MATH 102 Solutions to Midterm Exam

1. Use Cramer's rule to solve for x_2:

$$\begin{aligned} x_1 + 2x_2 \qquad\quad + 3x_4 &= 0 \\ x_1 + 3x_2 + x_3 + 3x_4 &= 1 \\ -x_1 - 2x_2 + 2x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 - 2x_4 &= 0 \end{aligned}$$

SOLUTION:

Denominator $= \det \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & 3 & 1 & 3 \\ -1 & -2 & 2 & 1 \\ 2 & 2 & 1 & -2 \end{pmatrix}$, $-R_1 + R_2$, $R_1 + R_3$, $-2R_1 + R_4$,

$\det \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -2 & 1 & -8 \end{pmatrix}$

Expand along column 1, get $\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ -2 & 1 & -8 \end{pmatrix}$, $2R_1 + R_3$, get $\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & -8 \end{pmatrix}$

$= \det \begin{pmatrix} 2 & 4 \\ 3 & -8 \end{pmatrix} = -16 - 12 = -28.$

Numerator $= \det \begin{pmatrix} 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 3 \\ -1 & 0 & 2 & 1 \\ 2 & 0 & 1 & -2 \end{pmatrix}$. Expand along column 2, get $\det \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$

$R_1 + R_2, -2R_1 + R_2 \det \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & -8 \end{pmatrix}$. Expand along 1st column, get $\det \begin{pmatrix} 2 & 4 \\ 1 & -8 \end{pmatrix} = -16 - 4 = -20.$

So $x_2 = -20/-28 = 5/7$.

2. SOLUTION: (a)

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & -2 & 3 & 3 & 2 & -2 & 10 \\ 1 & -1 & 1 & 1 & 2 & -2 & 5 \\ 1 & -1 & 1 & 1 & 1 & -1 & 4 \end{pmatrix} \underset{\substack{-R_1+R_3 \\ -R_1+R_4}}{\overset{-2R_1+R_2}{\sim}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 3 & 2 & -2 & 8 \\ 0 & 0 & 1 & 1 & 2 & -2 & 4 \\ 0 & 0 & 1 & 1 & 1 & -1 & 3 \end{pmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 & 4 \\ 0 & 0 & 3 & 3 & 2 & -2 & 8 \\ 0 & 0 & 1 & 1 & 1 & -1 & 3 \end{pmatrix} \underset{-R_2+R_4}{\overset{-3R_2+R_3}{\sim}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

$$\overset{-\frac{1}{4}R_3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix} \underset{R_3+R_4}{\overset{-2R_3+R_2}{\sim}} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, by letting $x_2 = s$, $x_4 = t$ and $x_6 = r$, we get $x_1 = 1 + s$, $x_3 = 2 - t$ and $x_5 = 1 + r$. So the solution is $\mathbf{x} = (1 + s,\ s,\ 2 - t,\ t,\ 1 + r,\ r)$.

SOLUTION: (b)

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & a \\ 1 & -1 & b & 0 \end{pmatrix} \underset{-R_1+R_3}{\overset{-R_1+R_2}{\sim}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & a-1 \\ 0 & -2 & b-1 & -1 \end{pmatrix} \overset{2R_2+R_3}{\sim} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & a-1 \\ 0 & 0 & b+1 & 2a-3 \end{pmatrix}$$

So there is a unique solution if $b \neq -1$ and $a \in \mathbb{R}$ and the system has no solution if $b = -1$ and $a \neq \frac{3}{2}$.

3. Start with a 3×3 matrix A, switch the first two rows, add twice the first row to the third, getting $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find A. Write each of A and A^{-1} as a product of elementary matrices.

SOLUTION: Let $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ correspond to the row-operations of switching the first two rows and adding the first row to the third, respectively. Now, from $E_2 E_1 A = B$, we get

$$A = E_1^{-1} E_2^{-1} \quad B = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ -2 & 0 & -3 \end{pmatrix}$$

where

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

Since

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can write A and A^{-1} as the product of the elementary matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here we used $A^{-1} = B^{-1} E_2 E_1$.

4. a. Determine whether or not the following vectors are linearly independent. If they are dependent, write one of them as a linear combination of the others: $(1,2,3)$, $(3,2,1)$, $(-1,1,2)$, $(1,1,3)$.

SOLUTION: Four vectors in R^3 must be linearly independent, hence $(1,2,3)$, $(3,2,1)$, $(-1,1,2)$, $(1,1,3)$ are linearly dependent. We can express

$$(1,1,3) = a(1,2,3) + b(3,2,1) + c(-1,1,2).$$

Using Gaussian Elimination, we solve $a = 11/4, b = -5/4, \text{and} c = -2$. Therefore

$$(1,1,3) = 11/4(1,2,3) - 5/4(3,2,1) - 2(-1,1,2).$$

b. Suppose that A is a 3×3 matrix, having the property that the system of equations

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

has a unique solution. Show that the columns of A must be linearly independent.

SOLUTION: Since the system has a unique solution, A is invertible (see Theorem 7, p54 in the text), and then $\det A \neq 0$ (see Theorem 2, p96 in the text), which implies that the columns of A must be linearly independent (see Theorem 3, p134 in the text).

5. SOLUTION: Let $\mathbf{v}_1 = (-2,3,2), \mathbf{v}_2 = (3,1,-2)$. Then a normal vector of the plane can be computed by

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 2 \\ 3 & 1 & -2 \end{vmatrix} = (-8, 2, -11).$$

Note that the point $\mathbf{p}_0 = (1,2,-1)$ on the first line must be on the plane. Hence an equation of the plane is

$$-8(x-1) + 2(y-2) - 11(z+1) = 0$$

or

$$-8x + 2y - 11z = 7.$$

6. For each of the following, determine whether or not V is a subspace of R^3;
a. $V = \{(x,y,z) \mid 3x = z \text{ and } y = 2z\}$
b. $V = \{(x,y,z) \mid x = y+1\}$

SOLUTION: a. Take two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, $\mathbf{v}_1 = (x_1, y_1, z_1)$, $\mathbf{v}_2 = (x_2, y_2, z_2)$. Since $\mathbf{v}_1$ is in V then $3x_1 = z_1$ and $y_1 = 2z_1$. Since $\mathbf{v}_2 \in V$, $3x_2 = z_2$ and $y_2 = 2z_2$. We have $\mathbf{v}_1 + \mathbf{v}_2 = (x_1+x_2, y+1+y_2, z_1+z_2)$. For this to be in V, we need $3(x_1+x_2) = z_1+z_2$ and $y_1+y_2 = 2(z_1+z_2)$. But $3x_1 = z_1$ and $3x_2 = z_2$ so $3(x_1+x_2) = 3x_1 + 3x_2 = z_1 + z_2$. Similarly $y_1 + y_2 = 2(z_1+z_2)$. So $\mathbf{v}_1 + \mathbf{v}_2$ is indeed in V. Next, multiply $\mathbf{v}_1$ by a scalar, say t, giving (tx_1, tx_2, tx_3). Since $3x_1 = z_1$, then $3tx_1 = tz_1$; since $y_1 = 2z_1$, then $ty_1 = 2tz_1$. So $t\mathbf{v}_1$ is in V. Therefore V is indeed a subspace of R^3.

SOLUTION: b. Take two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, $\mathbf{v}_1 = (x_1, y_1, z_1)$, $\mathbf{v}_2 = (x_2, y_2, z_2)$. Since $\mathbf{v}_1$ is in V then $x_1 = y_1 + 1$. Since $\mathbf{v}_2 \in V$, $x_2 = y_2 + 1$. We have $\mathbf{v}_1 + \mathbf{v}_2 = (x_1+x_2, y+1+y_2, z_1+z_2)$. For this to be in V, we need $x_1 + x_2 = y_1 + y_2 + 1$; but $x_1 + x_2 = y_1 + y_2 + 2$, so $\mathbf{v}_1 + \mathbf{v}_2$ is not in V, and V is not a subspace of R^3. (It is not hard to see that $t\mathbf{v}_1$ is also not in V if $t \neq 1$.)