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BOUNDEDNESS OF PROJECTION OPERATORS AND CESÀRO MEANS IN WEIGHTED L^p SPACE ON THE UNIT SPHERE

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ABSTRACT. For the weight function $\prod_{i=1}^{d+1}|x_i|^{2\kappa_i}$ on the unit sphere, sharp local estimates of the orthogonal projection operators are obtained and used to prove the convergence of the Cesàro (C,δ) means in the weighted L^p space for δ above the critical index. Similar results are also proved for corresponding weight functions on the unit ball and on the simplex.

1. Introduction

For spherical harmonic expansions on the unit sphere $S^d := \{(x_1,\cdots,x_{d+1}) : (x_1^2+\cdots+x_{d+1}^2)^{\frac12}=1\}$ of $\mathbb{R}^{d+1}$, it is well known that their Cesàro (C,δ) means are uniformly bounded, in terms of degrees, in the L^p norm for all $1\le p\le\infty$ if and only if $\delta\ge\frac{d-1}{2}$ ([3]). For δ below the critical index $\frac{d-1}{2}$, C. Sogge [14] proved a much deeper result that the (C,δ) means are uniformly bounded on $L^p(S^d)$ if

$$|\tfrac12-\tfrac1p|\ge\tfrac{1}{d+1}\quad\text{and}\quad\delta>\delta(p):=\max\{d|\tfrac1p-\tfrac12|-\tfrac12,0\}\tag{1.1}$$

for $d\ge 2$ and, moreover, the condition $|1/2-1/p|\ge 1/(d+1)$ is not needed in the case of $d=2$. The condition $\delta>\delta(p)$ is also known to be necessary ([3]). Later in [15], Sogge proved that the condition (1.1) ensures the boundededness of the Riesz means of eigenfunction expansions associated to the second-order elliptic differential operators on compact connected C^∞ manifolds of dimension d.

The purpose of the present paper is to establish analogous results for the Cesàro means of orthogonal expansions associated with the weight function $h_\kappa^2(x)$, where

$$h_\kappa(x):=\prod_{i=1}^{d+1}|x_i|^{\kappa_i},\quad\kappa:=(\kappa_1,\cdots,\kappa_{d+1}),\quad\min_{1\le i\le d+1}\kappa_i\ge 0,\tag{1.2}$$

on the unit sphere S^d, as well as for orthogonal expansions for related weight functions (see (2.6) and (2.12) below) on the unit ball and on the simplex.

The function h_κ in (1.2) is invariant under the group $\mathbb{Z}_2^d$ and it is the simplest example of weight functions invariant under reflection groups studied first by Dunkl [9]. Homogeneous polynomials that are orthogonal with respect to h_κ^2 on S^d are called h-harmonics and their restrictions on S^d are eigenfunctions of a second-order

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differential-difference operator, which plays the role of the ordinary Laplacian. For the theory of h-harmonics, we refer to [10] and the references therein. A brief account of what is needed in this paper is given in the following section.

The convergence of h-harmonic expansions has been studied recently. In [19] it was proved that Cesàro (C,δ) means converge uniformly if $\delta > |\kappa| + \frac{d-1}{2}$, where $|\kappa| = \kappa_1 + \ldots + \kappa_{d+1}$, and such a result holds for all other weight functions invariant under reflection groups. In the case of h_κ in (1.2) the critical index for the (C,δ) means in the uniform norm turned out to be ([11])

$$\delta > \sigma_\kappa := \tfrac{d-1}{2} + |\kappa| - \min_{1\le i\le d+1} \kappa_i. \tag{1.3}$$

This condition is also necessary for the almost everywhere convergence of the (C,δ) means ([21]). Our main result in this paper (see Theorem 3.1 in Section 3) shows that for h_κ in (1.2), the (C,δ) means of h-harmonic expansions converge in the $L^p(h_\kappa^2; S^d)$ norm if

$$|\tfrac{1}{2} - \tfrac{1}{p}| \ge \tfrac{1}{2\sigma_\kappa+2} \quad \text{and} \quad \delta > \delta_\kappa(p) := \max\{(2\sigma_\kappa+1)|\tfrac{1}{p} - \tfrac{1}{2}| - \tfrac{1}{2}, 0\} \tag{1.4}$$

and that the condition $\delta > \delta_\kappa(p)$ is also necessary. Note that (1.4) agrees with (1.1) when $\kappa = 0$, while $\delta_\kappa(p) > \delta(p)$ when $\kappa > 0$.

The reason that these sharp results can be established for h_κ in (1.2) lies in an explicit formula for the kernel $P_n(h_\kappa^2; \cdot, \cdot)$ of the orthogonal projection operator (definition in the next section), while no explicit formula for the kernel is known for other reflection invariant weight functions. For (1.2), we have

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n+\lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^{d+1}} C_n^{\lambda_\kappa}(u(x,y,t)) \prod_{i=1}^{d+1} (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \tag{1.5}$$

where C_n^λ is the Gegenbauer polynomial of degree n,

$$\lambda_\kappa := \tfrac{d-1}{2} + |\kappa|, \quad \text{and} \quad u(x,y,t) = x_1 y_1 t_1 + \ldots + x_{d+1} y_{d+1} t_{d+1}, \tag{1.6}$$

and c_κ is the normalization constant of the weight function $\prod_{i=1}^{d+1}(1+t_i)(1-t_i^2)^{\kappa_i-1}$. If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda\to 0} c_\lambda \int_{-1}^{1} g(t)(1+t)(1-t)^{\lambda-1} dt = g(1).$$

For the spherical harmonic expansions, this kernel is the familiar $P_n(x,y) := \frac{n+\lambda}{\lambda} C_n^\lambda(\langle x, y\rangle)$ with $\lambda = \frac{d-1}{2}$ (cf. [16]), which is also called a zonal harmonic.

The simple structure of the zonal harmonics means that one can derive various properties and estimates relatively easily. The structure of the kernel $P_n(h_\kappa^2; x, y)$ in (1.5) is far more complicated, making the derivation of any information from it more difficult. There is, however, a deeper reason that the study of h-harmonic expansion is more difficult than that of ordinary spherical harmonics. The zonal harmonics are invariant under the rotation group $O(d+1)$ in the sense that $P_n(x,y) = P_n(xg, yg)$ for all $g \in O(d+1)$, which reflects the fact that the sphere is a homogeneous space. The $P_n(h_\kappa^2; x, y)$ in (1.5) is invariant under $\mathbb{Z}_2^{d+1}$, a subgroup of $O(d+1)$, and we are in fact working with a weighted sphere that has singularity on the largest circles of the coordinate planes. In particular, we can no longer treat the sphere as a homogeneous space and many of our estimates of various kernels have to be local, depending on the location of the points.

The difficulty manifests itself acutely in the study of the L^p boundedness of Cesàro means of h-harmonic expansions. For the ordinary spherical harmonics, the proof of Sogge [14] relies on the sharp asymptotic bounds for the (L^p, L^2) norms of the orthogonal projection operators, and a result of Bonami-Clerc [3], which says that the sharp results for Cesàro summation on L^p can be deduced from these asymptotic estimates of orthogonal projections. (See also Sogge [15] for the case of general compact manifolds.) For our study, while we can obtain global sharp asymptotic bounds for the $L^p(h_\kappa^2; S^d) \to L^2(h_\kappa^2; S^d)$ norms of the orthogonal projection operators of h-harmonic expansions (see Theorem 3.3 in Section 3), which are in full analogy with those of Sogge [14] for ordinary spherical harmonics, seemingly, these global estimates are not enough for the proof of the uniform boundedness of Cesàro means on weighted L^p. (See Section 3.1 for the precise definition of $L^p(h_\kappa^2; S^d)$.) In order to obtain our main result on the boundedness of Cesàro means, we have to replace the norm of the orthogonal projection operator by a local estimate of the projection operator over a spherical cap. (See Theorem 3.4 in Section 3.) The latter local result is substantially more difficult to establish, since only a part of the proof can follow Sogge's strategy based on Stein's theorem on analytic interpolation and the rest has to rely on a sharp pointwise local estimate of the kernels.

Analogues of our main results also hold for orthogonal expansions on the unit ball and on the simplex for weight functions related to h_κ^2, including in particular the Lebesgue measure (see Section 2). In fact, they follow more or less from the results for h-harmonics. In particular, the same condition (1.3) guarantees the convergence of the Cesàro means in the corresponding weighted L^p space.

The paper is organized as follows: The next section contains preliminary results, whereas the main results are stated and discussed in Section 3. The local estimate of the projection operator is studied in Section 4. The proof of the main result for the projection operators on the sphere is given in Section 5, while the proof of the main result for the Cesàro means on the sphere is presented in Section 6. Finally, the results on the ball and on the simplex are proved in Section 7.

2. Preliminary results

2.1. **h-spherical harmonics.** We restrict our discussion to h_κ in (1.2). Unless otherwise stated, the main reference for the material in this section is [10]. An h-harmonic is a homogeneous polynomial P that satisfies the equation $\Delta_h P = 0$, where $\Delta_h := \mathcal{D}_1^2 + \ldots + \mathcal{D}_{d+1}^2$ and

$$\mathcal{D}_i f(x) := \partial_i f(x) + \kappa_i \frac{f(x) - f(x - 2x_i e_i)}{x_i}, \quad 1 \le i \le d+1,$$

$e_1, \cdots, e_{d+1}$ denoting the usual coordinate vectors in $\mathbb{R}^{d+1}$. The differential-difference operators $\mathcal{D}_1, \ldots, \mathcal{D}_{d+1}$ are the Dunkl operators, which commute. An h-harmonic is an orthogonal polynomial with respect to the weight function $h_\kappa^2(x)$ on S^d. Its restriction on the sphere is called a spherical h-harmonic. Let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of spherical h-harmonics of degree n on S^d. It is known that $\dim \mathcal{H}_n^d(h_\kappa^2) = \binom{n+d+1}{n} - \binom{n+d-1}{n-2}$. Let $L^2(h_k^2; S^d)$ denote the L^2 space with respect to h_k^2 on S^d; see Section 3.1 below for the precise definition. The Hilbert space theory shows that

$$L^2(h_k^2; S^d) = \sum_{n=0}^{\infty} \mathcal{H}_n^d(h_\kappa^2) : \qquad f = \sum_{n=0}^{\infty} \operatorname{proj}_n(h_\kappa^2; f),$$

where $\mathrm{proj}_n(h_\kappa^2) : L^2(h_\kappa^2; S^d) \mapsto \mathcal{H}_n^d(h_\kappa^2)$ is the projection operator, which can be written as an integral operator

$$\mathrm{proj}_n(h_\kappa^2; f, x) = a_\kappa \int_{S^d} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y), \quad x \in S^d,$$

where $d\omega(y)$ denotes the usual Lebesgue measure on S^d, a_κ is the normalization constant, $a_\kappa^{-1} = \int_{S^d} h_\kappa^2(y) d\omega(y)$ and $P_n(h_\kappa^2; \cdot, \cdot)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$. The kernel satisfies an explicit formula

$$P_n(h_\kappa^2; x, y) = \frac{n + \lambda_\kappa}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_\kappa}(\langle \cdot, y \rangle) \right](x), \qquad \lambda_\kappa = \frac{d-1}{2} + |\kappa|, \tag{2.1}$$

where C_n^λ is the Gegenbauer polynomial of degree n and V_κ is the so-called intertwining operator defined by

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^{d+1}} f(x_1 t_1, \ldots, x_{d+1} t_{d+1}) \prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt, \tag{2.2}$$

in which c_κ is a constant such that $V_\kappa 1 = 1$. If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 g(t)(1+t)(1-t)^{\lambda - 1} dt = g(1).$$

Clearly (2.1) is the same as (1.5). The operator V_κ is called an intertwining operator since it satisfies $\mathcal{D}_j V_\kappa = V_\kappa \partial_j$, $1 \le j \le d+1$.

Let $w_\lambda(t) := (1 - t^2)^{\lambda - 1/2}$ on $[-1, 1]$. The Gegenbauer polynomials are orthogonal with respect to w_λ. The intertwining operator can be used to define a convolution $f *_\kappa g$ for $f \in L^1(h_\kappa^2; S^d)$ and $g \in L^1(w_{\lambda_\kappa}; [-1, 1])$ ([20]):

$$f *_\kappa g(x) := a_\kappa \int_{S^d} f(y) V_\kappa [g(\langle x, \cdot \rangle)](y) h_\kappa^2(y) d\omega(y). \tag{2.3}$$

In particular, the projection operator $\mathrm{proj}_n(h_\kappa^2; f)$ can be written as

$$\mathrm{proj}_n(h_\kappa^2; f) = f *_\kappa Z_n^\kappa, \quad \text{where} \quad Z_n^\kappa(t) := \frac{n + \lambda_\kappa}{\lambda_k} C_n^{\lambda_k}(t). \tag{2.4}$$

This convolution satisfies the usual Young's inequality (see [20, p.6, Proposition 2.2]). For $\kappa = 0$, $V_\kappa = id$, it becomes the classical convolution on the sphere ([5]). For $f \in L^1(h_\kappa^2; S^d)$, we also have ([20])

$$\mathrm{proj}_n(h_\kappa^2; f *_\kappa g) = b_{\lambda_\kappa} \int_0^\pi \frac{C_n^{\lambda_\kappa}(\cos\theta)}{C_n^{\lambda_\kappa}(1)} g(\cos\theta)(\sin\theta)^{2\lambda_\kappa} d\theta \; \mathrm{proj}_n(h_\kappa^2; f), \tag{2.5}$$

where b_{λ_κ} is the normalization constant of $w_{\lambda_\kappa}(t)$ on $[-1, 1]$.

For $\delta > -1$, the Cesàro (C, δ) means of the h-harmonic expansion is defined by

$$S_n^\delta(h_\kappa^2; f, x) := (A_n^\delta)^{-1} \sum_{j=0}^n A_{n-j}^\delta \, \mathrm{proj}_j(h_\kappa^2; f, x), \qquad A_{n-j}^\delta = \binom{n - j + \delta}{n - j}.$$

The operator $S_n^\delta(h_\kappa^2)$ can be written as a convolution,

$$S_n^\delta(h_\kappa^2; f) = f *_\kappa K_n^\delta(w_{\lambda_\kappa}), \qquad K_n^\delta(w_{\lambda_\kappa}; t) := (A_n^\delta)^{-1} \sum_{j=0}^n A_{n-j}^\delta Z_j^\kappa(t).$$

Let $K_n^\delta(h_\kappa^2; x, y)$ denote the kernel of $S_n^\delta(h_\kappa^2)$; then

$$K_n^\delta(h_\kappa^2; x, y) = V_\kappa \left[K_n(w_{\lambda_\kappa}; \langle x, \cdot\rangle)\right](y).$$

2.2. **Orthogonal expansions on the unit ball.** We denote the usual Euclidean norm of $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ by $\|x\| := (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$. The weight functions we consider on the unit ball $B^d = \{x : \|x\| \le 1\} \subset \mathbb{R}^d$ are defined by

$$W_\kappa^B(x) := \prod_{i=1}^d |x_i|^{\kappa_i}(1 - \|x\|^2)^{\kappa_{d+1}-1/2}, \qquad \kappa_i \ge 0, \quad x \in B^d, \tag{2.6}$$

which is related to the h_κ in (1.2) by $h_\kappa^2(x, \sqrt{1 - \|x\|^2}) = W_\kappa^B(x)/\sqrt{1 - \|x\|^2}$, in which $1/\sqrt{1 - \|x\|^2}$ comes from the Jacobian of changing variables

$$\phi : x \in B^d \mapsto (x, \sqrt{1 - \|x\|^2}) \in S_+^d := \{y \in S^d : y_{d+1} \ge 0\}. \tag{2.7}$$

Furthermore, under the above changing variables, we have

$$\int_{S^d} g(y) d\omega(y) = \int_{B^d} \left[g(x, \sqrt{1 - \|x\|^2}\,) + g(x, -\sqrt{1 - \|x\|^2}\,)\right] \frac{dx}{\sqrt{1 - \|x\|^2}}. \tag{2.8}$$

The orthogonal structure is preserved under the mapping (2.7) and the study of orthogonal expansions for W_κ^B can be essentially reduced to that of h_κ^2. In fact, let $\mathcal{V}_n^d(W_\kappa^B)$ denote the space of orthogonal polynomials of degree n with respect to W_κ^B on B^d. The orthogonal projection, $\operatorname{proj}_n(W_\kappa^B; f)$, of $f \in L^2(W_\kappa^B; B^d)$ onto $\mathcal{V}_n^d(W_\kappa^B)$ can be expressed in terms of the orthogonal projection of $F(x, x_{d+1}) := f(x)$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$:

$$\operatorname{proj}_n(W_\kappa^B; f, x) = \operatorname{proj}_n(h_\kappa^2; F, X), \qquad \text{with} \quad X := (x, \sqrt{1 - \|x\|^2}). \tag{2.9}$$

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to W_κ^B from that of h-harmonic expansions.

For $d = 1$ the weight W_κ^B in (2.6) becomes the weight function

$$w_{\kappa_2,\kappa_1}(t) = |t|^{2\kappa_1}(1 - t^2)^{\kappa_2 - 1/2}, \qquad \kappa_i \ge 0, \quad t \in [-1, 1], \tag{2.10}$$

whose corresponding orthogonal polynomials, $C_n^{(\kappa_1,\kappa_2)}$, are called generalized Gegenbauer polynomials, and they can be expressed in terms of Jacobi polynomials,

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2-1), \end{aligned} \tag{2.11}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

2.3. **Orthogonal expansions on the simplex.** The weight functions we consider on the simplex $T^d = \{x : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |x| \ge 0\}$ are defined by

$$W_\kappa^T(x) := \prod_{i=1}^d x_i^{\kappa_i - 1/2}(1 - |x|)^{\kappa_{d+1}-1/2}, \qquad \kappa_i \ge 0, \tag{2.12}$$

where $|x| := x_1 + \cdots + x_d$. They are related to W_κ^B, hence to h_κ^2. In fact, W_κ^T is exactly the product of the weight function W_κ^B under the mapping

$$\psi : (x_1, \ldots, x_d) \in T^d \mapsto (x_1^2, \ldots, x_d^2) \in B^d \tag{2.13}$$

and the Jacobian of this change of variables. Furthermore, the change of variables shows

$$\int_{B^d} g(x_1^2,\ldots,x_d^2)dx = \int_{T^d} g(x_1,\ldots,x_d)\frac{dx}{\sqrt{x_1\cdots x_d}}. \tag{2.14}$$

The orthogonal structure is preserved under the mapping (2.13). Let $\mathcal{V}_n^d(W_\kappa^T)$ denote the space of orthogonal polynomials of degree n with respect to W_κ^T on T^d. Then $R \in \mathcal{V}_n^d(W_\kappa^T)$ if and only if $R\circ\psi \in \mathcal{V}_{2n}^d(W_\kappa^B)$. The orthogonal projection, $\operatorname{proj}_n(W_\kappa^T;f)$, of $f\in L^2(W_\kappa^T;T^d)$ onto $\mathcal{V}_n^d(W_\kappa^T)$ can be expressed in terms of the orthogonal projection of $f\circ\psi$ onto $\mathcal{V}_{2n}^d(W_\kappa^B)$:

$$\left(\operatorname{proj}_n(W_\kappa^T;f)\circ\psi\right)(x) = \frac{1}{2^d}\sum_{\varepsilon\in\mathbb{Z}_2^d}\operatorname{proj}_{2n}(W_\kappa^B;f\circ\psi,x\varepsilon). \tag{2.15}$$

The fact that $\operatorname{proj}_n(W_\kappa^T)$ of degree n is related to $\operatorname{proj}_{2n}(W_\kappa^B)$ of degree $2n$ suggests that some properties of the orthogonal expansions on B^d cannot be transformed directly to those on T^d. We will also need the explicit formula for the kernel, $P_n(W_\kappa^T;x,y)$, of $\operatorname{proj}_n(W_\kappa^T;f)$, which can be derived from (2.1) and the quadratic transform between Gegenbauer and Jacobi polynomials,

$$\begin{aligned} P_n(W_\kappa^T;x,y) &= \frac{(2n+\lambda_\kappa)\Gamma(\frac12)\Gamma(n+\lambda_k)}{\Gamma(\lambda_\kappa+1)\Gamma(n+\frac12)} \\ &\quad\times c_\kappa\int_{[-1,1]^{d+1}} P_n^{(\lambda_k-\frac12,-\frac12)}\left(2z(x,y,t)^2-1\right)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}dt, \end{aligned} \tag{2.16}$$

where $z(x,y,t)=\sqrt{x_1y_1}\,t_1+\ldots+\sqrt{x_dy_d}\,t_d+\sqrt{1-|x|}\sqrt{1-|y|}\,t_{d+1}$.

We will also denote the Cesàro means for orthogonal expansions with respect to a weight function W as $S_n^\delta(W;f)$ and denote their kernel as $K_n^\delta(W;x,y)$, where W is either W_κ^B or W_κ^T.

2.4. **Some estimates.** Throughout this paper we denote by c a generic constant that may depend on fixed parameters such as κ, d and p, whose value may change from line to line. Furthermore we write $A\sim B$ if $A\ge cB$ and $B\ge cA$.

Let $d(x,y) := \arccos\langle x,y\rangle$ denote the geodesic distance of $x,y\in S^d$. For $0\le\theta\le\pi$, the set

$$c(x,\theta) := \{y\in S^d : d(x,y)\le\theta\} = \{y\in S^d : \langle x,y\rangle\ge\cos\theta\}$$

is called the spherical cap centered at x with radius θ. It is shown in [7] that h_κ is a doubling weight and, furthermore, the following estimate holds:

Lemma 2.1. *For $0\le\theta\le\pi$ and $x=(x_1,\cdots,x_{d+1})\in S^d$,*

$$\int_{c(x,\theta)} h_\kappa^2(y)d\omega(y) \sim \theta^d\prod_{j=1}^{d+1}(|x_j|+\theta)^{2\kappa_j}, \tag{2.17}$$

where the constant of equivalence depends only on d and κ.

We refer to the remarkable paper [12] of Mastroianni and Totik for various polynomial inequalities with doubling weights.

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal with respect to the weight

$$w^{(\alpha,\beta)}(t) := (1-t)^\alpha(1+t)^\beta, \qquad t\in[-1,1].$$

We will need the following estimate from [17, p. 169]:

Lemma 2.2. *For $\alpha \ge \beta$ and $t \in [0,1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \le c n^{-1/2}(1-t+n^{-2})^{-(\alpha+1/2)/2}. \tag{2.18}$$

The estimate on $[-1,0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n P_n^{(\beta,\alpha)}(-t)$.

We will also need the estimate of the L^p norm for the Jacobi polynomials ([17, p. 391]: for $\alpha,\beta,\mu > -1$ and $p > 0$,

$$\int_0^1 \left|P_n^{(\alpha,\beta)}(t)\right|^p (1-t)^\mu dt \sim \begin{cases} n^{\alpha p-2\mu-2}, & p > p_{\alpha,\mu}, \\ n^{-\frac{p}{2}}\log n, & p = p_{\alpha,\mu}, \\ n^{-\frac{p}{2}}, & p < p_{\alpha,\mu}, \end{cases} \qquad p_{\alpha,\mu} := \frac{2\mu+2}{\alpha+\frac{1}{2}}. \tag{2.19}$$

Recall that $|\kappa| = \kappa_1 + \cdots + \kappa_{d+1}$ and V_κ is defined in (2.2). The following lemma was proved in [8, Theorem 3.1].

Lemma 2.3. *Assume $\alpha \ge \max\{\beta, |\kappa| - \frac{1}{2}\}$. Then for $x, y \in S^d$,*

$$\left| \int_{[-1,1]^{d+1}} P_n^{(\alpha,\beta)}(x_1y_1t_1 + \ldots + x_{d+1}y_{d+1}t_{d+1}) \prod_{j=1}^{d+1}(1+t_j)(1-t_j^2)^{\kappa_j-1}dt \right| \tag{2.20}$$
$$\le c n^{\alpha-2|\kappa|} \frac{\prod_{j=1}^{d+1}(|x_jy_j| + n^{-1}\|\bar{x}-\bar{y}\| + n^{-2})^{-\kappa_j}}{(1+nd(\bar{x},\bar{y}))^{\alpha+\frac{1}{2}-|\kappa|}},$$

where and throughout, $\bar{z} = (|z_1|, \ldots, |z_{d+1}|)$ for $z = (z_1, \cdots, z_{d+1}) \in S^d$.

This lemma plays an essential role in the proof of a sharp pointwise estimate for the kernel $K_n^\delta(h_\kappa^2; f)$ in [8]. For the present paper we will only need the pointwise estimate for $P_n(h_\kappa^2; x, y)$:

Lemma 2.4. *Let $x, y \in S^d$. Then*

$$|P_n(h_\kappa^2; x, y)| \le c \frac{\prod_{j=1}^{d+1}(|x_jy_j| + n^{-1}\|\bar{x}-\bar{y}\| + n^{-2})^{-\kappa_j}}{n^{-(d-1)/2}(\|\bar{x}-\bar{y}\| + n^{-1})^{(d-1)/2}}. \tag{2.21}$$

The kernel $P_n(W_\kappa^B)$ can be derived from (2.21) and will not be needed. We will need, however, the estimate for the kernel $P_n(W_\kappa^T)$, which is also proved in [8].

Lemma 2.5. *For $x = (x_1, \ldots, x_d) \in T^d$ and $y = (y_1, \ldots, y_d) \in T^d$,*

$$|P_n(W_\kappa^T; x, y)| \le c \frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j} + n^{-1}\|\xi-\zeta\| + n^{-2})^{-\kappa_j}}{n^{-(d-1)/2}(\|\xi-\zeta\| + n^{-1})^{(d-1)/2}}, \tag{2.22}$$

where $\xi := (\sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{x_{d+1}})$, $\zeta := (\sqrt{y_1}, \ldots, \sqrt{y_d}, \sqrt{y_{d+1}})$ with $x_{d+1} := 1-|x|$ and $y_{d+1} := 1-|y|$.

3. Main results

3.1. ***h*-harmonic expansions.** For h_κ defined in (1.2), we denote the L^p norm of $L^p(h_\kappa^2; S^d)$ by $\|\cdot\|_{\kappa,p}$,

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{S^d} |f(y)|^p h_\kappa^2(y) d\omega(y)\right)^{1/p}$$

for $1 \le p < \infty$ and with the usual understanding that it is the uniform norm on S^d when $p = \infty$. Recall that

$$\sigma_\kappa := \tfrac{d-1}{2} + |\kappa| - \kappa_{\min} \quad \text{with} \quad \kappa_{\min} := \min_{1\le j\le d+1} \kappa_j.$$

Our main results on the Cesàro summation of h-harmonic expansions are the following two theorems:

Theorem 3.1. *Suppose that $f \in L^p(h_\kappa^2; S^d)$, $1 \le p \le \infty$, $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\sigma_\kappa+2}$ and*

$$\delta > \delta_\kappa(p) := \max\{(2\sigma_\kappa+1)|\tfrac{1}{p} - \tfrac{1}{2}| - \tfrac{1}{2}, 0\}. \tag{3.1}$$

Then $S_n^\delta(h_\kappa^2; f)$ converges to f in $L^p(h_\kappa^2; S^d)$ and

$$\sup_{n\in\mathbb{N}} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Theorem 3.2. *Assume $1 \le p \le \infty$ and $0 < \delta \le \delta_\kappa(p)$. Then there exists a function $f \in L^p(h_\kappa^2; S^d)$ such that $S_n^\delta(h_\kappa^2; f)$ diverges in $L^p(h_\kappa^2; S^d)$.*

For $\kappa = 0$, $h_\kappa(x) \equiv 1$ and the spherical h-harmonic becomes the ordinary spherical harmonics. Hence Theorem 3.1 is the complete analogue of the Sogge theorem, while Theorem 3.2 is the analogue of [3, Theorem 5.2] for spherical harmonics.

For the projection operator $\operatorname{proj}_n(h_\kappa^2; f)$ we have the following theorem, which is a complete analogue of a theorem due to Sogge [14] for spherical harmonics.

Theorem 3.3. *Let $d \ge 2$ and $n \in \mathbb{N}$. Then*

(i) *for $1 \le p \le \frac{2(\sigma_k+1)}{\sigma_k+2}$,*

$$\left\|\operatorname{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le cn^{\delta_\kappa(p)}\|f\|_{\kappa,p},$$

with $\delta_k(p)$ given in (3.1);

(ii) *for $\frac{2(\sigma_k+1)}{\sigma_k+2} \le p \le 2$,*

$$\left\|\operatorname{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le cn^{\sigma_\kappa(\frac{1}{p}-\frac{1}{2})}\|f\|_{\kappa,p}.$$

Furthermore, the estimate (i) is sharp in terms of the order of n.

The estimate in (ii) is sharp if $\kappa = 0$ as shown in [14]. We expect that it is also sharp for $\kappa \ne 0$ but could not prove it at this moment. For further discussion on this point, see Remark 5.1 in Section 5.

For the spherical harmonics, the above theorem is enough for the proof of the boundedness of the Cesàro means. (See [3] and [15].) For h-harmonics, however, a stronger result is needed since $\delta_\kappa(p) > \delta(p) := \max\{d|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$.

Theorem 3.4. *Suppose that $1 \le p \le \frac{2\sigma_\kappa+2}{\sigma_k+2}$ and that f is supported in a spherical cap $c(\varpi, \theta)$ with $\theta \in (n^{-1}, \pi]$ and $\varpi \in S^d$. Then*

$$\left\|\operatorname{proj}_n(h_\kappa^2; f)\right\|_{\kappa,2} \le cn^{\delta_\kappa(p)}\theta^{\delta_\kappa(p)+\frac{1}{2}} \left[\int_{c(\varpi,\theta)} h_\kappa^2(x)\, d\omega(x)\right]^{\frac{1}{2}-\frac{1}{p}} \|f\|_{\kappa,p}.$$

The above theorems on the projection operators will be proved in Section 5 and the theorems on Cesàro means will be proved in Section 6.

3.2. **Orthogonal expansions on the ball and on the simplex.** Let Ω^d stand for either B^d or T^d and W_κ^Ω stand for either W_κ^B or W_κ^T, respectively. We denote the L^p norm of $L^p(W_\kappa^\Omega;\Omega^d)$ by $\|\cdot\|_{W_\kappa^\Omega,p}$,

$$\|f\|_{W_\kappa^\Omega,p} := \left(a_\kappa^\Omega \int_{\Omega^d} |f(y)|^p W_\kappa^\Omega(y)dy\right)^{1/p}$$

for $1 \le p < \infty$ and with the usual understanding that it becomes the uniform norm on Ω^d when $p = \infty$.

Our main results on the Cesàro summation of orthogonal expansions on B^d and T^d are the following two theorems:

Theorem 3.5. *Suppose that $f \in L^p(W_\kappa^\Omega;\Omega^d)$, $1 \le p \le \infty$, $|\frac{1}{p}-\frac{1}{2}| \ge \frac{1}{2\sigma_\kappa+2}$ and*

$$\delta > \delta_\kappa(p) := \max\{(2\sigma_\kappa+1)|\tfrac{1}{p}-\tfrac{1}{2}| - \tfrac{1}{2}, 0\}.$$

Then $S_n^\delta(W_\kappa^\Omega;f)$ converges to f in $L^p(W_\kappa^\Omega;\Omega^d)$ and

$$\sup_{n\in\mathbb{N}} \|S_n^\delta(W_\kappa^\Omega;f)\|_{W_\kappa^\Omega,p} \le c\|f\|_{W_\kappa^\Omega,p}.$$

Theorem 3.6. *Assume $1 \le p \le \infty$ and $0 < \delta \le \delta_\kappa(p)$. Then there exists a function $f \in L^p(W_\kappa^\Omega;\Omega^d)$ such that $S_n^\delta(W_\kappa^\Omega;f)$ diverges in $L^p(W_\kappa^\Omega;\Omega^d)$.*

For $d=1$ and $\Omega = T^1 = [0,1]$, these theorems become results for the Jacobi polynomial expansions ([6]). For $d = 1$ and $\Omega = B^1 = [-1,1]$, these theorems become results for the generalized Gegenbauer polynomial expansions with respect to w_{κ_1,κ_2} in (2.10), which appear to be new if $\kappa_1 \ne 0$ while the case $\kappa = 0$ corresponds to the Gegenbauer polynomial expansions [2, 3]. We state the result as follows:

Corollary 3.7. *Suppose that $1 \le p \le \infty$, $|\frac{1}{p}-\frac{1}{2}| \ge \frac{1}{2\max\{\lambda,\mu\}+2}$ and*

$$\delta > \delta(p) := \max\{(2\max\{\lambda,\mu\}+1)|\tfrac{1}{p}-\tfrac{1}{2}| - \tfrac{1}{2}, 0\}.$$

Then $S_n^\delta(w_{\lambda,\mu};f)$ converges to f in $L^p(w_{\lambda,\mu};[-1,1])$ and

$$\sup_{n\in\mathbb{N}} \|S_n^\delta(w_{\lambda,\mu};f)\|_{w_{\lambda,\mu},p} \le c\|f\|_{w_{\lambda,\mu},p}.$$

Furthermore, the condition $\delta > \delta(p)$ is sharp.

The result analogous to Theorem 3.3 also holds for the projection operator.

Theorem 3.8. *Let $d \ge 2$ and $n \in \mathbb{N}$. Then*

(i) *for $1 \le p \le \frac{2(\sigma_k+1)}{\sigma_k+2}$,*

$$\left\|\mathrm{proj}_n(W_\kappa^\Omega;f)\right\|_{W_\kappa^\Omega,2} \le cn^{\delta_\kappa(p)}\|f\|_{W_\kappa^\Omega,p};$$

(ii) *for $\frac{2(\sigma_k+1)}{\sigma_k+2} \le p \le 2$,*

$$\left\|\mathrm{proj}_n(W_\kappa^\Omega;f)\right\|_{W_\kappa^\Omega,2} \le cn^{\sigma_k(\frac{1}{p}-\frac{1}{2})}\|f\|_{W_\kappa^\Omega,p}.$$

Furthermore, the estimate in (i) is sharp.

An analogue of Theorem 3.4 also holds. We state only the one for W_κ^T for which we define a distance on T^d,

$$d_T(x,y) := \arccos(\sqrt{x_1y_1} + \ldots + \sqrt{x_dy_d} + \sqrt{1-|x|}\sqrt{1-|y|}),$$

where $|z| = |z_1| + \cdots + |z_d|$ for $z \in \mathbb{R}^d$. The analogue of the spherical cap on T^d is defined as $c_T(x,\theta) := \{y \in T^d : d_T(x,y) \le \theta\}$.

Theorem 3.9. *Suppose* $1 \le p \le \frac{2\sigma_\kappa+2}{\sigma_k+2}$ *and* f *is supported in the set* $c_T(x,\theta)$ *with* $\theta \in (n^{-1}, \pi]$ *and* $x \in T^d$. *Then*

$$\left\|\mathrm{proj}_n(W_\kappa^T; f)\right\|_{W_\kappa^T,2} \le c n^{\delta_\kappa(p)} \theta^{\delta_\kappa(p)+\frac12} \left[\int_{c_T(x,\theta)} W_\kappa^T(y)\,dy\right]^{\frac12-\frac1p} \|f\|_{W_\kappa^T,p}.$$

The analogous result for B^d holds with $c_T(x,\theta)$ replaced by $c_B(x,\theta)$ defined in terms of $d_B(x,y) := \arccos(\langle x,y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2})$.

These results will be proved in Section 7.

4. Local estimate of projection operator

The main effort in the proof of Theorem 3.4, given in the next section, lies in proving the following local estimate of the projection operator.

Theorem 4.1. *Let* $\nu := \frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$ *and* $\nu' := \frac{\nu}{\nu-1}$. *Let* f *be a function supported in a spherical cap* $c(\varpi,\theta)$ *with* $\theta \in (n^{-1}, 1/(8d)]$ *and* $\varpi \in S^d$. *Then*

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,\nu'} \le c n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} \theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}} \left[\int_{c(\varpi,\theta)} h_\kappa^2(x) d\omega(x)\right]^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu}.$$

Here χ_E denotes the characteristic function of the set E. Note the norm of the left-hand side is taken over $c(\varpi,\theta)$, so that the above estimate is a local one.

Throughout this section, we shall fix the spherical cap $c(\varpi,\theta)$. Without loss of generality, we may assume $\varpi = (\varpi_1, \cdots, \varpi_{d+1})$ satisfying $|\varpi_k| \ge 4\theta$ for $1 \le k \le v$ and $|\varpi_k| < 4\theta$ for $v < k \le d+1$. Accordingly, we define

$$\gamma = \gamma_\varpi := \begin{cases} 0, & \text{if } v = d+1; \\ \displaystyle\sum_{i=v+1}^{d+1} \kappa_i, & \text{if } v < d+1. \end{cases} \tag{4.1}$$

Since $\theta \in (0, 1/(8d)]$ and $\varpi \in S^d$, it follows that

$$0 \le \gamma \le |\kappa| - \min_{1\le i\le d+1} \kappa_i = \sigma_\kappa - \tfrac{d-1}{2}. \tag{4.2}$$

The proof of Theorem 4.1 consists of two cases, one for $\gamma < \sigma_\kappa - \frac{d-1}{2}$ and the other for $\gamma = \sigma_\kappa - \frac{d-1}{2}$, using different methods.

4.1. Proof of Theorem 4.1, case I: $\gamma < \sigma_\kappa - \frac{d-1}{2}$.

The proof is long and will be divided into several subsections.

4.1.1. *Decomposition of the projection operator.* Recall $\lambda_\kappa = \frac{d-1}{2} + |\kappa|$. Let $\xi_0 \in C^\infty[0,\infty)$ be such that $\chi_{[0,1/2]}(t) \le \xi_0(t) \le \chi_{[0,1]}(t)$, and define $\xi_1(t) := \xi_0(t/4) - \xi_0(t)$. Evidently $\mathrm{supp}\,\xi_1 \subset (1/2, 4)$ and $\xi_0(t) + \sum_{j=1}^\infty \xi_1(4^{-j+1}t) = 1$ whenever $t \in [0,\infty)$. Define, for $u \in [-1,1]$,

$$C_{n,0}(u) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u) \xi_0\left(n^2(1-u^2)\right),$$
$$C_{n,j}(u) := \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u) \xi_1\left(\frac{n^2(1-u^2)}{4^{j-1}}\right), \quad j = 1, 2, \ldots, L_n,$$

where $L_n := \lfloor \log_2 n \rfloor + 2$. By (2.4), $\mathrm{proj}_n(h_\kappa^2; f)$ can be decomposed as

$$\mathrm{proj}_n(h_\kappa^2; f) = \sum_{j=0}^{L_n} Y_{n,j} f, \quad \text{where} \quad Y_{n,j} f := f *_\kappa C_{n,j}. \tag{4.3}$$

By the definition of the convolution, the kernel of $Y_{n,j}$ is $V_\kappa[C_{n,j}(\langle x, \cdot \rangle)](y)$.

4.1.2. *Estimates of the kernels $V_\kappa\left[C_{n,j}\langle x, \cdot \rangle\right](y)$ and L^∞ estimate.*

Definition 4.2. Given $n, v \in \mathbb{N}_0$, and $\mu \in \mathbb{R}$, we say a continuous function $F : [-1,1] \to \mathbb{R}$ belongs to the class $\mathcal{S}_n^v(\mu)$ if there exist functions F_j, $j = 0, 1, \cdots, v$ on $[-1,1]$ such that $F_j^{(j)}(t) = F(t)$, $t \in [-1,1]$, $0 \le j \le v$, and

$$|F_j(t)| \le n^{-2j+\mu}\left(1 + n\sqrt{1-|t|}\right)^{-\mu-\frac{1}{2}+j}, \quad t \in [-1,1], \quad j = 0, 1, \cdots, v. \tag{4.4}$$

By (2.18) and the following well-known formula [17, (4.21.7)],

$$\frac{d}{dt} P_n^{(\alpha,\beta)}(t) = \tfrac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1,\beta+1)}(t), \tag{4.5}$$

it follows that $c_{v,\alpha} P_n^{(\alpha,\beta)} \in \mathcal{S}_n^v(\alpha)$ for all $v \in \mathbb{N}_0$ whenever $\alpha \ge \beta$.

Lemma 4.3. *Assume that $\delta = (\delta_1, \cdots, \delta_m) \in \mathbb{R}^m$ satisfying $\min_{1 \le j \le m} \delta_j > 0$ and $\mu \in \mathbb{R}$. Let $F \in \mathcal{S}_n^v(\mu)$ with v being an integer satisfying $v \ge 2m + \sum_{j=1}^m \delta_j + |\mu|$. Let ξ be a C^∞ function, supported in $[-8,8]$ and equal to a constant in a neighborhood of 0. For $\rho \in (n^{-1}, 4]$, define*

$$G(u) := F(u)\xi\left(\frac{1-u^2}{\rho^2}\right), \qquad u \in [-1,1].$$

Then for $s \in [-1,1]$ and $a = (a_1, \cdots, a_m) \in [-1,1]^m$ satisfying $\sum_{j=1}^m |a_j| + |s| \le 1$,

$$\left| \int_{[-1,1]^m} G\Big(\sum_{j=1}^m a_j t_j + s\Big) \prod_{j=1}^m (1 - t_j^2)^{\delta_j - 1}(1 + t_j)\, dt_j \right| \tag{4.6}$$
$$\le c n^{-\frac{1}{2}-|\delta|} \rho^{|\delta|-\mu-\frac{1}{2}} \prod_{j=1}^m (|a_j| + n^{-1}\rho)^{-\delta_j},$$

where $|\delta| = \sum_{j=1}^m \delta_j$.

Proof. Without loss of generality, we may assume that $|a_j| \ge n^{-1}\rho$ for $1 \le j \le m$, since otherwise we can modify the proof by replacing s with

$$s + \sum_{\{j : |a_j| < n^{-1}\rho\}} a_j t_j.$$

Let $\eta_0 \in C^\infty(\mathbb{R})$ be such that $\eta_0(t) = 1$ for $|t| \le \frac{1}{2}$ and $\eta_0(t) = 0$ for $|t| \ge 1$, and let $\eta_1(t) = 1 - \eta_0(t)$. Set $B_j := \frac{\rho}{n|a_j|}$, $j = 1, \cdots, m$. Given $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$, we define $\psi_\varepsilon : \ [-1,1]^m \to \mathbb{R}$ by

$$\psi_\varepsilon(t) := \xi\left(\frac{1 - (\sum_{j=1}^m a_j t_j + s)^2}{\rho^2}\right) \prod_{j=1}^m \eta_{\varepsilon_j}\Big(\frac{1 - t_j^2}{B_j}\Big)(1 + t_j)(1 - t_j^2)^{\delta_j - 1},$$

where $t = (t_1, \cdots, t_m)$. We then split the integral in (4.6) into a finite sum:

$$\sum_{\varepsilon\in\{0,1\}^m} \int_{[-1,1]^m} F\Big(\sum_{j=1}^m a_j t_j + s\Big)\psi_\varepsilon(t)\,dt =: \sum_{\varepsilon\in\{0,1\}^m} J_\varepsilon.$$

Thus, it is sufficient to prove that each term J_ε in the above sum satisfies the desired inequality. By symmetry and Fubini's theorem, we need only to consider the case when $\varepsilon_1 = \cdots = \varepsilon_{m_1} = 0$ and $\varepsilon_{m_1+1} = \cdots = \varepsilon_m = 1$ for some $0 \le m_1 \le m$.

Let m_1 and ε be fixed as in the last line. Fix $(t_1, \cdots, t_{m_1}) \in [-1,1]^{m_1}$ momentarily, and write $s_1 = \sum_{j=1}^{m_1} a_j t_j + s$. Define

$$\phi(t) := \xi\left(\frac{1-(\sum_{j=1}^m a_j t_j + s)^2}{\rho^2}\right) \prod_{j=m_1+1}^m \eta_1\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\delta_j-1}.$$

Since the support set of each $\eta_1\left(\frac{1-t_j^2}{B_j}\right)$ is a subset of $\{t_j : |t_j| \le 1 - \frac14 B_j\}$, we can use integration by parts $|\mathbf{l}| = \sum_{j=m_1+1}^m \ell_j$ times on the function $F = F_{|\mathbf{l}|}^{(|\mathbf{l}|)}$ as in Definition 4.2, where $\mathbf{l} = (\ell_{m_1+1}, \cdots, \ell_m) \in \mathbb{N}^{m-m_1}$ satisfies $\ell_j > \delta_j$ and $|\mathbf{l}| \ge \mu + \frac12$, which gives

$$\begin{aligned}
&\Big|\int_{[-1,1]^{m-m_1}} F\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\phi(t)\,dt\Big| \\
&= \prod_{j=m_1+1}^m |a_j|^{-\ell_j}\Big|\int_{[-1,1]^{m-m_1}} F_{|\mathbf{l}|}\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\,dt\Big| \\
&\le \prod_{j=m_1+1}^m |a_j|^{-\ell_j}\int_{[-1,1]^{m-m_1}} \Big|F_{|\mathbf{l}|}\Big(\sum_{j=m_1+1}^m a_j t_j + s_1\Big)\Big|\Big|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\Big|\,dt.
\end{aligned}$$

Since ξ is supported in $(-8,8)$, the integrand of the last integral is zero unless

$$\begin{aligned}
8\rho^2 &\ge 1 - \Big|\sum_{k=m_1+1}^m a_k t_k + s_1\Big| \\
&\ge 1 - \sum_{k=m_1+1}^m |a_k| - |s_1| + (1-|t_j|)|a_j| \ge |a_j|(1-|t_j|),
\end{aligned} \tag{4.7}$$

for all $m_1+1 \le j \le m$; that is, $\frac{|a_j|}{\rho^2} \le 8(1-|t_j|)^{-1}$ for $j = m_1+1, \cdots, m$. Also, recall that ξ is constant near 0. Hence, taking the k-th partial derivative with respect to t_j, the ξ part of ϕ is bounded by $c(1-t_j)^{-k}$. Also bounded is the same derivative of the η_1 part of ϕ since $B_j^{-1} \le (1-t_j^2)^{-1}$ in the support of η_1'. Consequently, by the Lebnitz rule, we conclude

$$\Big|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\Big| \le c\prod_{j=m_1+1}^m (1-|t_j|)^{\delta_j-\ell_j-1}$$

in the support of the integrand. Next, since $\rho \ge n^{-1}$ and $|\mathbf{l}| \ge \mu + \frac12$, (4.7) together with (4.4) implies

$$\Big|F_{|\mathbf{l}|}\Big(\sum_{k=m_1+1}^m a_k t_k + s_1\Big)\Big| \le cn^{-\frac12-|\mathbf{l}|}\rho^{-\mu-\frac12+|\mathbf{l}|}.$$

It follows that

$$
\begin{aligned}
\int_{[-1,1]^{m-m_1}} & \Big|F_{|\mathbf{l}|}\big(\sum_{j=m_1+1}^{m} a_j t_j + s_1\big)\Big|\Big|\frac{\partial^{|\mathbf{l}|}\phi(t)}{\partial^{\ell_{m_1+1}} t_{m_1+1}\cdots\partial^{\ell_m} t_m}\Big|\,dt \\
&\le c n^{-\frac12-|\mathbf{l}|}\rho^{-\mu-\frac12+|\mathbf{l}|}\prod_{j=m_1+1}^{m}\int_0^{1-\frac{B_j}{4}}(1-t_j)^{\delta_j-\ell_j-1}\,dt_j \\
&\le c n^{-\frac12-|\mathbf{l}|}\rho^{-\mu-\frac12+|\mathbf{l}|}\prod_{j=m_1+1}^{m} B_j^{\delta_j-\ell_j} \\
&\le c n^{-\frac12-\alpha}\rho^{\alpha-\mu-\frac12}\prod_{j=m_1+1}^{m}|a_j|^{\ell_j-\delta_j},
\end{aligned}
$$

where $\alpha = \sum_{j=m_1+1}^{m}\delta_j$. Thus, since

$$
\psi_\varepsilon(t) = \phi(t)\prod_{j=1}^{m_1}\eta_0\Big(\frac{1-t_j^2}{B_j}\Big)(1+t_j)(1-t_j^2)^{\delta_j-1},
$$

and $\eta_0\left(\frac{1-t_j^2}{B_j}\right)$ is supported in $\{t_j:\ 1-B_j\le |t_j|\le 1\}$, integrating with respect to $t_1,\cdots,t_{m_1}$ over $[-1,1]^{m_1}$ yields

$$
\begin{aligned}
J_\varepsilon \le & \int_{[-1,1]^{m_1}}\Big|\int_{[-1,1]^{m-m_1}} F\Big(\sum_{j=1}^{m} a_j t_j + s\Big)\phi(t)\,dt_{m_1+1}\cdots dt_m\Big| \\
& \times\prod_{j=1}^{m_1}\eta_0\left(\frac{1-t_j^2}{B_j}\right)(1+t_j)(1-t_j^2)^{\delta_j-1}dt_j \\
\le & \, c n^{-\frac12-\alpha}\rho^{\alpha-\mu-\frac12}\prod_{j=m_1+1}^{m}|a_j|^{-\delta_j}\prod_{j=1}^{m_1}\int_{1-B_j\le|t_j|\le 1}(1-|t_j|)^{\delta_j-1}\,dt_j \\
\le & \, c n^{-\frac12-|\delta|}\rho^{|\delta|-\mu-\frac12}\prod_{j=1}^{m}|a_j|^{-\delta_j},
\end{aligned}
$$

where we have used $|a_j|^{\ell_j}\le 1$ in the second step. This completes the proof. □

Using the relation between the Gegenbauer and the Jacobi polynomials, we have

$$
C_{n,j}(u) = a_n P_n^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(u)\xi\left(\frac{1-u^2}{(2^{j-1}/n)^2}\right),
$$

where $\xi=\xi_1$ or ξ_0, and $|a_n|\le c n^{\lambda_\kappa+\frac12}$. Hence, using the fact that $c_{v,\kappa}P_n^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)} \in \mathcal{S}_n^v(\lambda_\kappa-\frac12)$ for all $v\in\mathbb{N}$, Lemma 4.3 has the following corollary.

Corollary 4.4. *For $x,y\in S^d$ and $j=1,2,\ldots,L_n$,*

$$
\Big|V_\kappa\Big[C_{n,j}(\langle x,\cdot\rangle)\Big](y)\Big| \le c n^{d-1}2^{-j(d-1)/2}\prod_{i=1}^{d+1}\big(|x_i y_i| + 2^j n^{-2}\big)^{-\kappa_i}.
$$

Recall that $c(\varpi,\theta)$ is a fixed spherical cap, $\theta\in[n^{-1},\pi]$ and $\gamma=\gamma_\varpi$ is defined in (4.1). We are now in a position to prove the following L^∞ estimate:

Lemma 4.5. *If f is supported in $c(\varpi,\theta)$, then*

$$\sup_{x\in c(\varpi,\theta)}|Y_{n,j}(f)(x)|\le cn^{d-1+2\gamma}2^{-j(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}\Big[\int_{c(\varpi,\theta)}h_\kappa^2(x)\,d\varpi(x)\Big]^{-1}\|f\|_{\kappa,1}.$$

Proof. Note that if $x\in c(\varpi,\theta)$, then $|x_i-\varpi_i|\le\|x-\varpi\|\le d(x,\varpi)\le\theta$ so that $\frac{3}{4}|\varpi_i|\le|x_i|\le\frac{5}{4}|\varpi_i|$ for $1\le i\le v$, and $|x_i|\le5\theta$ for $v+1\le i\le d+1$. It follows from Corollary 4.4 that, for any $x,y\in c(\varpi,\theta)$,

$$\begin{aligned}\Big|V_\kappa\Big[C_{n,j}(\langle x,\cdot\rangle)\Big](y)\Big|&\le cn^{d-1}2^{-j(d-1)/2}\prod_{i=1}^{v}|\varpi_i|^{-2\kappa_i}\prod_{i=v+1}^{d+1}n^{2\kappa_i}2^{-j\kappa_i}\\&\le cn^{d-1}2^{-j(\frac{d-1}{2}+\gamma)}(n\theta)^{2\gamma}\prod_{i=1}^{d+1}(|\varpi_i|+\theta)^{-2\kappa_i}\\&\le cn^{d-1}2^{-j(\frac{d-1}{2}+\gamma)}(n\theta)^{2\gamma}\theta^d\Big[\int_{c(\varpi,\theta)}h_\kappa^2(z)\,d\omega(z)\Big]^{-1},\end{aligned}$$

where the last step follows from the relation (2.17). This implies that

$$\begin{aligned}\sup_{x\in c(\varpi,\theta)}|Y_{n,j}(f)(x)|&\le\sup_{x\in c(\varpi,\theta)}\int_{c(\varpi,\theta)}|f(y)|\Big|V_\kappa\Big[C_{n,j}\langle x,\cdot\rangle\Big](y)\Big|h_\kappa^2(y)\,d\omega(y)\\&\le cn^{d-1+2\gamma}2^{-j\left(\frac{d-1}{2}+\gamma\right)}\theta^{2\gamma+d}\Big[\int_{c(\varpi,\theta)}h_\kappa^2(x)\,d\omega(x)\Big]^{-1}\|f\|_{\kappa,1},\end{aligned}$$

which is the desired inequality. □

4.1.3. L^2 *estimates.* We prove the following estimate:

Lemma 4.6. *For any $f\in L^2(h_\kappa^2;S^d)$,*

$$\|Y_{n,j}(f)\|_{\kappa,2}\le c\,n^{-1}2^j\|f\|_{\kappa,2}.$$

Proof. For simplicity, we shall write $\xi_j=\xi_1$ for $j\ge1$. Also let $\lambda=\lambda_\kappa$ in this proof. From (2.5) and the definition of $Y_{n,j}$ in (4.3), it follows that each $Y_{n,j}$ is a multiplier operator,

$$Y_{n,j}(f)=\sum_{k=0}^{\infty}m_{n,j}(k)\operatorname{proj}_k(h_\kappa^2;f),$$

where the equality is understood in a distributional sense, and

$$m_{n,j}(k):=c_{n,k}\int_0^\pi C_n^\lambda(\cos t)C_k^\lambda(\cos t)\xi_j\left(\frac{n^2\sin^2t}{4^{j-1}}\right)\sin^{2\lambda}t\,dt$$

with $|c_{n,k}|\le cnk^{-2\lambda+1}$. Hence, it is enough to prove

$$\sup_k|m_{n,j}(k)|\le c\,n^{-1}2^j. \tag{4.8}$$

If $k \geq \frac{n}{4}$, then using the fact that $|(\sin\theta)^\lambda C_n^\lambda(\cos\theta)| \leq cn^{\lambda-1}$, a straightforward computation gives

$$\begin{aligned} |m_{n,j}(k)| &\leq |c_{n,k}| \int_0^\pi \left| C_n^\lambda(\cos t) C_k^\lambda(\cos t) \xi_j\left(\frac{n^2\sin^2 t}{4^{j-1}}\right)\right| \sin^{2\lambda} t\, dt \\ &\leq c\int_0^\pi \left|\xi_j\left(\frac{n^2\sin^2 t}{4^{j-1}}\right)\right| dt \leq c\frac{2^j}{n}, \end{aligned}$$

where the last step follows easily using the support of ξ_j.

For $k \leq \frac{n}{4}$, we shall use the following formula (cf. [1, p. 319, Theorem 6.8.2]):

$$C_k^\lambda(t) C_n^\lambda(t) = \sum_{i=0}^{\min\{k,n\}} a(i,k,n) C_{k+n-2i}^\lambda(t), \tag{4.9}$$

where

$$a(i,k,n) := \frac{(k+n+\lambda-2i)(\lambda)_i(\lambda)_{k-i}(\lambda)_{n-i}(2\lambda)_{k+n-i}}{(k+n+\lambda-i)i!(k-i)!(n-i)!(\lambda)_{k+n-i}} \frac{(k+n-2i)!}{(2\lambda)_{k+n-2i}}.$$

For $k \leq n/4$, it is easy to see that

$$\begin{aligned} |a(i,k,n)| &\sim \left(\frac{(i+1)(\min\{k,n\}-i+1)(k+n-i+1)}{k+n-2i+1}\right)^{\lambda-1} \\ &\sim (i+1)^{\lambda-1}(k-i+1)^{\lambda-1}. \end{aligned} \tag{4.10}$$

Consequently, it follows that for $k \leq n/4$,

$$\begin{aligned} |m_{n,j}(k)| &\leq cnk^{-2\lambda+1}\sum_{i=0}^{k}(i+1)^{\lambda-1} \\ &\quad \times (k-i+1)^{\lambda-1}\left|\int_0^\pi C_{k+n-2i}^\lambda(\cos t)\xi_j\left(\frac{n^2\sin^2 t}{4^{j-1}}\right)\sin^{2\lambda} t\, dt\right| \\ &\leq cn^{\lambda+\frac12}\max_{3n/4\leq m\leq 5n/4}\left|\int_{-1}^1 P_m^{(\lambda-\frac12,\lambda-\frac12)}(s)\xi_j\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac12}\, ds\right|. \end{aligned}$$

Then using the estimate (2.18) we obtain

$$m_{n,0}(k) \leq cn^{2\lambda}\int_{1-|s|\leq cn^{-2}}(1-|s|)^{\lambda-\frac12}\, ds \leq cn^{-1}.$$

If $j \geq 1$, then for all $\ell \in \mathbb{N}$, it follows that

$$\left|\frac{d^\ell}{ds^\ell}\left(\xi_1\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac12}\right)\right| \leq c\left(\frac{2^j}{n}\right)^{2\lambda-1-2\ell},$$

since $1-s^2 \sim (\frac{2^j}{n})^2$ in the support of ξ_1'; consequently, we obtain by integration by parts, (4.5) and (2.18) that

$$\begin{aligned} &m_{n,j}(k) \leq cn^{\lambda+\frac12-\ell} \\ &\quad\times \max_{3n/4\leq m\leq 5n/4}\left|\int_{-1}^1 P_{m+\ell}^{(\lambda-\frac12-\ell,\lambda-\frac12-\ell)}(s)\frac{d^\ell}{ds^\ell}\left(\xi_1\left(\frac{n^2(1-s^2)}{4^{j-1}}\right)(1-s^2)^{\lambda-\frac12}\right) ds\right| \\ &\leq c2^{j(\lambda-\ell)}2^jn^{-1} \leq c2^jn^{-1} \end{aligned}$$

upon choosing $\ell \geq \lambda$, Thus, in both cases, we get the desired estimate. $\square$

4.1.4. *Proof of Theorem* 4.1, *case I:* $\gamma < \sigma_\kappa - \frac{d-1}{2}$. Recall $\nu = \frac{2+2\sigma_\kappa}{2+\sigma_\kappa}$. We set, in this subsection,

$$A := \int_{c(\varpi,\theta)} h_\kappa^2(y)\, d\omega(y).$$

Recall the decomposition (4.3). For a generic f, we set

$$T_{n,j}f := Y_{n,j}(f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)}, \qquad 0 \le j \le L_n.$$

Clearly, if f is supported in $c(\varpi,\theta)$ and $x \in c(\varpi,\theta)$, then $T_{n,j}f(x) = Y_{n,j}f(x)$. Using Lemmas 4.5 and 4.6, we have

$$\begin{aligned} \|T_{n,j}f\|_\infty &\le cn^{2\gamma+d-1}2^{-j(\frac{d-1}{2}+\gamma)}\theta^{2\gamma+d}A^{-1}\|f\chi_{c(\varpi,\theta)}\|_{\kappa,1}, \\ \|T_{n,j}f\|_{\kappa,2} &\le c\,n^{-1}2^j\|f\chi_{c(\varpi,\theta)}\|_{\kappa,2}. \end{aligned} \tag{4.11}$$

Hence, by the Riesz-Thorin convexity theorem, we obtain (recall $\nu = \frac{2+2\sigma_\kappa}{2+\sigma_k}$)

$$\|T_{n,j}f\|_{\kappa,\nu'} \le cn^{-1}2^{j\left(1-(\frac{d+1}{2}+\gamma)\frac{1}{\sigma_\kappa+1}\right)}(n\theta)^{\frac{2\gamma+d}{\sigma_\kappa+1}}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}. \tag{4.12}$$

On the other hand, using (4.11), Hölder's inequality and (2.17), we obtain

$$\begin{aligned} \|T_{n,j}f\|_{\kappa,\nu'} &\le \|T_{n,j}f\|_\infty A^{1-\frac{1}{\nu}} \le cn^{2\gamma+d-1}2^{-j\left(\frac{d-1}{2}+\gamma\right)}\theta^{2\gamma+d}A^{-\frac{1}{\nu}}\|f\chi_{c(\varpi,\theta)}\|_{\kappa,1} \\ &\le cn^{-1}2^{-j\left(\frac{d-1}{2}+\gamma\right)}(n\theta)^{2\gamma+d}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}. \end{aligned} \tag{4.13}$$

Now assume that f is supported in $c(\varpi,\theta)$ and $\frac{2^{j_0-1}}{n} \le \theta \le \frac{2^{j_0}}{n}$ for some $1 \le j_0 \le L_n$. Using (4.3) and Minkowski's inequality, we have

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,\nu'} \le \sum_{j=0}^{2j_0}\|T_{n,j}f\|_{\kappa,\nu'} + \sum_{j=2j_0+1}^{L_n}\|T_{n,j}f\|_{\kappa,\nu'} =: \Sigma_1 + \Sigma_2.$$

For the first sum Σ_1, we use (4.12) to obtain

$$\begin{aligned} \Sigma_1 &\le cn^{-1}(n\theta)^{\frac{2\gamma+d}{\sigma_\kappa+1}}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}\sum_{j=0}^{2j_0}2^{j\left(1-(\frac{d+1}{2}+\gamma)\frac{1}{\sigma_\kappa+1}\right)} \\ &\le cn^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}\theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}, \end{aligned}$$

since $\gamma < \sigma_\kappa - \frac{d-1}{2}$ readily implies that $1-(\frac{d+1}{2}+\gamma)\frac{1}{\sigma_\kappa+1} > 0$. For the second sum Σ_2, we use (4.13) to obtain

$$\begin{aligned} \Sigma_2 &\le cn^{-1}(n\theta)^{2\gamma+d}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}\sum_{j=2j_0+1}^{\infty}2^{-j(\frac{d-1}{2}+\gamma)} \\ &\le cn^{-1}(n\theta)A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu} \le cn^{-1}(n\theta)^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu} \\ &= cn^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}\theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}}A^{1-\frac{2}{\nu}}\|f\|_{\kappa,\nu}, \end{aligned}$$

where in the third inequality we have used the fact that $n\theta \ge 1$.

Putting the above together proves Theorem 4.1 in the case $\gamma < \sigma_\kappa - \frac{d-1}{2}$. $\square$

4.2. **Proof of Theorem 4.1, case 2:** $\gamma = \sigma_\kappa - \frac{d-1}{2}$**.** Recall that $|\varpi_j| \ge 4\theta$ for $1 \le j \le v$, $|\varpi_j| < 4\theta$ for $v+1 \le j \le d+1$, and $\gamma = \gamma_\varpi = \sum_{j=v+1}^{d+1} \kappa_j$. In this case, either $v = 1$ and $|\varpi_1| = \max_{1\le j\le d+1} |\varpi_j| \ge \frac{1}{\sqrt{d+1}}$; or $v \ge 2$ and $\kappa_1 = \cdots = \kappa_v = 0$. Therefore, by (2.17), we have

$$\int_{c(\varpi,\theta)} h_\kappa^2(x)\, d\omega(x) \sim \theta^d \Big(\prod_{j=1}^{v} |\varpi_j|^{2\kappa_j}\Big)\theta^{2\gamma} \sim \theta^{2\sigma_\kappa+1}.$$

Hence, Theorem 4.1 in this case is equivalent to the following proposition:

Proposition 4.7. *Let f be supported in $c(\varpi,\theta)$ with $\theta \in (n^{-1}, 1/(8d)]$ and let $\nu := \frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$ and $\nu' := \frac{\nu}{\nu-1}$. Then*

$$\left\|\mathrm{proj}_n(h_\kappa^2; f)\chi_{c(\varpi,\theta)}\right\|_{\kappa,\nu'} \le c n^{\frac{\sigma_\kappa}{(1+\sigma_\kappa)}} \|f\|_{\kappa,\nu}.$$

To prove Proposition 4.7, we use the method of analytic interpolation [16]. For $z \in \mathbb{C}$, define

$$\mathcal{P}_n^z f(x) := (f *_\kappa G_n^z)(x) = a_\kappa \int_{S^d} f(y) V_\kappa\Big[G_n^z(\langle x, \cdot\rangle)\Big](y) h_\kappa^2(y)\, d\omega(y) \tag{4.14}$$

for $x \in S^d$, where

$$G_n^z(t) = (\sigma_\kappa+1)(1-z)\frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(t)(1-t^2+n^{-2})^{\frac{\sigma_\kappa-(\sigma_\kappa+1)z}{2}}. \tag{4.15}$$

From (2.4), it readily follows that

$$\mathcal{P}_n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} f = \mathrm{proj}_n(h_\kappa^2; f).$$

For the rest of this subsection, we shall use c_τ to denote a general constant satisfying $|c_\tau| \le c(1+|\tau|)^\ell$ for some inessential positive number ℓ.

4.2.1. *Estimate for $z = 1 + \mathrm{i}\tau$.*

Lemma 4.8. *For $\tau \in \mathbb{R}$,*

$$\|\mathcal{P}_n^{1+\mathrm{i}\tau} f\|_{\kappa,2} \le c_\tau \|f\|_{\kappa,2}.$$

Proof. From (4.14), (4.15) and (2.5), it follows that

$$\mathrm{proj}_k(h_\kappa^2; \mathcal{P}_n^{1+\mathrm{i}\tau} f) = J_n(k)\, \mathrm{proj}_k(h_\kappa^2; f), \qquad k = 0, 1, \cdots,$$

where

$$J_n(k) := \mathcal{O}(1) n k^{-2\lambda_\kappa+1}\tau' \int_0^\pi C_k^{\lambda_\kappa}(\cos t) C_n^{\lambda_\kappa}(\cos t)(\sin^2 t + n^{-2})^{-\frac12 + i\tau'}(\sin t)^{2\lambda_\kappa}\, dt$$

and $\tau' = -\frac{\sigma_\kappa+1}{2}\tau$. Therefore, it is sufficient to prove

$$|J_n(k)| \le c_\tau, \qquad \forall k, n \in \mathbb{N}. \tag{4.16}$$

For $k < \frac{n}{4}$, (4.16) can be shown as in the proof of Lemma 4.2. In fact, using (4.9) and (4.10), we obtain

$$|J_n(k)| \le c|\tau| n^{\lambda_\kappa+\frac12} \times \max_{3n/4 \le m \le 5n/4} \Big| \int_{-1}^1 P_m^{(\lambda_\kappa-\frac12, \lambda_\kappa-\frac12)}(s)(1-s^2+n^{-2})^{-\frac12+i\tau'}(1-s^2)^{\lambda_\kappa-\frac12}\, ds\Big|,$$

which is controlled by

$$\begin{aligned} &c_\tau + c_\tau n^{\lambda_\kappa+\frac12-\ell} \max_{3n/4\le m\le 5n/4} \int_{-1+n^{-2}}^{1-n^{-2}} \Big| P_{m+\ell}^{(\lambda_\kappa-\frac12-\ell,\lambda_\kappa-\frac12-\ell)}(s) \\ &\qquad \times \frac{d^\ell}{ds^\ell}\Big((1-s^2+n^{-2})^{-\frac12+i\tau'}(1-s^2)^{\lambda_\kappa-\frac12}\Big)\Big|\,ds \\ &\le c_\tau' \end{aligned}$$

using integration by parts $\ell > \lambda_\kappa$ times. This proves (4.16) for $k < \frac{n}{4}$.

For $k \ge \frac{n}{4}$, (4.16) can be established exactly as in [14, pp. 54–55] (see also [18, pp. 76–81]). For completeness, we sketch the proof as follows. Since $C_j^\lambda(t) = \mathcal{O}(1) j^{\lambda-\frac12} P_j^{(\lambda-\frac12,\lambda-\frac12)}(t)$ and $P_j^{(\alpha,\beta)}(-t) = (-1)^j P_j^{(\alpha,\beta)}(t)$, we can write

$$\begin{aligned} J_n(k) &= \mathcal{O}(1) k^{-\lambda_\kappa+\frac12} n^{\lambda_\kappa+\frac12}\tau' \Big[\int_0^{4n^{-1}} + \int_{4n^{-1}}^{\frac{\pi}{2}}\Big] P_k^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(\cos t) \\ &\quad \times P_n^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(\cos t)(\sin^2 t + n^{-2})^{-\frac12+i\tau'}(\sin t)^{2\lambda_\kappa}\,dt \\ &=: J_{n,1}(k) + J_{n,2}(k). \end{aligned}$$

Since $|P_j^{(\alpha,\alpha)}(t)| \le c j^\alpha$, a straightforward calculation shows that $|J_{n,1}(k)| \le c_\tau$. To estimate $J_{n,2}(k)$, we need the asymptotics of the Jacobi polynomials as given in [17, p. 198],

$$P_j^{(\alpha,\beta)}(\cos t) = \pi^{-\frac12} j^{-\frac12} (\sin\tfrac{t}{2})^{-\alpha-\frac12}(\cos\tfrac{t}{2})^{-\beta-\frac12}\left[\cos(N_j t + \tau_\alpha) + \mathcal{O}(1)(j\sin t)^{-1}\right]$$

for $j^{-1} \le t \le \pi - j^{-1}$, where $N_j = j + \frac{\alpha+\beta+1}{2}$ and $\tau_\alpha = -\frac{\pi}{2}(\alpha+\frac12)$. Applying this asymptotic formula with $\alpha=\beta=\lambda_\kappa - 1/2$, we obtain, for $k \ge \frac{n}{4}$ and $4n^{-1} \le t \le \frac{\pi}{2}$,

$$\begin{aligned} &k^{-\lambda_\kappa+\frac12} n^{\lambda_\kappa+\frac12} P_k^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(\cos t) P_n^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(\cos t)(\sin t)^{2\lambda_\kappa} \\ &= \mathcal{O}(1)\Big[\cos\big((k-n)t\big) + \cos\big((k+n+2\lambda_\kappa)t - \lambda_\kappa\pi\big)\Big] + \mathcal{O}\left(\frac{1}{nt}\right) \end{aligned}$$

using the cosine addition formula. Also, note that

$$\Big(\frac{1}{\sin^2 t + n^{-2}}\Big)^{\frac12 - i\tau'} = t^{-1+2i\tau'} + \mathcal{O}(t) + \mathcal{O}\left(n^{-2}t^{-3}\right), \quad 4n^{-1} \le t \le \frac{\pi}{2}.$$

It follows that

$$\begin{aligned} |J_{n,2}(k)| &\le c_\tau + c_\tau \sup_{\ell\in\mathbb{R}} \Big| 2\tau' \int_{4n^{-1}}^{\frac{\pi}{2}} t^{-1+2i\tau'} e^{i\ell t}\,dt\Big| \\ &\le c_\tau + c_\tau \sup_{a<b}\Big|\int_a^b e^{it}\,dt^{2i\tau'}\Big| \le c_\tau. \end{aligned}$$

This proves the desired inequality (4.16) for $k \ge \frac{n}{4}$. □

4.2.2. *Estimate for $z = \mathrm{i}\tau$.*

Lemma 4.9. *If $\tau\in\mathbb{R}$ and f is supported in $c(\varpi,\theta)$, then*

$$\sup_{x\in c(\varpi,\theta)} |\mathcal{P}_n^{\mathrm{i}\tau} f(x)| \le c_\tau n^{\sigma_\kappa}\|f\|_{1,\kappa}.$$

Proof. Since f is supported in $c(\varpi,\theta)$, we have

$$\sup_{x\in c(\varpi,\theta)} |\mathcal{P}_n^{\mathrm{i}\tau}(f)(x)| \le \sup_{x\in c(\varpi,\theta)} \int_{c(\varpi,\theta)} |f(y)| \left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right| h_\kappa^2(y)\,d\omega(y)$$
$$\le \|f\|_{1,\kappa} \sup_{x,y\in c(\varpi,\theta)} \left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right|.$$

Thus, it is sufficient to prove

$$\left|V_\kappa\left[G_n^{\mathrm{i}\tau}(\langle x,\cdot\rangle)\right](y)\right| \le c_\tau n^{\sigma_\kappa} \qquad \text{for all } x,y\in c(\varpi,\theta). \tag{4.17}$$

We note that (4.17) is trivial when $\kappa_{\min}=0$ since in this case $\|G_n^{i\tau}\|_\infty \le c_\tau n^{\lambda_\kappa} = c_\tau n^{\sigma_\kappa}$. So we shall assume $\kappa_{\min}>0$ for the rest of the proof.

To prove (4.17), we claim that it is enough to prove that

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)(1-t^2)^{\delta-1}(1+t)\,dt\right| \le c_\tau n^{\sigma_\kappa}, \tag{4.18}$$

whenever $|a|\ge\varepsilon_d>0$, $|a|+|s|\le 1$, $\delta\ge\kappa_{\min}$, where c_τ is independent of s.

To see this, let $x,y\in c(\varpi,\theta)$ and without loss of generality, assume $\varpi_1 = \max_{1\le j\le d+1}|\varpi_j|$. Then $\varpi_1\ge 1/\sqrt{1+d}$, which implies that $|x_1|,|y_1|\ge 1/\sqrt{d+1}-\theta\ge 1/\sqrt{d+1}-1/(8d)>0$, so that $|x_1y_1|\ge\varepsilon_d>0$. Thus, invoking (4.18) with $a=x_1y_1$, $\delta=\kappa_1$ and $s=\sum_{j=2}^{d+1} t_jx_jy_j$ gives

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}\Big(\sum_{j=1}^{d+1} x_jy_jt_j\Big)(1-t_1^2)^{\kappa_1-1}(1+t_1)\,dt_1\right| \le c_\tau n^{\sigma_\kappa}.$$

The desired inequality (4.17) then follows by the Fubini theorem and the integral representation of V_κ in (2.2). This proves the claim.

For the proof of (4.18), by symmetry, it is sufficient to prove

$$\left|\int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)(1-t)^{\delta-1}\xi(t)\,dt\right| \le c_\tau n^{\sigma_\kappa}, \tag{4.19}$$

where ξ is a C^∞ function supported in $[-\frac12,1]$, whenever $|a|\ge\varepsilon_d>0$, $|a|+|s|\le 1$ and $\delta\ge\kappa_{\min}$.

Let $\eta_0\in C^\infty(\mathbb{R})$ be such that $\chi_{[-\frac12,\frac12]}\le\eta_0\le\chi_{[-1,1]}$, and let $\eta_1(t):=1-\eta_0(t)$. Set, in this subsection,

$$B := \frac{n^{-1}+\sqrt{1-|a+s|}}{4n}.$$

We then split the integral in (4.19) into a sum $I_0(a,s)+I_1(a,s)$ with

$$I_j(a,s) := \int_{-1}^1 G_n^{\mathrm{i}\tau}(at+s)\eta_j\Big(\frac{1-t}{B}\Big)(1-t)^{\delta-1}\xi(t)\,dt, \qquad j=0,1.$$

It is easy to verify that $1+n\sqrt{1-|at+s|}\sim 1+n\sqrt{1-|a+s|}$ whenever $t\in[1-B,1]\cap[-1,1]$. Therefore, for $1-B\le t\le 1$, using (2.18),

$$|G_n^{\mathrm{i}\tau}(at+s)| \le cn^{\lambda_\kappa}(n^{-1}+\sqrt{1-|at+s|})^{-\lambda_\kappa+\sigma_\kappa} \le c\,n^{\sigma_\kappa}B^{-\kappa_{\min}},$$

which implies that

$$|I_0(a,s)| \le c\int_{\max\{1-B,-\frac12\}}^1 |G_n^{\mathrm{i}\tau}(at+s)|(1-t)^{\delta-1}\,dt \le cn^{\sigma_\kappa}B^{\delta-\kappa_{\min}} \le cn^{\sigma_\kappa}.$$

To estimate $I_1(a,s)$, we write

$$G_n^{\mathrm{i}\tau}(at+s)\eta_1\Big(\frac{1-t}{B}\Big)(1-t)^{\delta-1}\xi(t)=c_nP_n^{(\lambda_\kappa-\frac12,\lambda_\kappa-\frac12)}(at+s)\varphi(t),$$

where $|c_n|\le c_\tau n^{\lambda_\kappa+\frac12}$ and

$$\varphi(t):=\big(1-(at+s)^2+n^{-2}\big)^{\frac{\sigma_\kappa}{2}-\frac{\sigma_\kappa+1}{2}\mathrm{i}\tau}\eta_1\Big(\frac{1-t}{B}\Big)\xi(t)(1-t)^{\delta-1}.$$

Recall $|a|\ge\varepsilon_d>0$. Using integration by parts ℓ times gives

$$|I_1(a,s)|\le cn^{\lambda_\kappa+\frac12-\ell}\int_{-1}^{1}\Big|P_{n+\ell}^{(\lambda_\kappa-\frac12-\ell,\lambda_\kappa-\frac12-\ell)}(at+s)\Big|\,|\varphi^{(\ell)}(t)|\,dt.$$

If $-\frac12\le t\le 1-B/2$, then $1-|at+s|\ge 1-|a|-|s|+(1-|t|)|a|\ge c(1-t)\ge cB\ge cn^{-2}$, which implies, in particular, $\big(1-(at+s)^2+n^{-2}\big)^{-1}\le c(1-t)^{-1}$. Since φ is supported in $(-\frac12,1-\frac{B}{2})$, which gives $B^{-1}\le(1-t)^{-1}$, it follows from Leibniz' rule that

$$|\varphi^{(\ell)}(t)|\le c_\tau(1-|at+s|)^{\frac{\sigma_\kappa}{2}}(1-t)^{\delta-\ell-1}.$$

Therefore, choosing $\ell>2\delta$ and recalling that $\delta\ge\kappa_{\min}$, we have by (2.18) that

$$\begin{aligned}|I_1(a,s)|&\le c_\tau n^{\lambda_\kappa-\ell}\int_{-\frac12}^{1-\frac{B}{2}}(1-|at+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}(1-t)^{\delta-1-\ell}\,dt\\&\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\frac32|a|}(1-|a+s|+u)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,du.\end{aligned}$$

Using the fact that $(1-|a+s|+u)^\alpha\le c((1-|a+s|)^\alpha+u^\alpha)$ we break the last integral into a sum J_1+J_2, where

$$\begin{aligned}J_1&\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\infty}(1-|a+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,du\\&\le c_\tau n^{\lambda_\kappa-\ell}(1-|a+s|)^{\frac{\sigma_\kappa+\ell}{2}-\frac{\lambda_\kappa}{2}}B^{\delta-\ell}\le c_\tau n^{\lambda_\kappa-\ell}n^{\ell-\delta}(nB)^{\delta+\sigma_\kappa-\lambda_\kappa}\\&=c_\tau n^{\lambda_\kappa-\delta}(nB)^{\delta-\kappa_{\min}}\le c_\tau n^{\lambda_\kappa-\delta}\le c_\tau n^{\sigma_\kappa}\end{aligned}$$

and

$$\begin{aligned}J_2&\le c_\tau n^{\lambda_\kappa-\ell}\int_{\frac{B|a|}{2}}^{\infty}u^{\frac{\sigma+\ell}{2}-\frac{\lambda_\kappa}{2}}u^{\delta-1-\ell}\,du\le c_\tau n^{\lambda_\kappa-\ell}B^{\frac{\sigma_\kappa-\ell}{2}-\frac{\lambda_\kappa}{2}+\delta}\\&=c_\tau n^{\lambda_\kappa-\delta}(n^2B)^{\frac{\kappa_{\min}-\ell}{2}}(nB)^{\delta-\kappa_{\min}}\le c_\tau n^{\lambda_\kappa-\delta}\le c_\tau n^{\sigma_\kappa}.\end{aligned}$$

Putting the above together, we obtain the desired estimate (4.19) and complete the proof of Lemma 4.9. □

4.2.3. *Proof of Proposition* 4.7. Define

$$T^zf=n^{\sigma_\kappa(z-1)}\mathcal{P}_n^z(f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)},\quad 0\le\Re z\le 1.$$

By Lemmas 4.9 and 4.8, we have

$$\|T^{1+\mathrm{i}\tau}f\|_{\kappa,2}\le c_\tau\|f\|_{\kappa,2}\quad\text{and}\quad\|T^{\mathrm{i}\tau}f\|_\infty\le c_\tau\|f\|_{1,\kappa}.$$

These allow us to apply Stein's interpolation theorem [16, p. 205] to the analytic family of operators T^z, which yields

$$\|T^{\frac{\sigma_\kappa}{1+\sigma_\kappa}}f\|_{\kappa,\nu'}\le c\|f\|_{\kappa,\nu}.$$

Consequently, using the fact that

$$T^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} f = n^{-\frac{\sigma_\kappa}{1+\sigma_\kappa}} \mathcal{P}_n^{\frac{\sigma_\kappa}{1+\sigma_\kappa}} (f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)} = n^{-\frac{\sigma_\kappa}{1+\sigma_\kappa}} \operatorname{proj}_n(h_\kappa^2; f\chi_{c(\varpi,\theta)})\chi_{c(\varpi,\theta)},$$

we have proved Proposition 4.7. □

5. Boundedness of projection operator

The objective of this section is to prove Theorems 3.3 and 3.4.

5.1. **Proof of Theorem 3.4.** Assume that f is supported in a spherical cap $c(\varpi,\theta)$. Without loss of generality, we may assume $\theta < 1/(8d)$, since otherwise we can decompose f as a finite sum of functions supported on a family of spherical caps of radius $< 1/(8d)$.

We start with the case $p = 1$. By the definition of the projection operator, it follows from the integral version of the Minkowski inequality and orthogonality that

$$\begin{aligned}\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,2} &\le \sup_{y\in c(\varpi,\theta)} \left(\int_{S^d} \left|P_n(h_\kappa^2; x, y)\right|^2 h_\kappa^2(x)\, d\omega(x)\right)^{1/2} \|f\|_{\kappa,1} \\ &= \Big(\sup_{y\in c(\varpi,\theta)} P_n(h_\kappa^2; y, y)\Big)^{1/2} \|f\|_{\kappa,1}.\end{aligned}$$

Using the pointwise estimate of the kernel in (2.21) and the fact that $n\theta \ge 1$, we then obtain

$$\begin{aligned}\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,2} &\le cn^{\frac{d-1}{2}} \sup_{y\in c(\varpi,\theta)} \prod_{j=1}^{d+1} (|y_j| + n^{-1})^{-\kappa_j} \|f\|_{\kappa,1} \\ &\le cn^{\frac{d-1}{2}} (n\theta)^{\sigma_\kappa - \frac{d-1}{2}} \sup_{y\in c(\varpi,\theta)} \prod_{j=1}^{d+1} (|y_j| + \theta)^{-\kappa_j} \|f\|_{\kappa,1} \\ &\le cn^{\sigma_\kappa} \theta^{\sigma_\kappa + \frac{1}{2}} \Big(\int_{c(\varpi,\theta)} h_\kappa^2(y)\, d\omega(y)\Big)^{-\frac{1}{2}} \|f\|_{\kappa,1},\end{aligned}$$

where the last step follows from (2.17). This proves Theorem 3.4 for $p = 1$.

Next, we use Hölder's inequality and Theorem 4.1 to obtain

$$\begin{aligned}\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,2}^2 &= \int_{c(\varpi,\theta)} f(y) \operatorname{proj}_n(h_\kappa^2; f)(y) h_\kappa^2(y)\, d\omega(y) \\ &\le \|f\|_{\kappa,\nu} \Big(\int_{c(\varpi,\theta)} |\operatorname{proj}_n(h_\kappa^2; f, y)|^{\nu'} h_\kappa^2(y)\, d\omega(y)\Big)^{\frac{1}{\nu'}} \\ &\le cn^{\frac{\sigma_\kappa}{\sigma_\kappa+1}} \theta^{\frac{2\sigma_\kappa+1}{\sigma_\kappa+1}} \Big[\int_{c(\varpi,\theta)} h_\kappa^2(x)\, d\sigma(x)\Big]^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu}^2,\end{aligned}$$

which proves Theorem 3.4 for $p = \nu = \frac{2\sigma_\kappa+2}{\sigma_\kappa+2}$.

Finally, Theorem 3.4 for $1 \le p \le \nu$ follows by applying the Riesz-Thorin convexity theorem to the linear operator $g \mapsto \operatorname{proj}_n(h_\kappa^2; g\chi_{c(\varpi,\theta)})$. □

5.2. **Proof of Theorem 3.3.** Theorem 3.3 (i) follows directly by invoking Theorem 4.1 with $\theta = \pi$. Theorem 3.3 (ii) follows from the Riesz-Thorin convexity theorem applied to the boundedness of $f \mapsto \operatorname{proj}_n(h_\kappa^2; f)$ in (2,2) and in $(\nu, 2)$.

We now prove that the estimates are sharp. We start with a duality result whose proof is standard:

Lemma 5.1. *Assume $1 \le p \le 2 \le q \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following are equivalent:*

(i) $\|\operatorname{proj}_n(h_\kappa^2; f)\|_{k,2} \le A\|f\|_{k,p}$,
(ii) $\|\operatorname{proj}_n(h_\kappa^2; f)\|_{k,q} \le A\|f\|_{k,2}$.

To prove the sharpness of the estimates, we can assume without loss of generality that $\kappa_{\min} = \kappa_1$. For the case in Theorem 3.3 (i) we define

$$f_n(x) := P_n(h_\kappa^2; x, e), \qquad e = (1,0,0,\dots,0).$$

Since $f_n \in \mathcal{H}_n^{d+1}(h_\kappa^2)$, we have

$$\|\operatorname{proj}_n(h_\kappa^2; f_n)\|_{\kappa,q} = \|f_n\|_{\kappa,q} = \left(\int_{S^d} |P_n(h_\kappa^2; x, e)|^q h_\kappa^2(x) d\omega(x)\right)^{1/q}.$$

Thus, it is sufficient to show that, for n sufficiently large,

$$\|f_n\|_{k,q} \sim n^{\sigma_k - \frac{2\sigma_\kappa+1}{q}} \|f_n\|_{k,2} \qquad \text{for } q \ge \frac{2(\sigma_\kappa+1)}{\sigma_\kappa}. \tag{5.1}$$

Indeed, setting $p = q/(q-1)$ and using Lemma 5.1, (5.1) shows that

$$\|\operatorname{proj}_n(h_\kappa^2; f_n)\|_{\kappa,2} \sim c\, n^{\sigma_k - \frac{2\sigma_\kappa+1}{q}} \|f_n\|_{\kappa,p} = c\, n^{(2\sigma_k+1)\left(\frac{1}{p} - \frac{\sigma_\kappa+1}{2\sigma_k+1}\right)} \|f_n\|_{\kappa,p},$$

which proves the sharpness of (i).

Recall that $C_n^{(\lambda,\mu)}(t)$ denotes the generalized Gegenbauer polynomial. It is connected to C_n^λ by an integral formula, which implies by (1.5) that

$$P_n(h_\kappa^2; x, e) = \frac{n+\lambda_\kappa}{\lambda_k} C_n^{(\sigma_\kappa,\kappa_1)}(x_1).$$

Hence, using (2.11), in terms of Jacobi polynomials we have

$$P_{2n}(h_\kappa^2; x, e) = \mathcal{O}(1) n^{\sigma_\kappa+\frac{1}{2}} P_n^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(2x_1^2-1). \tag{5.2}$$

Since this is a function that only depends on x_1, a standard change of variables leads to

$$\begin{aligned}
\|f_{2n}\|_{\kappa,q} &\sim n^{\sigma_\kappa+\frac{1}{2}} \left(\int_0^\pi |P_n^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(2\cos^2\theta-1)|^q |\cos\theta|^{2\kappa_1}(\sin\theta)^{2\sigma_\kappa} d\theta\right)^{\frac{1}{q}} \\
&\sim n^{\sigma_\kappa+\frac{1}{2}} \left(\int_{-1}^1 |P_n^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(t)|^q w^{(\sigma_k-\frac{1}{2},\kappa_1-\frac{1}{2})}(t) dt\right)^{1/q} \\
&\sim n^{\sigma_\kappa} n^{\sigma_k - \frac{2\sigma_k+1}{q}},
\end{aligned}$$

where in the last step we have used (2.19) and the condition $q \ge 2(\sigma_k+1)/\sigma_\kappa > (2\sigma_k+1)/\sigma_\kappa$ to conclude that the integral on $[0,1]$ has the stated estimate, whereas the integral over $[-1,0]$, using $P_n^{(\alpha,\beta)}(t) = P_n^{(\beta,\alpha)}(-t)$, has an order dominated by the integral on $[0,1]$. For $q=2$, using (5.2), we get

$$\|f_{2n}\|_{\kappa,2} = \left(P_{2n}(h_\kappa^2; e, e)\right)^{\frac{1}{2}} \sim n^{\sigma_\kappa}.$$

Together, these two relations establish (5.1) for even n. The proof for odd n is similar. $\square$

Remark 5.1. For the ordinary spherical harmonics, the sharpness of part (ii) in Theorem 3.3 was proved in [14] with the help of the function $(x_1 + ix_2)^n$. For h-harmonics, it is then natural to consider the function

$$F_n(x) := V_\kappa[(x_1 + ix_2)^n], \qquad x \in \mathbb{R}^d,$$

where V_κ is the intertwining operator associated with h_κ^2 and $\mathbb{Z}_2^d$. Since the Dunkl operator commutes with V_κ, so is the h-Laplacian, which leads to $\Delta_h V_\kappa\,[(x_1 + ix_2)^n] = V_\kappa\,[\Delta(x_1 + ix_2)^n] = 0$, proving that $F_n(x)$ is an h-harmonic of degree n. Furthermore, since V_κ is a product form, it follows from [10, Prop. 5.6.10] that

$$\begin{aligned} F_n(x) &= a_n(x_1^2 + x_2^2)^{n/2} \\ &\times \left[\frac{n + 2\kappa_2 + \delta_n}{2\kappa_2 + 2\kappa_1} C_n^{(\kappa_2,\kappa_1)} \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right) + ix_2 C_{n-1}^{(\kappa_2+1,\kappa_1)} \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right) \right], \end{aligned}$$

where a_n is a constant given explicitly in [10] and $\delta_n = 2\kappa_2$ if n is even and $\delta_n = 0$ if n is odd. This explicit formula allows us to compute the norm $\|F_n\|_{\kappa,p}$ explicitly. For example, using [10, Lemma 3.8.9], we have immediately that

$$\int_{S^d} |F_n(x)|^q h_\kappa^2(x) d\omega(x) = \int_{B^2} |F_n(x)|^q |x_1|^{2\kappa_1} |x_2|^{2\kappa_2} (1 - \|x\|^2)^{|\kappa| - \kappa_1 - \kappa_2 + \frac{d-3}{2}} dx_1 dx_2,$$

where, since F_n depends only on (x_1, x_2), we have abused notation somewhat by using $F_n(x)$ in the right-hand side as well. The above integral can then be evaluated, by (2.11) and in polar coordinates, by using (2.19). The result, however, does not yield the sharpness of (ii) in Theorem 3.3 when $\kappa \neq 0$.

6. Boundedness of Cesàro means

In this section we prove Theorems 3.1 and 3.2. By a standard duality argument, it suffices to prove these theorems for $1 \le p \le 2$. We shall assume $1 \le p \le \nu := \frac{2\sigma_\kappa + 2}{\sigma_\kappa + 2}$ and $\delta > \delta_\kappa(p)$ for the rest of this section.

6.1. **Proof of Theorem 3.1.**
We follow essentially the approach of [15], although there are still several difficulties that need to be overcome.

6.1.1. *Decomposition.* Let $\varphi_0 \in C^\infty[0, \infty)$ be such that $\chi_{[0,1]} \le \varphi_0 \le \chi_{[0,2]}$, and let $\varphi(t) := \varphi_0(t) - \varphi_0(2t)$. Clearly, φ is a C^∞-function supported in $(\frac{1}{2}, 2)$ and satisfying $\sum_{v=0}^\infty \varphi(2^v t) = 1$ for all $t > 0$. Setting

$$\widehat{S}_{n,v}^\delta(j) := \varphi\Big(\frac{2^v(n-j)}{n}\Big) \frac{A_{n-j}^\delta}{A_n^\delta}, \tag{6.1}$$

we define

$$S_{n,v}^\delta f := \sum_{j=0}^n \widehat{S}_{n,v}^\delta(j)\, \mathrm{proj}_j(h_\kappa^2; f), \qquad v = 0, 1, \cdots, \lfloor \log_2 n \rfloor + 2.$$

Since $\sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} \varphi\left(\frac{2^v(n-j)}{n}\right) = 1$ for $0 \le j \le n-1$, it follows that the Cesàro means are decomposed as

$$S_n^\delta(h_\kappa^2; f) = \sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} S_{n,v}^\delta f + \frac{1}{A_n^\delta}\, \mathrm{proj}_n(h_\kappa^2; f). \tag{6.2}$$

Using Theorem 3.3 and the fact that $\delta > \delta_\kappa(p)$, we have

$$\frac{1}{A_n^\delta}\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,p} \le cn^{-\delta}\|\operatorname{proj}_n(h_\kappa^2; f)\|_{\kappa,2} \le cn^{\delta_\kappa(p)-\delta}\|f\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

On the other hand, using summation by parts $\ell \ge 1$ times shows that

$$S_{n,v}^\delta f = \sum_{j=0}^n \Delta^\ell \left(\widehat{S}_{n,v}^\delta(j)\right) A_j^{\ell-1} S_j^{\ell-1}(h_\kappa^2; f),$$

where Δ denotes the forward difference and $\Delta^{\ell+1} := \Delta\Delta^\ell$. Since $\widehat{S}_{n,v}^\delta(j) = 0$ whenever $n - j > \frac{n}{2^{v-1}}$ or $n - j < \frac{n}{2^{v+1}}$, it is easy to verify by the Leibniz rule that

$$\left|\Delta^\ell(S_{n,v}^\delta(j))\right| \le c2^{-v\delta}\left(\frac{2^v}{n}\right)^\ell, \qquad \forall \ell \in \mathbb{N},\ 0 \le j \le n. \tag{6.3}$$

Hence, choosing $\ell > \lambda_\kappa$ and using the fact that $S_n^\ell(h_\kappa^2; f)$ is bounded in $L^p(h_\kappa^2; S^d)$ for all $1 \le p \le \infty$ if $\ell > \lambda_\kappa$, we conclude that for $v = 0$ and 1,

$$\|S_{n,v}^\delta f\|_{\kappa,p} \le cn^{-\ell}\sum_{j=0}^n j^{\ell-1}\|S_j^{\ell-1}(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

Therefore, by (6.2), it is sufficient to prove that

$$\|S_{n,v}^\delta(f)\|_{\kappa,p} \le c2^{-v\varepsilon_0}\|f\|_{\kappa,p}, \quad v = 2, \cdots, \lfloor \log_2 n\rfloor + 2, \tag{6.4}$$

where ε_0 is a sufficiently small positive constant depending on δ and p, but independent of n and v.

6.1.2. *Estimate of the kernel of $S_{n,v}^\delta$.* Let

$$D_{n,v}^\delta(t) := \sum_{j=0}^n \widehat{S}_{n,v}^\delta(j)\frac{\lambda_\kappa + j}{\lambda_\kappa}C_j^{\lambda_\kappa}(t).$$

The definition shows that $S_{n,v}^\delta f = f *_\kappa D_{n,v}^\delta$, so that the kernel of $S_{n,v}^\delta f$ is defined by

$$K_{n,v}^\delta(x, y) := V_\kappa\Big[D_{n,v}^\delta(\langle x, \cdot\rangle)\Big](y).$$

Lemma 6.1. *Let $2 \le v \le \lfloor \log_2 n\rfloor + 2$. Then for any given positive integer ℓ,*

$$|K_{n,v}^\delta(x, y)|h_\kappa^2(y) \le cn^d 2^{v(\ell-1-\delta)}\left(1 + nd(\bar{x}, \bar{y})\right)^{-\ell-d+\lambda_\kappa+1},$$

where $\bar{z} = (|z_1|, \cdots, |z_{d+1}|)$ for $z = (z_1, \cdots, z_{d+1}) \in \mathbb{R}^{d+1}$.

Proof. We first define a sequence of functions $\{a_{n,v,\ell}(\cdot)\}_{\ell=0}^\infty$ by

$$a_{n,v,0}(j) = 2(j + \lambda_\kappa)\widehat{S}_{n,v}^\delta(j),$$
$$a_{n,v,\ell+1}(j) = \frac{a_{n,v,\ell}(j)}{2j + 2\lambda_\kappa + \ell} - \frac{a_{n,v,\ell}(j+1)}{2j + 2\lambda_\kappa + \ell + 2}, \qquad \ell \ge 0.$$

Following the proof of Lemma 3.3 of [4, pp. 413–414], we can write, for any integer $\ell \ge 0$,

$$D_{n,v}^\delta(t) = c_\kappa\sum_{j=0}^\infty a_{n,v,\ell}(j)\frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + \lambda_\kappa + \frac{1}{2})}P_j^{(\lambda_\kappa+\ell-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(t), \tag{6.5}$$

so that

$$K_{n,v}^{\delta}(x,y) = c_\kappa \sum_{j=0}^{\infty} a_{n,v,\ell}(j) \frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac{1}{2})} V_\kappa \left[P_j^{(\lambda_\kappa+\ell-\frac{1}{2},\lambda_\kappa-\frac{1}{2})}(\langle x,\cdot\rangle)\right](y).$$

Note that $a_{n,v,\ell}(j) = 0$ if $j+\ell \le (1-\frac{1}{2^{v-1}})n$ or $j \ge (1-\frac{1}{2^{v+1}})n$, so that the sum is over $j \sim n$. Furthermore, it follows from the definition, (6.3) and Leibniz rule that

$$\left|\triangle^i a_{n,v,\ell}(j)\right| \le c2^{-v\delta} n^{-\ell+1} \left(\frac{2^v}{n}\right)^{i+\ell}, \qquad i,\ell = 0,1,\cdots. \tag{6.6}$$

Consequently, using the pointwise estimate of (2.20), it follows by (6.6) that

$$\begin{aligned} |K_{n,v}^{\delta}(x,y)| &\le cn^{2\lambda_\kappa+2\ell-1-2|\kappa|} \sum_{\substack{j\sim n \\ n-j\sim \frac{n}{2^v}}} |a_{n,v,\ell}(j)| \frac{\prod_{i=1}^{d+1}(|x_iy_i| + n^{-1}d(\bar x,\bar y) + n^{-2})^{-\kappa_i}}{(1+nd(\bar x,\bar y))^{\lambda_\kappa+\ell-|\kappa|}} \\ &\le cn^d 2^{v(\ell-1-\delta)} \frac{\prod_{j=1}^{d+1}(|x_jy_j| + n^{-1}d(\bar x,\bar y) + n^{-2})^{-\kappa_j}}{(1+nd(\bar x,\bar y))^{\lambda_\kappa+\ell-|\kappa|}} \\ &\le cn^d 2^{v(\ell-1-\delta)} h_\kappa^{-2}(y)\,(1+nd(\bar x,\bar y))^{\lambda_\kappa-d+1-\ell}, \end{aligned}$$

where in the last inequality we have used the fact that

$$\prod_{j=1}^{d+1}(|x_jy_j| + n^{-1}d(\bar x,\bar y) + n^{-2})^{-\kappa_j} \le ch_\kappa^{-2}(y) d(\bar x,\bar y)^{|\kappa|},$$

which follows since if $|y_j| \ge 2d(\bar x,\bar y)$, then $|\bar x_j - \bar y_j| \le d(\bar x,\bar y) \le |y_j|/2$ so that $|y_j|^2 \le 2|x_jy_j|$, whereas if $|y_j| < 2d(\bar x,\bar y)$, then $|y_j|^2 \le 2(n^{-1}d(\bar x,\bar y))\cdot nd(\bar x,\bar y)$. This completes the proof of Lemma 6.1. □

Corollary 6.2. *For any $\gamma > 0$ there exists an $\varepsilon_0 > 0$ independent of n and v such that*

$$\sup_{x\in S^d} \int_{\{y:\ d(\bar x,\bar y) > 2^{(1+\gamma)v}/n\}} |K_{n,v}^{\delta}(x,y)|\, h_\kappa^2(y)\, d\omega(y) \le c2^{-v\varepsilon_0}.$$

Proof. Invoking Lemma 6.1 with $\ell > \lambda_\kappa + 1 + \frac{\lambda_\kappa-\delta}{\gamma}$, we see that the quantity to be estimated is bounded by

$$\begin{aligned} c \sup_{x\in S^d} n^d 2^{v(\ell-1-\delta)} &\int_{\{y:\ d(\bar x,\bar y)>2^{(1+\gamma)v}/n\}} \frac{1}{(1+nd(\bar x,\bar y))^{\ell+d-\lambda_k-1}} d\omega(y) \\ &\le c2^{v(\ell-1-\delta)} \int_{2^{(1+\gamma)v}/n}^{\pi} \frac{n(n\theta)^{d-1}}{(1+n\theta)^{\ell+d-\lambda_k-1}} d\theta \\ &\le c2^{v(\ell-1-\delta-(1+\gamma)(\ell-\lambda_\kappa-1))} = c2^{-v\varepsilon_0}, \end{aligned}$$

which proves the corollary. □

6.1.3. *Proof of* (6.4). Now we are in a position to prove (6.4). Recall that

$$S_{n,v}^{\delta} f = \sum_{(1-2^{-v+1})n\le j\le (1-2^{-v-1})n} \widehat{S}_{n,v}^{\delta}(j)\, \mathrm{proj}_j(h_\kappa^2; f). \tag{6.7}$$

Assume $\delta > \delta_\kappa(p)$, and let $\gamma > 0$ be sufficiently small so that $\delta > \delta_\kappa(p) + \gamma\left(\delta_\kappa(p)+\frac{1}{2}\right)$. Set $v_1 = v(1+\gamma)$. Let Λ be a maximal $\frac{2^{v_1}}{n}$-separable subset of S^d;

that is, $\min_{\varpi\neq\varpi'\in\Lambda} d(\varpi,\varpi')\geq \frac{2^{v_1}}{n}$ and $S^d\subset\bigcup_{\varpi\in\Lambda} c(\varpi,\frac{2^{v_1}}{n})$. Define

$$f_\varpi(x):=f(x)\chi_{c(\varpi,\frac{2^{v_1}}{n})}(x)[A(x)]^{-1},\qquad A(x):=\sum_{\varpi\in\Lambda}\chi_{c(\varpi,\frac{2^{v_1}}{n})}(x).$$

Then evidently $1\leq A(x)\leq c$, $x\in S^d$, $|f_\varpi|\leq c|f|$, and $f(x)=\sum_{\varpi\in\Lambda} f_\varpi(x)$. Using the Minkowski inequality, we obtain

$$\|S^\delta_{n,v}(f)\|_{\kappa,p}\leq\sum_{\varpi\in\Lambda}\|S^\delta_{n,v}(f_\varpi)\|_{\kappa,p}.$$

Thus, it is sufficient to show that for each $\varpi\in\Lambda$, we have

$$\|S^\delta_{n,v}(f_\varpi)\|_{\kappa,p}\leq c2^{-v\varepsilon_0}\|f_\varpi\|_{\kappa,p}. \tag{6.8}$$

To this end, we denote by $c^*(\varpi,2^{v_1+1}/n)$ the set

$$c^*(\varpi,\tfrac{2^{v_1+1}}{n})=\left\{x\in S^d:\ d(\bar{x},\bar{\varpi})\leq 2^{v_1+1}/n\right\}$$

and further define $J(v,n):=\{j:(1-2^{-v+1})n\leq j\leq(1-2^{-v-1})n\}$. Using (6.7) and orthogonality, we obtain

$$\|S^\delta_{n,v}(f_\varpi)\|_{\kappa,2}=\Big(\sum_{j\in J(v,n)}|\widehat{S}^\delta_{n,v}(j)|^2\|\operatorname{proj}_j(h_\kappa^2;f_\varpi)\|^2_{\kappa,2}\Big)^{\frac12}.$$

Hence, by Hölder's inequality, Theorem 3.4 and (2.17), and (6.3) with $\ell=0$,

$$\begin{aligned}
&\Big(\int_{c^*(\varpi,2^{v_1+1}/n)}|S^\delta_{n,v}(f_\varpi)(x)|^p\,h_\kappa^2(x)\,d\omega(x)\Big)^{\frac1p}\\
&\quad\leq c\Big(\int_{c(\varpi,2^{v_1+1}/n)}h_\kappa^2(x)\,d\omega(x)\Big)^{\frac1p-\frac12}\Big(\sum_{j\in J(v,n)}|\widehat{S}^\delta_{n,v}(j)|^2\|\operatorname{proj}_j(h_\kappa^2;f_\varpi)\|^2_{\kappa,2}\Big)^{\frac12}\\
&\quad\leq c2^{v_1(\delta_\kappa(p)+\frac12)}n^{-\frac12}\Big(\sum_{j\in J(v,n)}|\widehat{S}^\delta_{n,v}(j)|^2\Big)^{\frac12}\|f_\varpi\|_{\kappa,p}\\
&\quad\leq c2^{-v\left(\delta-\delta_\kappa(p)-\gamma(\delta_\kappa(p)+\frac12)\right)}\|f_\varpi\|_{\kappa,p}=c2^{-v\varepsilon_0}\|f_\varpi\|_{\kappa,p}.
\end{aligned}$$

Finally, using Hölder's inequality, we obtain, for $x\notin c^*(\varpi,2^{v_1+1}/n)$,

$$\begin{aligned}
|S^\delta_{n,v}(f_\varpi)(x)|^p&=\Big|\int_{\{y:d(\varpi,y)\leq 2^{v_1}/n\}}f_\varpi(y)K^\delta_{n,v}(x,y)h_\kappa^2(y)\,d\omega(y)\Big|^p\\
&\leq\Big(\int_{\{y:\ d(\bar{y},\bar{x})\geq 2^{v_1}/n\}}|f_\varpi(y)|^p|K^\delta_{n,v}(x,y)|h_\kappa^2(y)\,d\omega(y)\Big)\\
&\qquad\times\Big(\int_{\{y:\ d(\bar{y},\bar{x})\geq 2^{v_1}/n\}}|K^\delta_{n,v}(x,y)|h_\kappa^2(y)\,d\omega(y)\Big)^{p-1},
\end{aligned}$$

which, together with Corollary 6.2, implies

$$\begin{aligned}
&\Big(\int_{S^d\setminus c^*(\varpi,2^{v_1+1}/n)}|S^\delta_{n,v}(f_\varpi)(x)|^p\,h_\kappa^2(x)\,d\omega(x)\Big)^{\frac1p}\\
&\quad\leq c(2^{-v\varepsilon_0})^{1-\frac1p}\sup_{y\in S^d}\Big(\int_{\{x:\ d(\bar{x},\bar{y})\geq 2^{v_1}/n\}}|K^\delta_{n,v}(x,y)|h_\kappa^2(x)\,d\omega(x)\Big)^{\frac1p}\|f_\varpi\|_{\kappa,p}\\
&\quad\leq c2^{-v\varepsilon_0}\|f_\varpi\|_{\kappa,p}.
\end{aligned}$$

Putting the above together, we deduce the desired estimate (6.8), hence (6.4), and complete the proof of Theorem 3.1. $\square$

6.2. **Proof of Theorem 3.2.**

6.2.1. *Main body of the proof.* For the proof of Theorem 3.1 we follow the approach in [2], which can be traced back to [13]. We start with the following lemma.

Lemma 6.3. *If Q is a polynomial of degree n on $\mathbb{R}^{d+1}$, then for $1 \le p < \infty$,*

$$\|Q\|_\infty := \max_{x \in S^d} |Q(x)| \le c n^{(2\sigma_\kappa+1)/p} \|Q\|_{\kappa,p}.$$

Proof. Let $S_n(h_\kappa^2; f)$ denote the partial sum operator of the h-harmonic expansion. Then $S_n(h_\kappa^2; Q) = Q$. The kernel of $S_n(h_\kappa^2; f)$ is

$$K_n(h_\kappa^2; x, y) = \sum_{k=0}^{n} P_k(h_\kappa^2; x, y), \qquad P_k(h_\kappa^2; x, y) = \sum_j Y_j^k(x) Y_j^k(y),$$

where $\{Y_j^k\}_j$ forms an orthonormal basis of $\mathcal{H}_k^{d+1}(h_\kappa^2)$. If $y \in S^d$ and $|y_\ell| = \max_{1 \le j \le d+1} y_j$, then $|y_\ell| \ge 1/\sqrt{d+1}$. Hence, the pointwise estimate of (2.21) implies that $|P_n(h_\kappa^2; x, x)| \le c n^{2\sigma_\kappa}$. By the Cauchy-Schwarz inequality, we obtain

$$|K_n(h_\kappa^2; x, y)| \le \sum_{k=0}^{n} P_k(h_\kappa^2; x, x)^{\frac12} P_k(h_\kappa^2; y, y)^{\frac12} \le c \sum_{k=0}^{n} k^{2\sigma_\kappa} \le c n^{2\sigma_\kappa+1}.$$

Consequently, we conclude that

$$\|Q\|_\infty = \|S_n(h_\kappa^2; Q)\|_\infty = \left\| a_\kappa \int_{S^d} Q(y) K_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega \right\|_\infty \le c n^{2\sigma_\kappa+1} \|Q\|_{\kappa,1}.$$

Furthermore, we clearly have $\|Q\|_\infty \le \|Q\|_\infty$, and the case $1 < p < \infty$ follows immediately from interpolation of these two cases. □

Proof of Theorem 3.2. Our main objective is to show that

$$\sup_{n \in \mathbb{N}} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c \|f\|_{k,p} \tag{6.9}$$

does not hold if $1 \le p \le \frac{2\sigma_\kappa+1}{\sigma_\kappa+\delta+1}$ or $p \ge \frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta}$. Let

$$p_1 := \frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta} \qquad \text{and} \qquad q_1 := \frac{p_1}{p_1-1} = \frac{2\sigma_\kappa+1}{\sigma_k+1+\delta}.$$

It is sufficient to prove that (6.9) does not hold for p_1, since it then follows from the Riesz-Thorin convexity theorem that (6.9) fails for $p_1 \le p \le \infty$ and the fact that (6.9) fails for $1 \le p \le q_1$ follows by duality.

Let $e \in S^d$ be fixed. Define a linear functional $T_n^\delta : L^p \mapsto \mathbb{R}$ by

$$T_n^\delta f := S_n^\delta(h_\kappa^2; f, e) = a_\kappa \int_{S^d} f(x) K_n^\delta(h_\kappa^2; x, e) h_\kappa^2(x) d\omega(x).$$

Since this is an integral operator, a standard argument shows that

$$\|T_n^\delta\|_{\kappa,p} = \|K_n^\delta(h_\kappa^2; x, e)\|_{\kappa,q}, \qquad \frac1p + \frac1q = 1,$$

where $\|T_n^\delta\|_{\kappa,p} = \sup_{\|f\|_{\kappa,p}=1} |T_n^\delta f|$. On the other hand, by Lemma 6.3, if (6.9) holds, then we will have

$$\begin{aligned} |T_n^\delta f| = |S_n^\delta(h_\kappa^2; f, e)| &\le \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,\infty} \\ &\le c n^{(2\sigma_\kappa+1)/p} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c n^{(2\sigma_\kappa+1)/p} \|f\|_{\kappa,p}. \end{aligned}$$

Consequently, the above two equations show that we will have

$$\|K_n^\delta(h_\kappa^2;\cdot,e)\|_{\kappa,q} \le cn^{(2\sigma_\kappa+1)/p}, \qquad \frac{1}{p}+\frac{1}{q}=1. \tag{6.10}$$

To complete the proof of Theorem 3.2, we show that (6.10) does not hold for $p=p_1$.

For this purpose we use the explicit formula for $K_n^\delta(h_\kappa^2;x,e)$ for $e=e_j$ (see, for example, [11]), where $e_1,\dots,e_{d+1}$ denote the usual coordinate vectors in $\mathbb{R}^{d+1}$ and

$$K_n^\delta(h_\kappa^2;x,e_j) := K_n^\delta(w_{\lambda_\kappa-\kappa_j,\kappa_j};1,x_j),$$

where $K_n^\delta(w_{\lambda,\mu};s,t)$ denotes the (C,δ) kernel of the generalized Gegenbauer polynomials with respect to the weight function (2.10). Hence, we have

$$\int_{S^d}\left|K_n^\delta(h_\kappa^2;x,e_j)\right|^q h_\kappa^2(x)d\omega(x) = c\int_{-1}^1\left|K_n^\delta(w_{\lambda_\kappa-\kappa_j,\kappa_j};1,t)\right|^q w_{\lambda_\kappa-\kappa_j,\kappa_j}(t)dt.$$

Consequently, choosing j such that $\kappa_j=\kappa_{\min}$, we see that the proof of Theorem 3.2 follows by applying Proposition 6.4 below to $\sigma=\sigma_\kappa$ and $\mu=\kappa_{\min}$. $\square$

Proposition 6.4. *Let $w_{\sigma,\mu}$ be the weight function in* (2.10) *and $\sigma\ge\mu\ge 0$. Define*

$$\Phi_{n,q}^\delta(w_{\sigma,\mu},s) := \int_{-1}^1 |K_n^\delta(w_{\sigma,\mu};s,t)|^q w_{\sigma,\mu}(t)dt.$$

Then for $q_1=\frac{2\sigma+1}{\sigma+1+\delta}$ and $p_1=\frac{2\sigma+1}{\sigma-\delta}$,

$$\Phi_{n,q_1}^\delta(w_{\sigma,\mu},1) \ge \Phi_{n,q_1}^\delta(w_{\mu,\sigma},0) \ge cn^{(2\sigma+1)q_1/p_1}\log n.$$

The proof of this proposition is given in the following subsection.

6.2.2. *Proof of Proposition* 6.4. The case of $q=1$ and $\delta=\sigma$ has already been established in [8, 11]. We follow the approach in [8] and briefly sketch the proof.

Using the sufficiency part of Corollary 3.7, we can follow exactly the deduction in [11, p. 293] to obtain

$$\begin{aligned}&\Phi_{n,q_1}^\delta(w_{\sigma,\mu};1) \ge \Phi_{n,q_1}^\delta(w_{\mu,\sigma};0) = cn^{(\sigma+\mu-\delta+\frac12)q_1}\\ &\quad\times\int_0^1\left|\int_{-1}^1 P_n^{(\sigma+\mu+\delta+\frac12,\sigma+\mu-\frac12)}(st)(1-s^2)^{\mu-1}ds\right|^{q_1} t^{2\mu}(1-t^2)^{\sigma-\frac12}dt+\mathcal{O}(1).\end{aligned}$$

As a result, we see that Proposition 6.4 is a consequence of the lower bound of a double integral of the Jacobi polynomial given in the next proposition.

Proposition 6.5. *Assume $\sigma,\mu\ge 0$ and $0\le\delta\le\sigma+\mu$. Let $a=\sigma+\mu+\delta$, $b=\sigma+\mu-1$ and $q_1=\frac{2\sigma+1}{\sigma+1+\delta}$. Then*

$$\begin{aligned}\int_0^1\left|\int_{-1}^1 P_n^{(a+\frac12,b+\frac12)}(st)(1-s^2)^{\mu-1}ds\right|^{q_1} t^{2\mu}(1-t^2)^{\sigma-1/2}dt&\\ \ge c\,n^{-(\mu+1/2)q_1}\log n.&\end{aligned} \tag{6.11}$$

Proof. Denote the left-hand side of (6.11) by I_{n,q_1}. First assume that $0<\mu<1$. Following the proof of [8], we can conclude that

$$I_{n,q_1} \ge cn^{-q_1/2}\int_{n^{-1}}^{\pi/4}|M_n(\phi)|^{q_1}(\sin\phi)^{2\sigma}d\phi - \mathcal{O}(1)E_{n,q_1},$$

where

$$E_{n,q_1} := n^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \left[\int_{\phi}^{\pi-\phi} \frac{(\cos^2\phi - \cos^2\theta)^{\mu-1}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}} d\theta \right]^{q_1} (\sin\phi)^{2\sigma} d\phi$$

and $M_n(\phi) := K_n(\phi) + G_n(\phi)$ satisfies

$$K_n(\phi) \geq cn^{-\mu}\phi^{-\sigma-\delta-1}\left(1 + \cos(2N\phi + 2\gamma)\right), \qquad n^{-1} \leq \phi \leq \varepsilon,$$

for a sufficiently small absolute constant $\varepsilon > 0$, where $N = n + \frac{a+b}{2} + 1$ and $\gamma = -\frac{\pi}{2}(a + 1 - \mu)$, and

$$|G_n(\phi)| \leq cn^{-1}\phi^{\mu-\sigma-\delta-2}, \qquad n^{-1} \leq \phi < \pi/4.$$

From these estimates and the fact that $q_1(\sigma + 1 + \delta) = 2\sigma + 1$, it follows that

$$\begin{aligned}
&\int_{n^{-1}}^{\varepsilon} |K_n(\phi)|^{q_1} (\sin\phi)^{2\sigma} d\phi \geq cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{2\sigma - q_1(\sigma+1+\delta)}(1 + \cos(2N\phi + 2\gamma))^{q_1} d\phi \\
&= cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{-1}(1 + \cos(2N\phi + 2\gamma))^{q_1} d\phi \geq cn^{-\mu q_1} \log n,
\end{aligned}$$

where in the last step we used $(1+A)^{q_1} \geq 1 + q_1 A$ for $A \in [-1, 1]$ and the fact that $\int_{n^{-1}}^{\varepsilon} \phi^{-1}\cos(2N\phi + 2\gamma))d\phi \leq c$ upon using integration by parts once. Furthermore,

$$\begin{aligned}
\int_{n^{-1}}^{\varepsilon} |G_n(\phi)|^{q_1} (\sin\phi)^{2\sigma} d\phi &\leq cn^{-q_1} \int_{n^{-1}}^{\varepsilon} \phi^{2\sigma + q_1(\mu-\sigma-\delta-2)} d\phi \\
&= cn^{-q_1} \int_{n^{-1}}^{\varepsilon} \phi^{q_1(\mu-1)-1} d\phi \leq cn^{-q_1\mu}.
\end{aligned}$$

Together, these estimates yield that for $0 < \mu < 1$,

$$\int_{n^{-1}}^{\pi/4} |M_n(\phi)|^{q_1} (\sin\phi)^{2\sigma} d\phi \geq cn^{-q_1\mu} \log n.$$

Moreover, the remainder E_{n,q_1} term can be estimated as follows:

$$\begin{aligned}
E_{n,q_1} &\leq cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \left[\int_{\phi}^{\pi/4} \theta^{\mu-a-2}(\theta - \phi)^{\mu-1} d\theta \right]^{q_1} \phi^{2\sigma} d\phi \\
&\leq cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \phi^{(\mu-\sigma-\delta-2)q_1 + 2\sigma} d\phi \\
&\leq cn^{-\frac{3}{2}q_1 - q_1(\mu-1)} = cn^{-(\mu+\frac{1}{2})q_1},
\end{aligned}$$

where in the second step, we divided the inner integral into two parts, over $[\phi, 2\phi]$ and over $[2\phi, \pi/2]$, respectively, to derive the stated estimate.

Putting these two terms together, we conclude the proof for the case $0 < \mu < 1$. The case $\mu = 1$ can be derived similarly upon taking an integration by parts for the inner integral in (6.11). The case $\mu > 1$ reduces to the case $0 < \mu < 1$ upon taking integration by parts $\lfloor \mu \rfloor$ times as in [8]. $\square$

7. Proof of theorems on the ball and on the simplex

7.1. **Proof of results on the unit ball.** Under the mapping $\phi : x \in B^d \mapsto (x, \sqrt{1-\|x\|^2}) \in S^d_+$ in (2.7), orthogonal polynomials with respect to W_κ^B in (2.6) can be deduced from h-spherical harmonics that are even in x_{d+1}. Moreover, the connection (2.9) shows that the (C,δ) means $S_n^\delta(W_\kappa^B; f)$ is related to $S_n^\delta(h_\kappa^2; F)$ by

$$S_n^\delta(W_\kappa^B; f, x) = S_n^\delta(h_\kappa^2; F, X), \quad X := (x, \sqrt{1-\|x\|^2}),$$

where $F(x, x_{d+1}) := f(x)$ for $x \in B^d$ and $(x, x_{d+1}) \in S^d$. Consequently, by (2.8), Theorem 3.5 with $\Omega = B$ follows immediately from Theorem 3.1.

The proof of Theorem 3.6 follows almost exactly as that of Theorem 3.2. We have in this case ([11, pp. 287-288])

$$\begin{aligned} K_n(W_\kappa^B; x, e_j) &= K_n^\delta(w_{\lambda_\kappa - \kappa_j, \kappa_j}; 1, x_j), \qquad 1 \le j \le d, \\ K_n(W_\kappa^B; x, 0) &= K_n^\delta(w_{k_{d+1}, \lambda_\kappa - \kappa_{d+1}}; \|x\|, 0). \end{aligned}$$

Hence, by (2.8), we can again reduce the proof of Theroem 3.6 to the lower bound of $\Phi_{n,q}^\delta(w_{\sigma,\mu}, 1)$ and $\Phi_{n,q}^\delta(w_{\mu,\sigma}, 0)$, which follows again from Proposition 6.4.

7.2. **Proof of results on the simplex.**

7.2.1. *Projection operator.* Under the mapping $\psi : x \in T^d \mapsto (x_1^2, \ldots, x_d^2) \in B^d$, orthogonal polynomials with respect to W_κ^T on T^d and those with respect to W_κ^B on B^d are related. In particular, we have the connection between $\mathrm{proj}_n(W_\kappa^T; f)$ and $\mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi)$ given in (2.15), from which the result on the projection operators can be readily derived.

In fact, if $\|\mathrm{proj}_n(W_\kappa^B; f)\|_{W_\kappa^B, p} \le A_n \|f\|_{W_\kappa^B;p}$, then by (2.15) and (2.14),

$$\begin{aligned} \|\mathrm{proj}_n(W_\kappa^T; f)\|_{W_\kappa^T;p} &= \|\mathrm{proj}_n(W_\kappa^T; f) \circ \psi\|_{W_\kappa^B;p} \\ &= \frac{1}{2^d} \Big\| \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi, \cdot \varepsilon) \Big\|_{W_\kappa^B;p} \\ &\le \|\mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi)\|_{W_\kappa^B;p} \\ &\le A_{2n} \|f \circ \psi\|_{W_\kappa^B;p} = A_{2n} \|f\|_{W_\kappa^T;p}, \end{aligned}$$

from which Theorem 3.8 for $\Omega = T$ follows immediately from the case $\Omega = B$. Furthermore, since the distance $d_T(x,y)$ on T^d is related to the geodesic distance on S^d by

$$d_T(\psi(x), \psi(y)) = d(X, Y), \qquad X = \left(x, \sqrt{1-\|x\|^2}\right), \quad Y = \left(y, \sqrt{1-\|y\|^2}\right),$$

from (2.8) and (2.14) it follows readily that

$$\int_{c_T(x,\theta)} W_\kappa^T(x) dx = \int_{c(X,\theta)} h_\kappa^2(y) d\omega(y),$$

where $X = \left(\sqrt{x_1}, \cdots, \sqrt{x_d}, \sqrt{1-|x|}\right)$. Consequently, we conclude that Theorem 3.9 follows from Theorem 3.4.

7.2.2. *Cesàro means.* Since the connection (2.15) relates the projection operator of degree n for W_κ^T to the projection operator of degree $2n$ for W_κ^B, we cannot deduce results for the Cesàro means $S_n^\delta(W_\kappa^T;f)$ from those of $S_n^\delta(W_\kappa^B;f)$ directly. We can, however, follow the proof of the theorems for the h-harmonics. Below we give a brief outline on how this will work.

Proof of Theorem 3.5 *with* $\Omega = T$. We follow the decomposition in the subsection 6.1.1 to define

$$S_{n,v}^\delta(W_\kappa^T;f) = \sum_{j=0}^n \widehat{S}_{n,v}^\delta(j)\operatorname{proj}_j(W_\kappa^T;f), \qquad v = 1,2,\ldots,\lfloor \log_2 n\rfloor + 2.$$

The same argument shows that it suffices to prove the analogue of (6.4),

$$\|S_{n,v}^\delta(W_\kappa^T;f)\|_{W_\kappa^T,p} \le c2^{-v\varepsilon}\|f\|_{W_\kappa^T;p}, \qquad v = 2,\ldots,\lfloor \log_2 n\rfloor + 2. \tag{7.1}$$

Denote the kernel of $S_{n,v}^\delta(W_\kappa^T;f)$ by $K_{n,v}^\delta(W_\kappa^T;x,y)$. Then we have by (2.16) that

$$K_{n,v}^\delta(W_\kappa^T;x,y) := c_\kappa \int_{[-1,1]^{d+1}} D_{n,v}^\delta(W_\kappa^T; 2z(x,y,t)^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}dt,$$

where

$$D_{n,v}^\delta(W_\kappa^T;t) := \sum_{j=0}^n \widehat{S}_{n,v}^\delta(j)\frac{(2j+\lambda_\kappa)\Gamma(\frac12)\Gamma(j+\lambda_k)}{\Gamma(\lambda_\kappa+1)\Gamma(j+\frac12)}P_j^{(\lambda_k-\frac12,\frac12)}(t).$$

Consequently, defining analogues of $a_{n,v,\ell}$ by

$$\begin{aligned} a_{n,v,0}^T(j) &= (2j+\lambda_k)\widehat{S}_{n,v}^\delta(j),\\ a_{n,v,\ell+1}^T(j) &= \frac{a_{n,v,\ell}(j)}{2j+2\lambda_\kappa+\ell} - \frac{a_{n,v,\ell}(j+1)}{2j+2\lambda_\kappa+\ell+2},\end{aligned}$$

we can then write, again following the proof of Lemma 3.3 of [4], that

$$D_{n,v}^\delta(W_\kappa^T;t) = c\sum_{j=0}^n a_{n,v,\ell}^T(j)\frac{\Gamma(j+2\lambda_\kappa+\ell)}{\Gamma(j+\lambda_\kappa+\frac12)}P_j^{(\lambda_k+\ell-\frac12,-\frac12)}(t).$$

The estimate (6.6) holds exactly for $a_{n,v,\ell}(j)$. Thus, to follow the proof of Lemma 6.1, we need to estimate

$$\int_{[-1,1]^{d+1}} P_j^{(\lambda_k+\ell-\frac12,-\frac12)}(2z(x,y,t)^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}dt. \tag{7.2}$$

Using the fact that $P_n^{(\alpha,-1/2)}(2t^2-1)$ can be written in terms of $P_n^{(\alpha,\alpha)}$ ([17, (4.1.5)]), we can estimate (7.2) again by Lemma 2.20. The result is

$$|K_{n,v}^\delta(W_\kappa^T;x,y)|\left[W_\kappa^T(y)\right]^{-1} \le cn^d 2^{v(\ell-1-\delta)}\left(1+nd(\bar{x},\bar{y})\right)^{-\lambda_\kappa-\ell+d-1},$$

which implies that the analogue of Corollary 6.2 holds; that is, for any $\gamma>0$ there is an $\varepsilon_0>0$ such that

$$\sup_{x\in T^d}\int_{\{y:d_T(x,y)\ge 2^{(1+v)\gamma}/n\}} |K_{n,v}(W_\kappa^T;x,y)|W_\kappa^T(y)dy \le c2^{-v\varepsilon_0}.$$

In order to prove (7.1), we then define Λ to be a maximal separate subset of T^d exactly as the one we defined in Subsection 6.1.3, except with $d(\varpi, \varpi')$ replaced by $d_T(x, x')$. Define

$$f_y(x) := f(x)\chi_{c_T(y, \frac{2v_1}{n})}(x)[A(x)]^{-1}, \qquad A(x) = \sum_{y\in\Lambda} \chi_{c_T(y, \frac{2v_1}{n})}(x).$$

Then the same argument shows that it suffices to show that

$$\|S^\delta_{n,v}(W^T_\kappa; f_y)\|_{W^T_\kappa, p} \le c2^{-v\varepsilon_0}\|f_y\|_{W^T_\kappa; p}.$$

This last inequality can be established exactly as in (6.8) and there is no need to introduce the additional set $c^*(\varpi, \theta)$. □

Proof of Theorem 3.6. We first note that the analogue of Lemma 6.3 holds; that is, for $1 \le p < \infty$,

$$\|Q\|_\infty := \max_{x\in T^d} |Q(x)| \le cn^{(2\sigma_\kappa+1)/p}\|Q\|_{W^T_\kappa, p}$$

for any polynomial of degree n on $\mathbb{R}^d$. Furthermore, following the proof of Theorem 3.2, it is sufficient to prove that

$$\|K^\delta_n(W^T_\kappa; \cdot, y)\|_{\kappa,q} \le cn^{(2\sigma_\kappa+1)/p}, \qquad \tfrac{1}{p} + \tfrac{1}{q} = 1, \tag{7.3}$$

where $y \in T^d$ is fixed, does not hold for $p = p_1 := \frac{2\sigma_\kappa+1}{\sigma_\kappa-\delta}$. To proceed, we then express $K^\delta_n(W^T_\kappa)$ in terms of the kernel for the Jacobi polynomial expansions ([11, p. 290])

$$K^\delta_n\left(W^T_\kappa; x, e_j\right) = K^\delta_n\left(w^{(\lambda_\kappa-\kappa_j-\frac{1}{2}, \kappa_j-\frac{1}{2})}; 1, 2x_j - 1\right), \quad 1 \le j \le d,$$
$$K^\delta_n(W^T_\kappa; x, 0) = K^\delta_n\left(w^{(\lambda_\kappa-\kappa_j-\frac{1}{2}, \kappa_j-\frac{1}{2})}; 1, 1-2|x|\right),$$

from which we can deduce by changing variables that

$$\int_{T^d} \left|K^\delta_n\left(W^T_\kappa; e_j, y\right)\right|^q W^T_\kappa(y)dy$$
$$= c\int_{-1}^1 \left|K^\delta_n\left(w^{(\lambda_\kappa-\kappa_j-\frac{1}{2}, \kappa_j-\frac{1}{2})}; 1, t\right)\right|^q w^{(\lambda_\kappa-\kappa_j-\frac{1}{2}, \kappa_j-\frac{1}{2})}(t)dt$$

for $1 \le j \le d$, and also for $j = d+1$ if we agree that $e_{d+1} = 0$. As a result, we have reduced the problem to that of Jacobi polynomial expansions, so that the desired result follows from [6]. □

References

[1] G. Andrews, R. Askey, and R. Roy, *Special functions*, Cambridge Univ. Press, Cambridge, 1999. MR1688958 (2000g:33001)

[2] R. Askey and I. I. Hirschman, Jr., Mean summability for ultraspherical polynomials, *Math Scand.* **12** (1963), 167-177. MR0164198 (29:1497)

[3] A. Bonami and J-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183** (1973), 223–263. MR0338697 (49:3461)

[4] G. Brown and F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, *J. Funct. Anal.* **220** (2005), no. 2, 401–423. MR2119285 (2005m:41054)

[5] A. P. Calderón and A. Zygmund, On a problem of Mihlin, *Trans. Amer. Math. Soc.* **78** (1955), 209-224. MR0068028 (16:816f)

[6] S. Chanillo and B. Muckenhoupt, Weak type estimates for Cesàro sums of Jacobi polynomial series, *Mem. Amer. Math. Soc.* **102**(487) 1993. MR1132609 (93g:42018)

[7] F. Dai, Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights, *J. Funct. Anal.* **235** (2006), no. 1, 137–170. MR2216443 (2007f:41010)
[8] Feng Dai and Yuan Xu, Cesàro means of orthogonal expansions in several variables, *Construct. Approx.*, accepted for publication.
[9] C. F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183. MR951883 (90k:33027)
[10] C. F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Cambridge Univ. Press, Cambridge, 2001. MR1827871 (2002m:33001)
[11] Zh.-K. Li and Yuan Xu, Summability of orthogonal expansions of several variables, *J. Approx. Theory* **122** (2003), 267-333. MR1988305 (2004e:42041)
[12] G. Mastroianni and V. Totik, Weighted polynomial inequalities with doubling and A_∞ weights, *Constr. Approx.* **16** (2000), no. 1, 37–71. MR1848841 (2002j:26010)
[13] J. Newman and W. Rudin, Mean convergence of orthogonal series, *Proc. Amer. Math. Soc.* **3** (1952), 219-222. MR0047811 (13:936b)
[14] C. D. Sogge, Oscillatory integrals and spherical harmonics, *Duke Math. J.* **53**(1986),43–65. MR835795 (87g:42026)
[15] C. D. Sogge. On the convergence of Riesz means on compact manifolds, *Ann. of Math. (2)***126** (1987), 439–447. MR908154 (89b:35126)
[16] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1971. MR0290095 (44:7280)
[17] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. Vol. 23, Providence, 4th edition, 1975.
[18] K. Y. Wang and L. Q. Li, *Harmonic Analysis and Approximation on the Unit Sphere*, Sciencepress, Beijing, 2000.
[19] Yuan Xu, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, *Proc. Amer. Math. Soc.* **125** (1997), 2963–2973. MR1402890 (97m:33004)
[20] Yuan Xu, Weighted approximation of functions on the unit sphere, *Constr. Approx.* **21** (2005), 1-28. MR2105389 (2006c:42027)
[21] H. Zhou, Divergence of Cesàro means of spherical h-harmonic expansions, *J. Approx. Theory* **147** (2007), 215–220. MR2344321 (2008g:42028)

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