MATHEMATICS 113/114

MIDTERM EXAMINATION

VERSION 2

DATE: October 24, 2012 TIME: 50 MINUTES

LAST NAME: Solutions

FIRST NAME:____________________

ID#:________________

Course	Section	Instructor	Please mark with X
Math 113	H1	E. K. Leonard	
Math 114	F1	Y.-S. Wang	

INSTRUCTIONS:

- Show all your work to receive full credit.
- Books, notes, calculators or other electronic aids are not permitted.
- Make sure that the exam has 7 pages in total.

Question	Value	Mark
1	10	
2	9	
3	8	
4	11	
5	12	
Total	50	

1. $(3+3+4=10$ points) Evaluate the following limits. Show steps. If the limit does not exist, explain why not. No marks will be given if l'Hospital's rule is used.

(a) $\lim_{x\to\infty} \dfrac{\sqrt{2x^2-5}}{7x+1}$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2-5}{x^2}}}{\frac{7x+1}{x}} = \lim_{x\to\infty} \frac{\sqrt{2-\frac{5}{x^2}}}{7+\frac{1}{x}} = \frac{\sqrt{2-0}}{7+0} = \frac{\sqrt{2}}{7}$$

(b) $\lim_{x\to 1} \dfrac{\sqrt{x^2+1}-\sqrt{2}}{x^2+2x-3}$

$$= \lim_{x\to 1} \frac{(\sqrt{x^2+1}-\sqrt{2})(\sqrt{x^2+1}+\sqrt{2})}{(x^2+2x-3)(\sqrt{x^2+1}+\sqrt{2})}$$

$$= \lim_{x\to 1} \frac{x^2+1-2}{(x+3)(x-1)\sqrt{x^2+1}+\sqrt{2})}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}(\sqrt{x^2+1}+\sqrt{2})} = \frac{2}{4(\sqrt{2}+\sqrt{2})} = \frac{1}{2\cdot 2\sqrt{2}} = \frac{1}{4\sqrt{2}} \overset{\text{or}}{=} \frac{\sqrt{2}}{8}$$

(c) $\lim_{x\to 3} \dfrac{|x-3|}{9x-x^3}$

$$\lim_{x\to 3^-} \frac{|x-3|}{9x-x^3} = \lim_{x\to 3^-} \frac{-(x-3)}{x(9-x^2)} = \lim_{x\to 3^-} \frac{3-x}{x(3-x)(3+x)} = \lim_{x\to 3^-} \frac{1}{x(\underbrace{3+x}_{6})} = \frac{1}{18}$$

$$\lim_{x\to 3^+} \frac{|x-3|}{9x-x^3} = \lim_{x\to 3^+} \frac{x-3}{x(9-x^2)} = \lim_{x\to 3^+} \frac{x-3}{x(3-x)(3+x)} = \lim_{x\to 3^+} \frac{-1}{x(\underbrace{3+x}_{6})} = -\frac{1}{18}$$

Since the one-sided limits are not equal, $\lim_{x\to 3} \dfrac{|x-3|}{9x-x^3}$ does not exist.

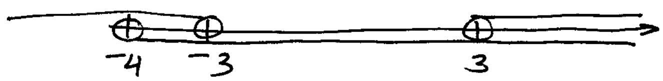
2. (4 + 5 = 9 points)

(a) Find the domain of the function $f(x) = \ln(x^2 - 9) + \dfrac{3}{\sqrt{x+4}}$ in interval form.

We need $x^2 - 9 > 0 \Rightarrow x^2 > 9 \Rightarrow |x| > 3$

$\therefore \quad x < -3$ or $x > 3$

Also, $x + 4 > 0 \Rightarrow x > -4$



$\therefore$ the domain of f is $(-4, -3) \cup (3, \infty)$.

(b) Let $g(x) = \cos[h(\sin x)]$ where h is a differentiable function. Calculate $g'(\pi/6)$ given that $h(\frac{1}{2}) = \frac{\pi}{2}$ and $h'(\frac{1}{2}) = \frac{3}{5}$.

$$g'(x) = -\sin(h(\sin x))\,(h'(\sin x))\,(\cos x)$$

$$g'(\tfrac{\pi}{6}) = -\sin(h(\sin\tfrac{\pi}{6}))\,(h'(\sin\tfrac{\pi}{6}))\,(\cos\tfrac{\pi}{6})$$

$$= -\sin(h(\tfrac{1}{2}))\,(h'(\tfrac{1}{2}))\,(\tfrac{\sqrt{3}}{2})$$

$$= -(\sin\tfrac{\pi}{2})(\tfrac{3}{5})(\tfrac{\sqrt{3}}{2})$$

$$= -1 \cdot \frac{3\sqrt{3}}{10} = -\frac{3\sqrt{3}}{10}$$

3. (8 points) Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x < 3 \\ k + 2b, & x = 3 \\ \sqrt{bx - k} & x > 3. \end{cases}$$

Determine the values of b and k so that the function f is continuous at $x = 3$.

We need $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \quad (*)$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3^-} \frac{(x-3)(x+2)}{x-3} = \lim_{x \to 3^-} (x+2) = 5$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{bx - k} = \sqrt{3b - k}$$

$\therefore$ By $(*)$

$$\sqrt{3b - k} = 5 \Rightarrow 3b - k = 25 \Rightarrow k = 3b - 25$$

Also by $(*)$, $f(3) = 5$

$\therefore \; k + 2b = 5$

$$\Rightarrow 3b - 25 + 2b = 5$$

$$5b = 30$$

$$\boxed{b = 6}$$

$$\Rightarrow k = 3(6) - 25 = 18 - 25 = -7 \quad \therefore \boxed{k = -7}$$

4. ($7 + 4 = 11$ points)

(a) Write down the definition of the derivative as a function and use it to find $f'(x)$ if $f(x) = \frac{3}{x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{(x+h)^2} - \frac{3}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 - 3(x+h)^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{3x^2 - 3x^2 - 6xh - 3h^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}(2x+h)}{(x+h)^2 x^2 \cancel{h}}$$

$$= \lim_{h \to 0} \frac{-3(2x+h)}{(x+h)^2 x^2} = \frac{-3(2x)}{x^2 x^2} = -\frac{6}{x^3}$$

(b) Find an equation of the tangent line to the curve in part (a) at $x = -2$.

The slope of tang. line at $x = -2$ is

$$f'(-2) = \frac{-6}{(-2)^3} = -\frac{6}{-8} = \frac{3}{4}$$

at $x = -2$, $y = \frac{3}{(-2)^2} = \frac{3}{4}$

$\therefore\ y - \frac{3}{4} = \frac{3}{4}(x+2)$ or $y = \frac{3}{4}x + \frac{9}{4}$

5. $(3+3+3+3 = 12$ points) Find y'. Do not simplify your answers.

(a) $y = \left(\dfrac{x^3 - 7\sin x}{3x+5}\right)^{\frac{2}{3}}$

$$y' = \frac{2}{3}\left(\frac{x^3 - 7\sin x}{3x+5}\right)^{-\frac{1}{3}}\left[\frac{(3x^2 - 7\cos x)(3x+5) - (x^3 - 7\sin x)(3)}{(3x+5)^2}\right]$$

(b) $y = \tan^5(6 - \cos^2 x)$

$$y' = \left(5\tan^4(6-\cos^2 x)\right)\left(\sec^2(6-\cos^2 x)\right)(-2\cos x)(-\sin x)$$

(c) $y = x^5 \sin(x^7)$

$$y' = 5x^4 \sin(x^7) + x^5 \cos(x^7)(7x^6)$$

(d) $y = e^{3x^3 - x^2 + 9} + (3 - 4x^2)^{10}$

$$y' = e^{3x^3 - x^2 + 9} \cdot (9x^2 - 2x) + 10(3 - 4x^2)^9(-8x).$$