$\frac{1}{2}\ln(1+y^2) = \frac{1}{2}e^{-x^2} + C.$$

So the general solution is

$$\ln(1+y^2) = e^{-x^2} + C_1 \qquad \text{(where } C_1 = 2C\text{)}$$

or

$$1 + y^2 = C_2 e^{e^{-x^2}}.$$

2. (**2.2.18**) Solve the I.V.P. $\dfrac{dy}{dx} = (1 + y^2)\tan x, \quad y(0) = \sqrt{3}.$

Solution:

This equation is also separable

$$\frac{dy}{1+y^2} = \tan x\, dx.$$

Integrating

$$\tan^{-1} y = -\ln|\cos x| + C.$$

Using the initial condition $y(0) = \sqrt{3}$, we get

$$\tan^{-1}\sqrt{3} = C,$$

that is,

$$C = \frac{\pi}{3}.$$

Therefore, the solution is

$$\tan^{-1} y = -\ln|\cos x| + \frac{\pi}{3}.$$

3. (**2.2.34**) **Newton's Law of Cooling**. According to Newton's law of cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M - T$. This gives the differential equation

$$\frac{dT}{dt} = k(M - T).$$

(a) Solve the differential equation for T.

(b) A thermometer reading 100° is placed in a medium having a constant temperature of 70°. After 6 minutes, the thermometer reads 80°. What is the reading after 20 min?

Solution:

(a) Separation of variables gives us

$$\frac{dT}{M - T} = k\,dt.$$

Integrating

$$\begin{aligned} -\ln|M - T| &= kt + C_1 \\ |M - T| &= C_2 e^{-kt} \\ M - T &= C_3 e^{-kt} \end{aligned}$$

So the general solution is

$$T = M - C_3 e^{-kt} \qquad \text{or} \qquad T = M + Ce^{-kt}.$$

(b) Given: $M = 70, \quad T(0) = 100, \quad T(6) = 80$, we wish to find $T(20)$. $T(0) = 100 \implies 100 = 70 - C_3$. So $C_3 = -30$. Hence,

$$T(t) = 70 + 30e^{-kt}.$$

$T(6) = 80 \implies 80 = 70 + 30e^{-6k} \implies 30e^{-6k} = 10 \implies e^{-6k} = 1/3$. Hence,

$$k = -\tfrac{1}{6}\ln\left(\tfrac{1}{3}\right) = \tfrac{1}{6}\ln 3.$$

So, $T(t) = 70 + 30e^{-\frac{\ln 3}{6}t}$ and $T(20) = 70 + 30(3^{-\frac{10}{3}})$ ($\approx 70.8^\circ$)

4. (**2.3.14**) Solve the equation $x\dfrac{dy}{dx} + 3y + 2x^2 = x^3 + 4x$.

Solution:

We rewrite the equation as

$$\frac{dy}{dx} + \frac{3}{x}y = x^2 - 2x + 4.$$

This equation is linear in y. Look for an integrating factor $\mu(x)$.

$$\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3.$$

Multiplying the equation by x^3 we obtain

$$\begin{aligned} x^3y' + 3x^2y &= x^5 - 2x^4 + 4x^3 \\ \frac{d(x^3y)}{dx} &= x^5 - 2x^4 + 4x^3 \\ \Longrightarrow \qquad x^3y &= \frac{x^6}{6} - \frac{2}{5}x^5 + x^4 + C \end{aligned}$$

So the general solution is

$$y = \frac{1}{6}x^3 - \frac{2}{5}x^2 - x + Cx^{-3}.$$

5. (**2.3.22**) Solve the I.V.P. $\sin x \dfrac{dy}{dx} + y\cos x = x\sin x, \quad y(\pi/2) = 2.$

Solution:

Note that the left hand side of the equation is $\dfrac{(d(\sin x)y)}{dx}$. So

$$d((\sin x)y) = x\sin x\, dx.$$

Integrating,

$$\begin{aligned}(\sin x)y &= \int x\sin x\, dx \\ &= -x\cos x + \int \cos x\, dx \qquad \text{(using integration by parts)} \\ &= -x\cos x + \sin x + C\end{aligned}$$

Hence, the general solution is

$$y = -x\frac{\cos x}{\sin x} + 1 + \frac{C}{\sin x},$$

or

$$y = -x\cot x + 1 + C\csc x.$$

Using the initial value $y(\frac{\pi}{2}) = 2$, we have

$$\begin{aligned} & 2 = -\frac{\pi}{2}\cot\frac{\pi}{2} + 1 + C\csc\frac{\pi}{2} \\ \therefore \qquad & 2 = 1 + C \qquad (\cot\frac{\pi}{2} = 0, \quad \csc\frac{\pi}{2} = 1) \\ \text{and} \qquad & C = 1.\end{aligned}$$

The solution is

$$y = -x\cot x + 1 + \csc x.$$

6. (2.3.28) Constant Multiples of Solution

(a) Show that $y = e^{-x}$ is a solution of the linear equation

$$\frac{dy}{dx} + y = 0, \tag{16}$$

and $y = x^{-1}$ is a soloution of the nonlinear equation

$$\frac{dy}{dx} + y^2 = 0. \tag{17}$$

(b) Show that for any constant C, Ce^{-x} is a solution of equation (16), while Cx^{-1} is a solution of equation (17) only when $C = 0$ or 1.

Solution:

(a) To verify that $y = e^{-x}$ is a solution of (16), note that $y' = -e^{-x}$. Then using (16) $LHS = -e^{-x} + e^{-x} = 0 = RHS$.
To verify that $y = x^{-1}$ is a solution of (17), note that $y' = -x^{-2}$. Then using (17) $LHS = -x^{-2} + (x^{-1})^2 = -x^{-2} + x^{-2} = 0 = RHS$.

(b) $y = Ce^{-x} \implies y' = -Ce^{-x}$. Eqn (16) gives $LHS = -Ce^{-x} + Ce^{-x} = 0 = RHS$. So $y = Ce^{-x}$ is a solution of (16).
$y = Cx^{-1} \implies y' = -Cx^{-2}$. Eqn (17) gives $LHS = -Cx^{-2} + (Cx^{-1})^2 = (C^2 - C)x^{-2} = 0 \iff C^2 - C = 0$, i.e., $C(C-1) = 0$. So $y = Cx^{-1}$ is a solution of (17) iff $C = 0$ or 1

7. (2.4.14) Solve the equation $e^t(y-t)dt + (1+e^t)dy = 0$.

Solution:

Note that $M_y = e^t$ and $N_t = e^t$. So the equation is exact. So we look for a solution of the form $F(x,y) = C$. Then

$$\frac{\partial F}{\partial t} = e^t(y-t).$$

Integrating with respect to t

$$\begin{aligned} F(t,y) &= ye^t - \int te^t\,dt \\ &= ye^t - (te^t - e^t) + g(y) \qquad \text{(using int. by parts)} \end{aligned}$$

Differentiating with respect to y we get

$$\frac{\partial F}{\partial y} = e^t + g'(y).$$

Then since $\frac{\partial F}{\partial y} = N(x,y)$

$$\begin{aligned} e^t + g'(y) &= 1 + e^t \\ g'(y) &= 1 \\ g(y) &= y. \end{aligned}$$

Hence $F(x,y) = ye^t - te^t + e^t + y$, and the solution is

$$ye^t - te^t + e^t + y = C.$$

8. (**2.4.18.**) Solve $(2x + y^2 - \cos(x + y))dx + (2xy - \cos(x + y) - e^y)dy = 0.$

Solution:

Note $M_y = 2y + \sin(x + y)$ and $N_x = 2y + \sin(x + y)$. So exact. So solution is of the form $F(x, y) = C$. Now

$$\frac{\partial F}{\partial x} = 2x + y^2 - \cos(x + y).$$

Integrating with respct to x we have

$$F(x, y) = x^2 + y^2 x - \sin(x + y) + f(y).$$

So

$$\frac{\partial F}{\partial y} = 2xy - \cos(x + y) + f'(y).$$

Therefore,

$$\begin{aligned} 2xy - \cos(x + y) + f'(y) &= 2xy - \cos(x + y) - e^y \\ \Longrightarrow \qquad f'(y) &= -e^y \\ \Longrightarrow \qquad f(y) &= -e^y. \end{aligned}$$

So, $F(x, y) = x^2 + xy^2 - \sin(x + y) - e^y$ and the solution is

$$x^2 + xy^2 - \sin(x + y) - e^y = C.$$

9. (**2.4.34**) Use the method described in Problem 32 to show that the orthogonal trajectories to the family of curves $x^2 + y^2 = kx$, k a parameter, satisfy

$$(2yx^{-1})dx + (y^2x^{-2} - 1)dy = 0.$$

Find the orthogonal trajectories by solving the above equation.

Solution:

So the given family of curves is $x^2 + y^2 = kx$ and $F(x, y) = x + y^2/x$. So

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{(1 - \frac{y^2}{x^2})}{\frac{2y}{x}} \\ &= -\frac{(1 - x^{-2}y^2)}{2x^{-1}y} \end{aligned}$$

So the O.T. have the differential equation

$$\frac{dy}{dx} = \frac{2x^{-1}y}{1 - x^{-2}y^2}$$

i.e.,

$$(1 - x^{-2}y^2)dy - 2x^{-1}ydx = 0,$$

i.e.,

$$2x^{-1}ydx + (x^{-2}y^2 - 1)dy = 0.$$

To solve the equation we note that this is not an exact equation. We multiply the equation by $x^m y^n$ and find n and m so that the equation will become exact. We have

$$(2x^{m-1}y^{n+1})dx + (x^{m-2}y^{n+2} - x^m y^n)dy = 0.$$

So $M_y = 2(n+1)x^{m-1}y^n$ and $N_x = (m-2)x^{m-3}y^{n+2} - mx^{m-1}y^n$. Then $M_y = N_x \iff m - 2 = 0$ and $2(n+1) = -m$ Solving the equations we get $m = 2$ and $n = -2$. So if we multiply the original equation (for O.T.) by x^2y^{-2} we have an exact equation, i.e.,

$$2xy^{-1}dx + (1 - x^2y^{-2})dy = 0.$$

Look for a solution of the form $G(x,y)=C$. Then $\partial G/\partial x = 2xy^{-1}$. Integrating with respect to x, $G(x,y) = x^2y^{-1} + A(y)$. Differentiating w.r.t. y, $\partial G/\partial y = -x^2y^{-2} + A'(y)$. Hence,

$$\begin{aligned} -x^2y^{-2} + A'(y) &= 1 - x^2y^{-2} \\ A'(y) &= 1 \\ A(y) &= y. \end{aligned}$$

So $x^2y^{-1} + y = C$ is a general solution (O.T.).

10. (2.6.12) Solve using "Homogeneous equations" method

$$(x^2+y^2)dx + 2xydy = 0.$$

Solution:

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} = -\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right).$$

Let $v = \frac{y}{x}$. So $y' = v + x\frac{dv}{dx}$. Then the d.e. becomes

$$\begin{aligned} x\frac{dv}{dx} + v &= -\frac{1}{2}\left(\frac{1}{v}+v\right) \\ x\frac{dv}{dx} &= -\frac{1}{2}\frac{1}{v} - \frac{v}{2} - v \\ &= -\frac{1}{2}\left(\frac{1+3v^2}{v}\right). \end{aligned}$$

Separting variables,

$$\frac{2v}{1+3v^2}\,dv = -\frac{1}{x}\,dx.$$

Integrating,

$$\begin{aligned} 2\cdot\frac{1}{6}\ln(1+3v^2) &= -\ln|x| + C \\ 1+3v^2 &= C_1|x|^3 \\ 1+3\left(\frac{y}{x}\right)^2 &= C_2x^3 \end{aligned}$$

Simplifying we get the general solution

$$x^2+3y^2 = C_2x^5.$$

11. (**2.6.28**) Solve the Bernoulli equation $y' + y^3x + y = 0$.

Solution:

Rewriting the equation to $y' + y = -xy^3$ we see that this is a Bernoulli equation with $n = 3$. Let $v = y^{1-n} = y^{-2}$. Then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\frac{1}{2}y^3\frac{dv}{dx}$$

So the differential equation becomes

$$-\frac{1}{2}y^3\frac{dv}{dx} + y = -xy^3,$$

i.e.,

$$\frac{dv}{dx} - 2v = 2x \qquad \text{(a linear equation in } v\text{)}$$

So $\mu(x) = e^{-\int 2\,dx} = e^{-2x}$ is an integrating factor. Then

$$\begin{aligned}
e^{-2x}v' - 2e^{-2x}v &= 2xe^{-2x} \\
d(e^{-2x}v) &= 2xe^{-2x}\,dx \\
e^{-2x}v &= \int 2xe^{-2x}\,dx \\
&= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \\
\text{so,} \qquad v &= -\frac{1}{2}x - \frac{1}{4} + C = -\frac{1}{2}x + C_1
\end{aligned}$$

Using $v = y^{-2}$ we have $y^{-2} = -\frac{1}{2}x + C_1$ or $y^2 = \frac{1}{C_1 - \frac{1}{2}x}$. The general solution is

$$y^2 = \frac{1}{C_1 - \frac{1}{2}x} \quad \text{or} \quad y \equiv 0.$$