Math 113/114, Midterm Exam, Version I
Solutions

1. Evaluate the limit or explain why it does not exist:

a) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

b) $\lim_{x\to 3} \frac{|3-x|}{x^2-x-6}$

Solution:

$$a) \lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}} \cdot \frac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}} = \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2} =$$

$$= \lim_{x\to 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5}{4}.$$

b) Since $|3-x| = 3-x$ if $x \leq 3$ and $|3-x| = -(3-x)$ if $x > 3$, we consider one sided limits at $x = 3$.

$$\lim_{x\to 3^-} \frac{|3-x|}{x^2-x-6} = \lim_{x\to 3^-} \frac{3-x}{(x-3)(x+2)} = \lim_{x\to 3^-} \frac{-1}{x+2} = -\frac{1}{5}$$

$$\lim_{x\to 3^+} \frac{|3-x|}{x^2-x-6} = \lim_{x\to 3^+} \frac{-(3-x)}{(x-3)(x+2)} = \lim_{x\to 3^-} \frac{1}{x+2} = \frac{1}{5}$$

Since $\lim_{x\to 3^-} \frac{|3-x|}{x^2-x-6} \neq \lim_{x\to 3^+} \frac{|3-x|}{x^2-x-6}$, the limit $\lim_{x\to 3} \frac{|3-x|}{x^2-x-6}$ does not exist.

2. Find constants a and b so that the given function is continuous at $x = -2$.

$$f(x) = \begin{cases} 3a + b + \frac{x^2-4}{x+2} & \text{if } x < -2 \\ -7 & \text{if } x = -2 \\ 3x^2 + bx + 1 & \text{if } x > -2 \end{cases}$$

Solution: First note, that f is continuous at $x = -2$ if and only if

$$f(-2) = \lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^+} f(x).$$

Now,

$$f(-2) = -7$$

$$\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^-} (3a+b+\frac{x^2-4}{x+2}) = 3a+b+\lim_{x\to(-2)^-} \frac{x^2-4}{x+2} = 3a+b+\lim_{x\to(-2)^-} (x-2) = 3a+b-4$$

$$\lim_{x\to(-2)^+} f(x) = \lim_{x\to(-2)^+} (3x^2 + bx + 1) = 13 - 2b.$$

Thus,

$$3a + b - 4 = -7 = 13 - 2b,$$

and f is continuous at $x = -2$ if and only if $a = \frac{-13}{3}$ and $b = 10$.

3. Let $f(x) = \frac{2}{x^2+2}$.

a) Use only the definition of the derivative as the limit to find $f'(x)$. No marks will be given if the definition is not used.

b) Find an equation of the tangent line to the graph of f at the point $P(2, \frac{1}{3})$.

Solution:

(a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{(x+h)^2+2} - \frac{2}{x^2+2}}{h} =$$

$$\lim_{h\to 0} \frac{h(-4x - 2h)}{h[(x+h)^2 + 2](x^2 + 2)} = \lim_{h\to 0} \frac{-4x - 2h}{[(x+h)^2 + 2](x^2 + 2)} = \frac{-4x}{(x^2 + 2)^2}.$$

(b) The equation of the tangent line to the graph of f at the point $P(2, \frac{1}{3})$ is:

$$y - \frac{1}{3} = f'(2)(x - 2).$$

Since $f'(2) = \frac{-2}{9}$, an equation of the tangent line is:

$$y - \frac{1}{3} = \frac{-2}{9}(x - 2),$$

or $y = \frac{-2}{9}x + \frac{7}{9}$.

4.

(a) Let $g(x) = e^{3x+1} f(x^2)$. Find $g'(2)$ if $f(4) = 1$, $f'(4) = 2$.

(b) Solve the equation $e^{\ln(1+e^{-x})} = e^3$.

Solution:

(a) By the Chain Rule:

$$g'(x) = e^{3x+1}3f(x^2) + e^{3x+1}f'(x^2)2x = e^{3x+1}(3f(x^2) + 2xf'(x^2)).$$

Thus,

$$g'(2) = e^7(3f(4) + 4f'(4)) = 11e^7.$$

(b)

$$e^{\ln(1+e^{-x})} = e^3 \leftrightarrow \ln(1+e^{-x}) = 3 \leftrightarrow 1+e^{-x} = e^3 \leftrightarrow e^{-x} = e^3-1 \leftrightarrow x = -\ln(e^3-1).$$

5. Differentiate.Do not simplify your result.

a) $y = (1 + \cos^2 x)^{1/3} + \tan^3(5x)$.

b) $y = \left(\frac{x^2-1}{x^2+1}\right)^3$

c) $y = \left(\dfrac{7^x}{\cos x + 2}\right)^{10}$.

Solution:

$$a)\ \frac{dy}{dx} = (1/3)(1 + \cos^2 x)^{(-2/3)} \cdot 2\cos x \cdot (-\sin x) + 3\tan^2(5x) \cdot \sec^2(5x) \cdot 5$$

$$b)\ \frac{dy}{dx} = 3\left(\frac{x^2-1}{x^2+1}\right)^2 \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2}$$

$$c)\ y' = 10\left(\frac{7^x}{\cos x + 2}\right)^9 \frac{7^x \ln 7(\cos x + 2) - 7^x(-\sin x)}{(\cos x + 2)^2}.$$