MATHEMATICS 113/114

MIDTERM EXAMINATION

VERSION 1

DATE: October 24, 2012 TIME: 50 MINUTES

LAST NAME: Solutions

FIRST NAME:

ID#:

Course	Section	Instructor	Please mark with X
Math 113	H1	E. K. Leonard	
Math 114	F1	Y.-S. Wang	

INSTRUCTIONS:

- Show all your work to receive full credit.
- Books, notes, calculators or other electronic aids are not permitted.
- Make sure that the exam has 7 pages in total.

Question	Value	Mark
1	12	
2	10	
3	11	
4	9	
5	8	
Total	50	

1. $(3+3+3+3=12$ points) Find y'. Do not simplify your answers.

(a) $y = x^3 \sin(x^4)$

$$y' = 3x^2 \sin(x^4) + x^3 \cos(x^4)(4x^3)$$

(b) $y = e^{x^3-5x^2+7} + (1-3x^2)^{15}$

$$y' = e^{x^3-5x^2+7} \cdot (3x^2 - 10x) + 15(1-3x^2)^{14}(-6x)$$

(c) $y = \left(\dfrac{x^5 - 5\cos x}{2x+1}\right)^{\frac{1}{3}}$

$$y' = \frac{1}{3}\left(\frac{x^5 - 5\cos x}{2x+1}\right)^{-\frac{2}{3}} \left[\frac{(5x^4 + 5\sin x)(2x+1) - (x^5 - 5\cos x)(2)}{(2x+1)^2}\right]$$

(d) $y = \tan^3(3 - \cos^2 x)$

$$y' = 3\left(\tan^2(3-\cos^2 x)\right)\left(\sec^2(3-\cos^2 x)\right)(-2\cos x)(-\sin x)$$

2. $(3+4+3=10 \text{ points})$ Evaluate the following limits. Show steps. If the limit does not exist, explain why not. No marks will be given if l'Hospital's rule is used.

(a) $\lim_{x\to 1} \frac{\sqrt{x^2+1}-\sqrt{2}}{x^2+2x-3}$

$$= \lim_{x\to 1} \frac{(\sqrt{x^2+1}-\sqrt{2})(\sqrt{x^2+1}+\sqrt{2})}{(x^2+2x-3)(\sqrt{x^2+1}+\sqrt{2})}$$

$$= \lim_{x\to 1} \frac{\overbrace{x^2+1-2}^{x^2-1}}{(x+3)(x-1)(\sqrt{x^2+1}+\sqrt{2})}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x-1)}(\sqrt{x^2+1}+\sqrt{2})} = \frac{\overset{1}{\cancel{2}}}{\underset{2}{\cancel{4}}(\sqrt{2}+\sqrt{2})} = \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$$

or $= \frac{\sqrt{2}}{8}$

(b) $\lim_{x\to 2} \frac{|x-2|}{4x-x^3}$

$$\lim_{x\to 2^-} \frac{|x-2|}{4x-x^3} = \lim_{x\to 2^-} \frac{-(x-2)}{x(4-x^2)} = \lim_{x\to 2^-} \frac{2-x}{x(2-x)(2+x)} = \lim_{x\to 2^-} \frac{1}{x(2+x)}$$

$$= \frac{1}{2(4)} = \frac{1}{8}$$

$$\lim_{x\to 2^+} \frac{|x-2|}{4x-x^3} = \lim_{x\to 2^+} \frac{x-2}{x(4-x^2)} = \lim_{x\to 2^+} \frac{x-2}{x(2-x)(2+x)}$$

$$= \lim_{x\to 2^+} \frac{-1}{x(2+x)} = -\frac{1}{2(4)} = -\frac{1}{8}$$

Since $\lim_{x\to 2^-} \frac{|x-2|}{4x-x^3} \neq \lim_{x\to 2^+} \frac{|x-2|}{4x-x^3}$, $\lim_{x\to 2} \frac{|x-2|}{4x-x^3}$ does not exist.

(c) $\lim_{x\to\infty} \frac{\sqrt{3x^2-4}}{4x+1}$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2-4}{x^2}}}{\frac{4x+1}{x}} = \lim_{x\to\infty} \frac{\sqrt{3-\frac{4}{x^2}}}{4+\frac{1}{x}} = \frac{\sqrt{3-0}}{4+0} = \frac{\sqrt{3}}{4}.$$

3. $(7+4=11 \text{ points})$

(a) Write down the definition of the derivative as a function and use it to find $f'(x)$ if $f(x) = \dfrac{2}{x^2}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 - 2(x+h)^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h\to 0} \frac{2x^2 - 2x^2 - 4xh - 2h^2}{(x+h)^2 x^2 h}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}(2x+h)}{(x+h)^2 x^2 \cancel{h}}$$

$$= \lim_{h\to 0} \frac{-2(2x+h)}{(x+h)^2 x^2}$$

$$= -\frac{2(2x)}{x^2 x^2} = -\frac{4}{x^3}$$

(b) Find an equation of the tangent line to the curve in part (a) at $x = -3$.

The slope of the tang. line at $x=-3$ is

$$f'(-3) = -\frac{4}{(-3)^3} = -\frac{4}{-27} = \frac{4}{27}$$

at $x=-3$, $y = \frac{2}{(-3)^2} = \frac{2}{9}$

$$\therefore\ y - \frac{2}{9} = \frac{4}{27}(x+3) \text{ or } y = \frac{4}{27}x + \frac{2}{3}$$

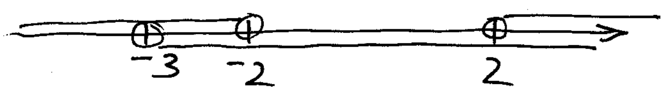
4. ($4+5=9$ points)

(a) Find the domain of the function $h(x) = \ln(x^2-4) + \dfrac{1}{\sqrt{x+3}}$ in <u>interval form</u>.

$$x^2 - 4 > 0 \implies x^2 > 4 \implies |x| > 2$$

$$\therefore\ x < -2 \text{ or } x > 2$$

$$\text{Also, } x+3 > 0 \implies x > -3$$



$$\therefore \text{ the domain of } h \text{ is } (-3,-2) \cup (2,\infty)$$

(b) Let $f(x) = \cos[g(\sin x)]$ where g is a differentiable function. Calculate $f'(\pi/6)$ given that $g(\frac{1}{2}) = \frac{\pi}{2}$ and $g'(\frac{1}{2}) = \frac{5}{8}$.

$$f'(x) = -\sin(g(\sin x))\,(g'(\sin x))\,(\cos x)$$

$$f'(\tfrac{\pi}{6}) = -\sin(g(\sin\tfrac{\pi}{6}))\,(g'(\sin\tfrac{\pi}{6}))\,(\cos\tfrac{\pi}{6})$$

$$= -\sin(g(\tfrac{1}{2}))\,(g'(\tfrac{1}{2}))\,(\tfrac{\sqrt{3}}{2})$$

$$= -(\sin\tfrac{\pi}{2})\,(\tfrac{5}{8})(\tfrac{\sqrt{3}}{2})$$

$$= -1 \cdot \frac{5\sqrt{3}}{16} = -\frac{5\sqrt{3}}{16}.$$

5. (8 points) Let

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x < 2 \\ 4a - 3m, & x = 2 \\ \sqrt{ax - m} & x > 2. \end{cases}$$

Determine the values of a and m so that the function f is continuous at $x = 2$.

We need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+1)}{\cancel{x-2}} = 3.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \sqrt{ax - m} = \sqrt{2a - m}.$$

$\therefore \sqrt{2a - m} = 3 \Rightarrow 2a - m = 9 \Rightarrow m = 2a - 9$

Also $\lim_{x \to 2^-} f(x) = f(2)$

$\therefore 4a - 3m = 3$

$\therefore 4a - 3(2a - 9) = 3$

$-2a = 3 - 27 = -24$

$\therefore a = 12$

$\Rightarrow m = 24 - 9 = 15$