MATHEMATICS 113/114

MIDTERM EXAMINATION

VERSION 3

DATE: October 24, 2012 TIME: 50 MINUTES

LAST NAME: Solutions

FIRST NAME:________________

ID#:________________

Course	Section	Instructor	Please mark with X
Math 113	H1	E. K. Leonard	
Math 114	F1	Y.-S. Wang	

INSTRUCTIONS:

- Show all your work to receive full credit.
- Books, notes, calculators or other electronic aids are not permitted.
- Make sure that the exam has 7 pages in total.

Question	Value	Mark
1	11	
2	12	
3	9	
4	8	
5	10	
Total	50	

1. $(7 + 4 = 11 \text{ points})$

(a) Write down the <u>definition of the derivative as a function</u> and use it to find $f'(x)$ if $f(x) = \dfrac{5}{x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{5}{(x+h)^2} - \frac{5}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 - 5(x+h)^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{5x^2 - 5x^2 - 10xh - 5h^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{-5h(2x+h)}{(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{-5(2x+h)}{(x+h)^2 x^2}$$

$$= \frac{-5(2x)}{x^2 x^2} = -\frac{10}{x^3}$$

(b) Find an equation of the tangent line to the curve in part (a) at $x = -4$.

The slope of tang. line at $x = -4$ is

$$f'(-4) = -\frac{10}{(-4)^3} = -\frac{10}{-64} = +\frac{5}{32}$$

At $x = -4$, $y = \dfrac{5}{(-4)^2} = \dfrac{5}{16}$

$$\therefore\ y - \frac{5}{16} = +\frac{5}{32}(x+4) \quad \text{or} \quad y = \frac{5}{32}x + \frac{15}{16}.$$

2. $(3+3+3+3=12$ points) Find y'. Do not simplify your answers.

(a) $y = e^{2x^4-x^2+5} + (2-5x^2)^9$

$$y' = e^{2x^4-x^2+5}(8x^3-2x) + 9(2-5x^2)^8(-10x)$$

(b) $y = \left(\dfrac{x^4-3\sin x}{7x+3}\right)^{\frac{1}{4}}$

$$y' = \frac{1}{4}\left(\frac{x^4-3\sin x}{7x+3}\right)^{-\frac{3}{4}}\left[\frac{(4x^3-3\cos x)(7x+3)-(x^4-3\sin x)(7)}{(7x+3)^2}\right]$$

(c) $y = \tan^4(9-\cos^2 x)$

$$y' = (4\tan^3(9-\cos^2 x))(\sec^2(9-\cos^2 x))(-2\cos x)(-\sin x).$$

(d) $y = x^6\sin(x^5)$

$$y' = 6x^5\sin(x^5) + x^6\cos(x^5)(5x^4)$$

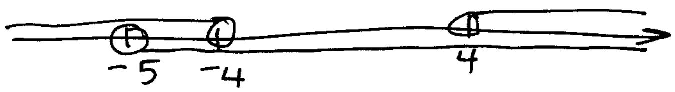
3. $(4 + 5 = 9$ points)

(a) Find the domain of the function $f(x) = \ln(x^2 - 16) + \dfrac{4}{\sqrt{x+5}}$ in interval form.

We need $x^2 - 16 > 0 \Rightarrow x^2 > 16 \Rightarrow |x| > 4$

$\therefore \; x < -4$ or $x > 4$

Also, $x + 5 > 0 \Rightarrow x > -5$



$\therefore$ the domain of f is $(-5, -4) \cup (4, \infty)$.

(b) Let $h(x) = \cos[f(\sin x)]$ where f is a differentiable function. Calculate $h'(\pi/6)$ given that $f(\frac{1}{2}) = \frac{\pi}{2}$ and $f'(\frac{1}{2}) = \frac{4}{7}$.

$$h'(x) = -\sin\left(f(\sin x)\right)\left(f'(\sin x)\right)(\cos x)$$

$$h'\left(\tfrac{\pi}{6}\right) = -\sin\left(f\left(\sin \tfrac{\pi}{6}\right)\right)\left(f'\left(\sin \tfrac{\pi}{6}\right)\right)\left(\cos \tfrac{\pi}{6}\right)$$

$$= -\sin\left(f\left(\tfrac{1}{2}\right)\right)\left(f'\left(\tfrac{1}{2}\right)\right)\left(\tfrac{\sqrt{3}}{2}\right)$$

$$= -\left(\sin \tfrac{\pi}{2}\right)\left(\tfrac{4}{7} \cdot \tfrac{\sqrt{3}}{2}\right)$$

$$= -1 \cdot \frac{4\sqrt{3}}{7(2)} = -\frac{2\sqrt{3}}{7}$$

4. (8 points) Let

$$f(x) = \begin{cases} \dfrac{x^2+x-2}{x-1}, & x < 1 \\ 2c - 5n, & x = 1 \\ \sqrt{cx - n}, & x > 1. \end{cases}$$

Determine the values of c and n so that the function f is continuous at $x = 1$.

We need $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1)$ $(*)$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \frac{x^2+x-2}{x-1} = \lim_{x\to 1-} \frac{(x+2)(x-1)}{x-1} = \lim_{x\to 1-}(x+2) = 3$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \sqrt{cx-n} = \sqrt{c-n}$$

$\therefore$ By $(*)$, $\sqrt{c-n} = 3 \implies c - n = 9 \implies c = 9 + n$

Also, $f(1) = 2c - 5n$, so by $(*)$

$2c - 5n = 3$

$\therefore$ $2(9+n) - 5n = 3$

$2n - 5n = 3 - 18$

$-3n = -15$

$\boxed{n = 5}$

$\implies c = 9 + 5 = 14$ $\therefore \boxed{c = 14}$

5. $(4+3+3=10$ points) Evaluate the following limits. Show steps. If the limit does not exist, explain why not. No marks will be given if l'Hospital's rule is used.

(a) $\displaystyle\lim_{x\to 4}\frac{|x-4|}{16x-x^3}$

$$\lim_{x\to 4^-}\frac{|x-4|}{16x-x^3}=\lim_{x\to 4^-}\frac{-(x-4)}{x(16-x^2)}=\lim_{x\to 4^-}\frac{4-x}{x(4-x)(4+x)}=\lim_{x\to 4^-}\frac{1}{x(4+x)}$$

$$=\frac{1}{4(8)}=\frac{1}{32}$$

$$\lim_{x\to 4^+}\frac{|x-4|}{16x-x^3}=\lim_{x\to 4^+}\frac{x-4}{x(16-x^2)}=\lim_{x\to 4^+}\frac{x-4}{x(4-x)(4+x)}$$

$$=\lim_{x\to 4^+}\frac{-1}{x(4+x)}=-\frac{1}{4(8)}=-\frac{1}{32}$$

Since the limits are not equal, $\displaystyle\lim_{x\to 4}\frac{|x-4|}{16x-x^3}$ does not exist.

(b) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{5x^2-3}}{6x+7}$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{5x^2-3}{x^2}}}{\frac{6x+7}{x}}=\lim_{x\to\infty}\frac{\sqrt{5-\frac{3}{x^2}}}{6+\frac{7}{x}}=\frac{\sqrt{5-0}}{6+0}=\frac{\sqrt{5}}{6}.$$

(c) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{x^2+2x-3}$

$$=\lim_{x\to 1}\frac{(\sqrt{x^2+1}-\sqrt{2})(\sqrt{x^2+1}+\sqrt{2})}{(x^2+2x-3)(\sqrt{x^2+1}+\sqrt{2})}$$

$$=\lim_{x\to 1}\frac{\overbrace{x^2+1-2}^{x^2-1}}{(x+3)(x-1)(\sqrt{x^2+1}+\sqrt{2})}$$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}(\sqrt{x^2+1}+\sqrt{2})}=\lim_{x\to 1}\frac{x+1}{(x+3)(\sqrt{x^2+1}+\sqrt{2})}=\frac{2}{4(2\sqrt{2})}=\frac{1}{4\sqrt{2}}\text{ or }\frac{\sqrt{2}}{8}$$