Math 113/114–Solutions to Final Exam Sample Problems

1. Find y'.

(a) $y' = (-1-3x^2)\sec^2(x^2-1) + (1-x-x^3)\cdot 2\sec^2(x^2-1)\tan(x^2-1)(2x)$.

(b) $\ln y = \ln x \ln(\tan(3x)) \implies \frac{1}{y}\,y' = \dfrac{1}{x}\ln(\tan(3x)) + \ln x \cdot \dfrac{3\sec^2 3x}{\tan 3x}$

$\therefore \quad y' = (\tan 3x)^{\ln x}\left[\dfrac{\ln(\tan 3x)}{x} + \dfrac{3\ln x \sec^2 3x}{\tan 3x}\right]$

(c) $y' = 3(\tan^2\sqrt{x^2-\sin x})(\sec^2\sqrt{x^2-\sin x})\cdot \frac{1}{2}(x^2-\sin x)^{-\frac{1}{2}}(2x-\cos x)$

(d) $\ln y = 3\ln(\sin x) + \frac{1}{2}\ln(3x^5+1) - x^3\ln e - 5\ln(2-5x^2)$

$\implies \dfrac{1}{y}y' = \dfrac{3\cos x}{\sin x} + \dfrac{1}{2}\dfrac{15x^4}{(3x^5+1)} - 3x^2 - \dfrac{5(-10x)}{2-5x^2}$

$\therefore \quad y' = \dfrac{\sin^3 x\sqrt{5x^5+1}}{e^{x^3}(2-5x^2)^5}\left[3\cot x + \dfrac{15x^4}{2(3x^5+1)} - 3x^2 + \dfrac{50x}{2-5x^2}\right]$

(e) $y' = [\sec^2(x^2-\sin\sqrt{1-x^2})^4](4(x^2-\sin\sqrt{1-x^3})^3)\{2x - (\cos\sqrt{1-x^3})\frac{1}{2}(1-x^3)^{-\frac{1}{2}}(-3x^2)\}$

(f) Diff. implicitly, $[\cos(x-y^2)](1-2yy') = 2x - y + xy' \implies$

$(-2y\cos(x-y^2)+x)y' = 2x - y - \cos(x-y^2) \implies y' = \dfrac{2x-y-\cos(x-y^2)}{x-2y\cos(x-y^2)}$

(g) We use logarithmic differentiation: taking natural log, we get $\ln y = (\tan x)\ln(\sec x)$

$\implies \dfrac{1}{y}\,y' = (\sec^2 x)\ln(\sec x) + \tan x\dfrac{1}{\sec x}\sec x\tan x \implies y' = (\sec x)^{\tan x}[(\sec^2 x)\ln(\sec x) + \tan^2 x]$

(h) Diff. implicitly. $e^{xy}(y+xy') - 2yy' = \frac{1}{x} \implies$

$ye^{xy} + xe^{xy}y' - 2yy' = \frac{1}{x} \implies (xe^{xy}-2y)y' = \frac{1}{x} - ye^{xy} \implies y' = \dfrac{\frac{1}{x} - ye^{xy}}{xe^{xy}-2y}$

2. Evaluate the limits.

(a)

$$\begin{aligned}
\lim_{x\to\infty}(x-\sqrt{x^2+x+1}) &= \lim_{x\to\infty}\frac{(x-\sqrt{x^2+x+1})(x+\sqrt{x^2+x+1})}{x+\sqrt{x^2+x+1}} \\
&= \lim_{x\to\infty}\frac{x^2-x^2-x-1}{x+\sqrt{x^2+x+1}} = \lim_{x\to\infty}\frac{x(-1-\frac{1}{x})}{x+\sqrt{x^2(1+\frac{1}{x}+\frac{1}{x^2})}} \\
&= \lim_{x\to\infty}\frac{x(-1-\frac{1}{x})}{x\big(1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}\big)} = \frac{-1}{1+1} = -\frac{1}{2} \quad (\sqrt{x^2}=x,\ x>0).
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x\sin 2x} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x\sin 2x(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x\sin 2x(1+\cos x)} \\
&= \lim_{x\to 0}\frac{\sin^2 x}{x2\sin x\cos x(1+\cos x)} = \lim_{x\to 0}\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1}{1+\cos x} = \frac{1}{2}\cdot 1\cdot 1\cdot\frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

(c)

$$\lim_{x\to 0}\frac{\sin 3x}{\tan 2x} = \lim_{x\to 0}\frac{\sin 3x}{\sin 2x}\cos 2x = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3x}{2x}\cdot\frac{2x}{\sin 2x}\cdot\cos 2x = 1\cdot\frac{3}{2}\cdot 1\cdot 1 = \frac{3}{2}.$$

(d)

$$\lim_{x\to 2}\frac{\sqrt{2x}-x}{x-2} = \lim_{x\to 2}\frac{\sqrt{2x}-x)(\sqrt{2x}+x)}{(x-2)(\sqrt{2x}+x)} = \lim_{x\to 2}\frac{2x-x^2}{(x-2)(\sqrt{2x}+x)}$$
$$= \lim_{x\to 2}\frac{x(2-x)}{-(2-x)(\sqrt{2x}+x)} = \lim_{x\to -2}\frac{x}{-(\sqrt{2x}+x)} = \frac{2}{-(2+2)} = -\frac{1}{2}.$$

(e)

$$\lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+1}} = \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2(1+\frac{1}{x^2})}} = \lim_{x\to-\infty}\frac{2x}{-x\sqrt{1+\frac{1}{x^2}}} = \frac{2}{-1} = -2,$$

(since $\sqrt{x^2} = -x,\ x<0$).

(f)

$$\lim_{x\to-\infty}\frac{1-4x+x^3}{3x^2-x+4} = \lim_{x\to-\infty}\frac{x^3(\frac{1}{x^3}-\frac{4}{x^2}+1)}{x^2(3-\frac{1}{x}+\frac{4}{x^2})} = \lim_{x\to-\infty}\frac{x(\frac{1}{x^3}-\frac{4}{x^2}+1)}{3-\frac{1}{x}+\frac{4}{x^2}} = -\infty,$$

since $1/x^r \to 0$ as $x\to-\infty$.

3. Evaluate the integrals.

(a)

$$\int\frac{1-x^2-3x^2}{x^{\frac{2}{3}}}\,dx = \int(x^{-\frac{2}{3}}-4x^{\frac{4}{3}})\,dx = 3x^{\frac{1}{3}} - 4\cdot\frac{3}{7}x^{\frac{7}{3}}+C = 3x^{\frac{1}{3}}-\frac{12}{7}x^{\frac{7}{3}}+C.$$

(b)

$$\int x\sin(1-2x^2)\,dx = -\frac{1}{4}\int\sin u\,du = \frac{1}{4}\cos u + C = \frac{1}{4}\cos(1-2x^2)+C \quad (u=1-2x^2,\ du=-4xdx).$$

(c) Let $u = 1+\cos 2x,\ du = -2\sin 2x dx.\ x=0 \implies u=2,\ x=\frac{\pi}{3} \implies u=\frac{1}{2}.$

$$\int_0^{\frac{\pi}{3}}\frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx = -\frac{1}{2}\int_2^{\frac{1}{2}}\frac{du}{\sqrt{u}} = \frac{1}{2}\int_{\frac{1}{2}}^2 u^{-\frac{1}{2}}\,du = \frac{1}{2}\cdot 2u^{\frac{1}{2}}\Big|_{\frac{1}{2}}^2 = \sqrt{2}-\frac{1}{\sqrt{2}} = \frac{2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

(d)

$$\int_0^2|3x-1|\,dx = \int_0^{\frac{1}{3}}-(3x-1)\,dx + \int_{\frac{1}{3}}^2(3x-1)\,dx = \left[x-\frac{3x^2}{2}\right|_0^{\frac{1}{3}} + \left[\frac{3x^2}{2}-x\right|_{\frac{1}{3}}^2$$
$$= \frac{1}{3}-\frac{3}{2}\cdot\frac{1}{9}+\frac{3}{2}\cdot 4-2-\frac{3}{2}\cdot\frac{1}{9}+\frac{1}{3} = \frac{13}{3}.$$

(e)

$$\int_0^1|2x^2+x-1|\,dx = \int_0^1|(2x-1)(x+1)|\,dx = -\int_0^{\frac{1}{2}}(2x^2+x-1)\,dx + \int_{\frac{1}{2}}^1(2x^2+x-1)\,dx$$
$$= -\left[2\frac{x^3}{3}+\frac{x^2}{2}-x\right|_0^{\frac{1}{2}} + \left[2\frac{x^3}{3}+\frac{x^2}{2}-x\right|_{\frac{1}{2}}^1$$
$$= -\left[\frac{1}{12}+\frac{1}{8}-\frac{1}{2}\right] + \left[\frac{2}{3}+12-1-\left(\frac{1}{12}+\frac{1}{8}-\frac{1}{2}\right)\right] = \frac{7}{24}+\frac{1}{6}+\frac{7}{24} = \frac{3}{4}.$$

(f) Let $u = 2 + \cos^2\theta$, $du = -2\cos\theta\sin\theta\, d\theta$.

$$\int \sin\theta\cos\theta(2+\cos^2\theta)^2\, d\theta = -\frac{1}{2}\int u^2\, du = -\frac{1}{2}\cdot\frac{u^3}{3} + C = -\frac{1}{6}(2+\cos^2\theta)^3 + C.$$

(g) Let $u = \tan 7x$, $du = 7\sec^2 7x\, dx$. Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{e^{\tan 7x}}{\cos^2 7x}\, dx &= \int_0^{\frac{\pi}{4}} \sec^2 7x\, e^{\tan 7x}\, dx \\ &= \frac{1}{7}\int_0^{-1} e^u\, du \quad \left(x = 0 \implies u = \tan 0 = 0;\ x = \frac{\pi}{4} \implies u = \tan\frac{7\pi}{4} = -1\right) \\ &= -\frac{1}{7}\int_{-1}^{0} e^u\, du = -\frac{1}{7}e^u\Big|_{-1}^{0} = -\frac{1}{7}(e^0 - e^{-1}) = -\frac{1}{7}\left(1 - \frac{1}{e}\right).\end{aligned}$$

(h) Note that $f(x) = \dfrac{x^3\sin^4 x}{1+x^4}$ is an odd function since $f(-x) = \dfrac{-x^3\sin^4 x}{1+x^4} = -f(x)$. So

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^3\sin^4 x}{1+x^4}\, dx = 0.$$

(i) Let $u = x^3 + 1$, $du = 3x^2\, dx$.

$$\begin{aligned}\int x^5\sqrt{x^3+1}\, dx &= \int x^3\sqrt{x^3+1}\, x^2\, dx = \frac{1}{3}\int (u-1)\sqrt{u}\, du \\ &= \frac{1}{3}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\, du = \frac{1}{3}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + C = \frac{2}{15}(x^3+1)^{\frac{5}{2}} - \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C.\end{aligned}$$

(j) Let $u = 2 - x$, $du = -dx$.

$$\begin{aligned}\int x\sqrt{2-x}\, dx &= -\int (2-u)\sqrt{u}\, du = -\int (2u^{\frac{1}{2}} - u^{\frac{3}{2}})\, du \\ &= -\left(\frac{4}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C = -\frac{4}{3}(2-x)^{\frac{3}{2}} + \frac{2}{5}(2-x)^{\frac{5}{2}} + C.\end{aligned}$$

(k) Let $u = 3 + 7\tan 5x$. So $du = 35\sec^2 5x\, dx$. Then

$$\int \frac{\sec^2 5x}{3 + 7\tan 5x}\, dx = \frac{1}{35}\int \frac{du}{u} = \frac{1}{35}\ln|u| + C = \frac{1}{35}\ln|3 + 7\tan 5x| + C.$$

(l)

$$\begin{aligned}\int (1 + e^{\cos x})\sin x\, dx &= \int (\sin x + e^{\cos x}\sin x)\, dx = -\cos x - \int e^u\, du \quad (u = \cos x,\ du = -\sin x\, dx) \\ &= -\cos x - e^u + C = -\cos x - e^{\cos x} + C.\end{aligned}$$

(m) First let $u = \pi\ln x$, $du = \frac{\pi}{x}\, dx$. Then

$$\begin{aligned}\int_1^e \frac{\cos^2(\pi\ln x)\sin(\pi\ln x)}{x}\, dx &= \frac{1}{\pi}\int_0^{\pi} \cos^2 u\sin u\, du \\ &= -\frac{1}{\pi}\int_1^{-1} t^2\, dt \quad (t = \cos u,\ dt = -\sin u\, du \text{ and } \cos 0 = 1,\ \cos\pi = -1) \\ &= \frac{1}{\pi}\int_{-1}^{1} t^2\, dt = \frac{2}{\pi}\int_0^1 t^2\, dt = \frac{2}{\pi}\frac{t^3}{3}\Big|_0^1 = \frac{2}{3\pi} - 0 = \frac{2}{3\pi}.\end{aligned}$$

4.

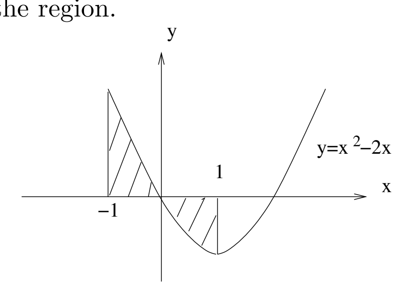
(a) $f(x) = \int_{\tan x}^{2} \sqrt{1+t^4}\, dt.$

$$f'(x) = \frac{d}{dx}\int_{\tan x}^{2} \sqrt{1+t^4}\, dt = -\frac{d}{dx}\int_{2}^{\tan x} \sqrt{1+t^4}\, dt = -\frac{d}{du}\int_{2}^{u} \sqrt{1+t^4}\, dt \cdot \frac{du}{dx}$$
$$\overset{\text{F.T.C.}}{=} -\sqrt{1+u^4}\, \sec^2 x = -\sqrt{1+\tan^4 x}\, \sec^2 x.$$

(b) $f(x) = \int_{\cot^2(1-\tan x)}^{0} \cos t^2\, dt.$

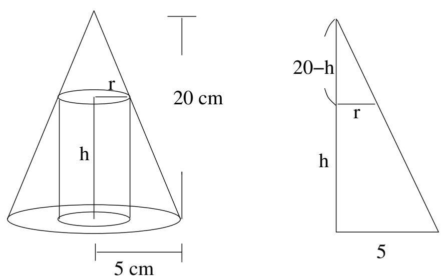
$$\begin{aligned} f'(x) &= \frac{d}{dx}\int_{\cot^2(1-\tan x)}^{0} \cos t^2\, dt = -\frac{d}{dx}\int_{0}^{\cot^2(1-\tan x)} \cos t^2\, dt \\ &= -\frac{d}{du}\int_0^u \cos t^2\, dt \cdot \frac{du}{dx} \qquad (\text{where } u = \cot^2(1-\tan x)) \\ &\overset{\text{F.T.C.}}{=} -(\cos u^2) 2(\cot(1-\tan x))(-\csc^2(1-\tan x))(-\sec^2 x) \\ &= -2\cos(\cot^4(1-\tan x))\cot(1-\tan x)(\csc^2(1-\tan x))\sec^2 x. \end{aligned}$$

5. We first graph the function and the region.



$$A = \int_{-1}^{1} |x^2-2x|\, dx = \int_{-1}^{0} (x^2-2x)\, dx - \int_0^1 (x^2-2x)\, dx = -\left(\frac{x^3}{3} - x^2\right|_{-1}^{0} - \left(\frac{x^3}{3} - x^2\right|_0^1$$
$$= -\frac{-1}{3} + 1 - \frac{1}{3} + 1 = \frac{1}{3} + 1 - \frac{1}{3} + 1 = 2.$$

6. We wish to maximize the volume of the cylinder, i.e., $V = \pi r^2 h$.

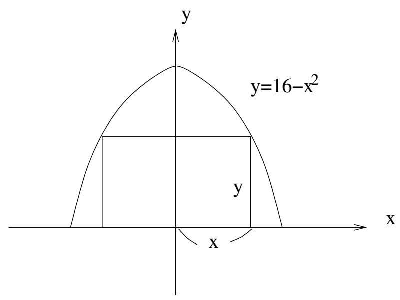


From the diagram we have

$$\frac{20-h}{r} = \frac{20}{5} \implies 20-h = 4r \implies h = 20-4r.$$

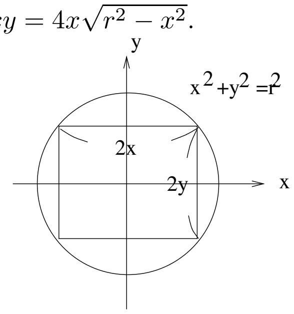
Then
$$V = \pi r^2(20-4r) = 4\pi r^2(5-r) = 4\pi(5r^2 - r^3), \quad 0 \le r \le 5$$
Differentiating,
$$\frac{dV}{dr} = 4\pi(10r - 3r^2) = 4\pi r(10-3r).$$
So the critical points are $r = 0,\ 10/3$. We will use the Extreme Value Theorem to verify:
$$V(0) = 0, \quad V(5) = 0, \quad V\left(\frac{10}{3}\right) = 4\pi\left(\frac{10}{3}\right)^2\left(5 - \frac{10}{3}\right) = 4\pi \cdot \frac{100}{9} \cdot \frac{5}{3} = \frac{2000\pi}{27}.$$
So the maximum volume of the cylinder is $\dfrac{2000\pi}{27}$ cm^3.

7.



We wish to maximize the area $A = 2xy$. Then
$$A = 2xy = 2x(16-x^2) = 2(16x - x^3) \implies \frac{dA}{dx} = 2(16-3x^2).$$
To find the critical points let $dA/dx = 0$ so that $x^2 = 16/3 \implies x = 4/\sqrt{3}$. Let's use the first derivative test to verify:
$$x < \frac{4}{\sqrt{3}} \implies \frac{dA}{dx} > 0 \implies A \text{ increasing}$$
$$x > \frac{4}{\sqrt{3}} \implies \frac{dA}{dx} < 0 \implies A \text{ decreasing.}$$
So the area is a maximum at $x = 4/\sqrt{3}$. Then $y = 16 - (16/3) = 32/3$. The dimensions of the rectangle are $8/\sqrt{3}$ by $32/3$.

8. We wish to maximize the area $A = 4xy = 4x\sqrt{r^2 - x^2}$.



$$\frac{dA}{dx} = 4\sqrt{r^2-x^2} + \frac{4x}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x) = 4(r^2-x^2)^{-\frac{1}{2}}(r^2 - x^2 - x^2) = \frac{4(r^2-2x^2)}{\sqrt{r^2-x^2}}.$$
Setting $dA/dx = 0$, we get $x = r/\sqrt{2}$. Let's use the first derivative test to verify: if $0 < x < r/\sqrt{2}$, then $dA/dx > 0$ and if $r/\sqrt{2} < x$, then $dA/dx < 0$. So A is increasing on the left $r/\sqrt{2}$ and decreasing on the right of $r/\sqrt{2}$, so A is a maximum when $x = r/\sqrt{2}$.

Then $y=\sqrt{r^2-\frac{r^2}{2}}=\sqrt{\frac{r^2}{2}}=\frac{r}{\sqrt{2}}$. So it is a square with $2r/\sqrt{2}$ by $2r/\sqrt{2}$.

9. The statement of the Mean Value Theorem: If a function f is continuous on $[a,b]$ and differentiable on (a,b), then there is a number $c\in(a,b)$ such that $f'(c)=[f(b)-f(a)]/(b-a),\ a\neq b$.

Then since $1\le f'(x)\le 4$ for all $x\in(2,5)$, f is differentiable on $(2,5)$, then f is continuous on $[2,5]$. Then by the Mean Value Theorem there exists a number $c\in(2,5)$ such that $f'(c)=\frac{f(5)-f(2)}{3}$. But $1\le f'(c)\le 4$. So $1\le\frac{f(5)-f(2)}{3}\le 4\implies 3\le f(5)-f(2)\le 12$.

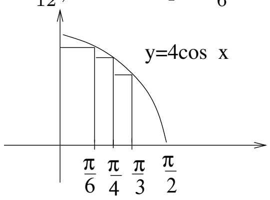
10. Step 1: Let $f(x)=3x+2\cos x+5$. The function is continuous everywhere since it is the sum of everywhere-continuous polynomial and sine functions. We note that

$$f(-\pi)=3(-\pi)+2\cos(-\pi)+5=-3\pi-2+5=-3\pi+3<0,$$
$$f(0)=0+2\cos 0+5=7>0,$$

so that $f(-\pi)<0<f(0)$. Then by the Intermediate Value Theorem, there is a number $c\in(-\pi,0)$ such that $f(c)=0$, i.e., there is a root $x=c$ of the equation.

We now show that there is exactly one real root. Assume that there are two roots, a and b such that $a<b$. Then $f(a)=0=f(b)$. By Step 1, f is continuous on $[a,b]$. $f'(x)=3-2\cos x$ so f is differentiable on (a,b). Then by Rolle's Theorem, there is a number $d\in(a,b)$ such that $f'(d)=0$, i.e. $3-2\cos d=0$. But this equation has no solution since $-1\le\cos d\le 1$ and hence $f'(d)>0$. This is a contradiction to Rolle's Thm. Therefore, there cannot be two roots of the given equation.

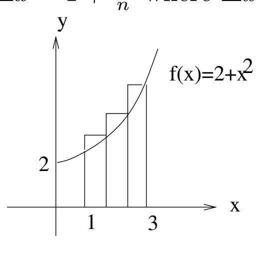
11. $f(x)=4\cos x$ on $[0,\frac{\pi}{2}]$. Using $x_i^*=\frac{\pi}{6},\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}$, Note that the width Δx is not the same for all the subintervals. $\Delta x_1=\frac{\pi}{6}$, $\Delta x_2=\frac{\pi}{12}$, $\Delta x_3=\frac{\pi}{12}$, and $\Delta x_4=\frac{\pi}{6}$.



So

$$A\approx f(x_1^*)\Delta x_1+f(x_2^*)\Delta x_2+f(x_3^*)\Delta x_3+f(x_4^*)\Delta x_4=f\left(\frac{\pi}{6}\right)\left(\frac{\pi}{6}\right)+f\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right)+f\left(\frac{\pi}{3}\right)\left(\frac{\pi}{12}\right)+f\left(\frac{\pi}{2}\right)\left(\frac{\pi}{6}\right)$$
$$=\frac{\pi}{6}\left(4\cos\frac{\pi}{6}\right)+\frac{\pi}{12}\left(4\cos\frac{\pi}{4}\right)+\frac{\pi}{12}\left(4\cos\frac{\pi}{3}\right)+\frac{\pi}{6}\left(4\cos\frac{\pi}{2}\right)=\frac{2\pi}{3}\cdot\frac{\sqrt{3}}{2}+\frac{\pi}{3}\cdot\frac{\sqrt{2}}{2}+\frac{\pi}{3}\frac{1}{2}+0$$
$$=\frac{2\sqrt{3}}{6}\pi+\frac{\sqrt{2}}{6}\pi+\frac{\pi}{6}=\left(\frac{1+2\sqrt{3}+\sqrt{2}}{6}\right)\pi.$$

12. $f(x)=2+x^2$ on $[1,3]$. $x_i^*=x_i=1+i\Delta x=1+\frac{i2}{n}$ where $\Delta x=\frac{2}{n}$.



$$\begin{aligned}
\int_1^3 (2+x^2)\,dx &= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f\left(1+\frac{i2}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n\left[2+\left(1+\frac{i2}{n}\right)^2\right]\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n\left[3+\frac{4i}{n}+\frac{4i^2}{n^2}\right] = \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^n 3+\frac{4}{n}\sum_{i=1}^n i+\frac{4}{n^2}\sum_{i=1}^n i^2\right]\\
&= \lim_{n\to\infty}\frac{2}{n}\left[3n+\frac{4}{n}\cdot\frac{n(n+1)}{2}+\frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]\\
&= \lim_{n\to\infty}\left[6+\frac{4(n+1)}{n}+\frac{4(n+1)(2n+1)}{3n\cdot n}\right]\\
&= \lim_{n\to\infty}\left[6+4\left(1+\frac{1}{n}\right)+\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 6+4+\frac{4}{3}(2) = \frac{38}{3}.
\end{aligned}$$

13.

(a) $y = \dfrac{x}{(x-1)^2}$.

$$\begin{aligned}
y' &= \frac{(x-1)^2 - x\cdot 2(x-1)}{(x-1)^4} = \frac{(x-1)[x-1-2x]}{(x-1)^4} = \frac{-1-x}{(x-1)^3} = \frac{-(x+1)}{(x-1)^3}\\
y'' &= \frac{-(x-1)^3+(x+1)\cdot 3(x-1)^2}{(x-1)^6} = \frac{(x-1)^2[-(x-1)+3(x+1)]}{(x-1)^6}\\
&= \frac{-x+1+3x+3}{(x-1)^4} = \frac{2x+4}{(x-1)^4} = \frac{2(x+2)}{(x-1)^4}.
\end{aligned}$$

(b) $y = x\sqrt{9-x^2}$.

$$\begin{aligned}
y' &= \sqrt{9-x^2}+\frac{x}{2}(9-x^2)^{-\frac12}(-2x) = (9-x^2)^{-\frac12}(9-x^2-x^2) = \frac{9-2x^2}{\sqrt{9-x^2}},\\
y'' &= \frac{-4x\sqrt{9-x^2}-(9-2x^2)\frac12(9-x^2)^{-\frac12}(-2x)}{9-x^2}\\
&= \frac{-x(9-x^2)^{-\frac12}[4(9-x^2)-(9-2x^2)]}{9-x^2} = \frac{-x(36-4x^2-9+2x^2)}{(9-x^2)^{\frac32}} = \frac{x(2x^2-27)}{(9-x^2)^{\frac32}}.
\end{aligned}$$

14. $y = x\sqrt{9-x^2}$. So the domain is $[-3,3]$ and the function is continuous there. So by the Extreme Value Thm, there is an absolute max. and absolute min. on $[-3,3]$. Using the result of #12(b), the critical points are $x = \pm\frac{3}{\sqrt2}$, ±3.

$$\begin{aligned}
&\text{At } x=-3, && y=0,\\
&\text{at } x=3, && y=0,\\
&\text{at } x=\frac{3}{\sqrt2}, && y=\frac{3}{\sqrt2}\sqrt{9-\frac92}=\frac{3}{\sqrt2}\sqrt{\frac92}=\frac92,\\
&\text{at } x=-\frac{3}{\sqrt2}, && y=-\frac{3}{\sqrt2}\sqrt{9-\frac92}=-\frac92.
\end{aligned}$$

So the absolute min. value is $-\frac92$ at $x=-\frac{3}{\sqrt2}$ and absolute max. value is $\frac92$ at $x=\frac{3}{\sqrt2}$.

15. The domain of f is $x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$. Using the results of 12(a), the critical point is $x = -1$.

	$-(x+1)$	$(x-1)^3$	y'	y
$(-\infty, -1)$	$+$	$-$	$-$	decreasing
$(-1,\ 1)$	$-$	$-$	$+$	increasing
$(1, \infty)$	$-$	$+$	$-$	decreasing

So y is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$. A local min. at $x = -1,\ y = -1/4$. No local maximum. From y'' above possible inflection point is at $x = -2$.

	$x+2$	$(x-1)^4$	y''	y
$(-\infty, -2)$	$-$	$+$	$-$	conc. down
$(-2,\ 1)$	$+$	$+$	$+$	conc. up
$(1, \infty)$	$+$	$+$	$+$	conc. up.

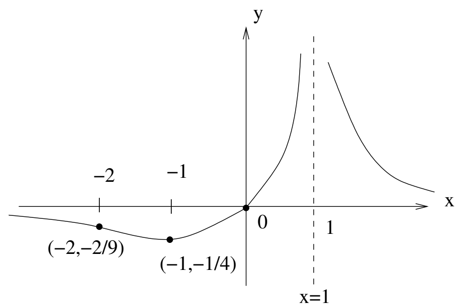
Hence, the inflection points occur at $x = -2,\ y = -2/9$.
We now look for asymptotes.

$$\lim_{x \to \pm\infty} \frac{x}{(x-1)^2} = \lim_{x \to \pm\infty} \frac{\frac{x}{x^2}}{(\frac{x-1}{x})^2} = \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{(1 - \frac{1}{x})^2} = \frac{0}{(1-0)^2} = 0.$$

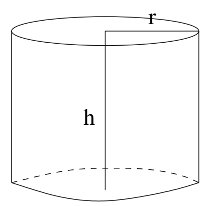
So $y = 0$ is a horizontal asymptote.

$$\lim_{x \to 1^-} \frac{x}{(x-1)^2} = \infty \quad \text{(since } x \to 1 \text{ and } (x-1)^2 \to 0 \text{ pos.)}$$
$$\lim_{x \to 1^+} \frac{x}{(x-1)^2} = \infty \quad \text{(since } x \to 1 \text{ and } (x-1)^2 \to 0 \text{ pos.).}$$

So $x = 1$ is a vertical asymptote.



16. See the figure below. Then we are given that $dV/dt = 2\ \text{cm}^3/\text{min}$, $dr/dt = 1$ cm/min. We wish to find dh/dt when $r = 5$ and $V = 60$.



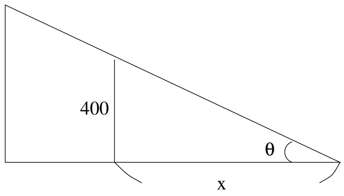
$$V = \pi r^2 h \quad \Longrightarrow \quad \frac{dV}{dt} = \pi \left[2rh\frac{dr}{dt} + r^2 \frac{dh}{dt} \right]$$
$$\Longrightarrow \quad \pi r^2 \frac{dh}{dt} = \frac{dV}{dt} - 2\pi rh \frac{dr}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \frac{1}{r^2 \pi} \left[\frac{dV}{dt} - 2\pi rh \frac{dr}{dt} \right]$$

Now when $r = 5$ and $V = 60$, $h = 12/5\pi$, then

$$\frac{dh}{dt} = \frac{1}{25\pi}\left[2 - 2\pi(5)\left(\frac{12}{5\pi}\right)(1)\right] = \frac{2-24}{25\pi} = -\frac{22}{25\pi}.$$

So the height is decreasing at a rate of $\frac{22}{25\pi}$ cm/min.

17. See the figure below. Let x be the length of the shadow cast by the building and θ the angle of elevation of the sun.



We are given $\frac{d\theta}{dt} = -0.25$ rad/h and wish to find $\frac{dx}{dt}$ when $\theta = \pi/6$.

$$\tan\theta = \frac{400}{x} \implies \sec^2\theta\frac{d\theta}{dt} = -\frac{400}{x^2}\frac{dx}{dt}$$
$$\therefore \frac{dx}{dt} = -\frac{x^2}{400}\sec^2\theta\,\frac{d\theta}{dt}.$$

We will now find x when $\theta = \pi/6$,

$$\tan\frac{\pi}{6} = \frac{400}{x} \implies x = \frac{400}{\tan\frac{\pi}{6}} = \frac{400}{\frac{1}{\sqrt{3}}} = 400\sqrt{3}.$$

Then

$$\frac{dx}{dt} = -\frac{400^2\cdot 3}{400}\sec^2\frac{\pi}{6}(-0.25) = 400(3)\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{4}\right) = 400 \text{ ft/h}.$$