Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 143 (2006) 135–149
Journal of Approximation Theory
www.elsevier.com/locate/jat

An extrapolation theorem for nonlinear approximation and its applications ☆

Feng Dai

Department of Mathematical and Statistical Sciences, University of Alberta, CAB 632, Edmonton, Alta., Canada T6G 2G1

Received 19 November 2004; accepted 6 April 2006

Communicated by V.N. Temlyakov
Available online 5 June 2006

Abstract

We prove an extrapolation theorem for the nonlinear m-term approximation with respect to a system of functions satisfying very mild conditions. This theorem allows us to prove endpoint $L^p - L^q$ estimates in nonlinear approximation. As a consequence, some known endpoint estimates can be deduced directly and some new estimates are also obtained. Finally, applications of these new estimates are given to spherical m-widths and m-term approximation of the weighted Besov classes.
© 2006 Elsevier Inc. All rights reserved.

MSC: Primary 41A46; 41A10

Keywords: Nonlinear m-term approximation; Extrapolation; Spherical widths; Weighted Besov classes

1. Introduction

In this Introduction, we shall describe our main results with a minimum of definitions. We refer to the survey articles [4,15] for the background information on nonlinear approximation. We also refer to the recent impressive paper [16] by Temlyakov for the motivations of the problems considered in this paper.

Let $\{\phi_j\}_{j=0}^{\infty}$ be a sequence of L_∞-functions on a probability space $(\Omega, \mathcal{F}, dm)$. Given an integer $n \geqslant 0$, we put

$$\Pi_n = \left\{ \sum_{j=0}^{n} c_j \phi_j : c_0, c_1, \ldots, c_n \in \mathbb{C} \right\},$$

☆ The work was supported in part by the NSERC Canada under Grant G121211001.
E-mail address: dfeng@math.ualberta.ca.

0021-9045/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2006.04.001

$$\Sigma_n = \left\{ \sum_{k=1}^{n} c_k \phi_{j_k} : 0 \leqslant j_1 < j_2 < \cdots < j_n, c_1, c_2, \ldots, c_n \in \mathbb{C} \right\},$$

and define

$$\begin{aligned} B_p^n &= \left\{ f \in \Pi_n : \|f\|_p \leqslant 1 \right\}, \quad 1 \leqslant p \leqslant \infty, \\ \sigma_0(f)_p &:= \|f\|_p, \quad \sigma_n(f)_p := \inf_{g \in \Sigma_n} \|f-g\|_p, \quad n \geqslant 1 \ \text{ for } f \in L_p(\Omega), \\ \sigma_n(\mathcal{B})_p &= \sup_{f \in \mathcal{B}} \sigma_n(f)_p \quad \text{for a function class } \mathcal{B} \subset L_p(\Omega). \end{aligned}$$

We assume that the following condition is satisfied:

(A) There exists a sequence of linear operators V_n on $L_1(\Omega)$ such that $V_n(f) \in \Pi_{2n}$ for $f \in L_1(\Omega)$, $V_n(f) = f$ for $f \in \Pi_n$, $V_n(\Sigma_v) \subset \Sigma_v$ for $v = 1, 2, \ldots,$ and

$$\sup_n \|V_n(f)\|_p \leqslant K_1 \|f\|_p \quad \text{for any } f \in L_p(\Omega), \quad p = 1, \infty \ \ (K_1 > 1).$$

We point out that a condition similar to condition (A) was previously used in many papers (see [1,16,15,13]).

One of our main purposes in this paper is to show the following extrapolation theorem.

Theorem 1.1. (i) *For* $1 \leqslant r_1 \leqslant r_2 \leqslant p \leqslant \infty$ *and* $1 \leqslant m \leqslant n$,

$$\sigma_m(B_{r_2}^n)_p \leqslant 2K_1 \left(\sigma_m(B_{r_1}^{2n})_p \right)^{(\frac{1}{r_2} - \frac{1}{p})/(\frac{1}{r_1} - \frac{1}{p})}.$$

(ii) *For* $1 \leqslant r < p \leqslant q \leqslant \infty$ *and* $1 \leqslant m \leqslant n$, *we have*

$$\sigma_m(B_r^n)_q \leqslant C \left(\widetilde{\sigma}_m(B_r^n)_p \right)^{\frac{p(q-r)}{q(p-r)}}$$

provided that

$$\sup_{N \geqslant 1} N^{-\alpha} \sigma_0(B_r^N)_q \leqslant K_2 \quad \textit{for some } \alpha \geqslant 0,$$

and that $\left\{ \widetilde{\sigma}_j(B_r^N)_p : 0 \leqslant j \leqslant N < \infty \right\}$ *is a sequence of positive numbers satisfying*

$$\sigma_j(B_r^N)_p \leqslant \widetilde{\sigma}_j(B_r^N)_p, \quad 0 \leqslant j \leqslant N < \infty$$

and

$$\widetilde{\sigma}_{[\frac{j}{2}]}(B_r^{2N})_p \leqslant K_3 \widetilde{\sigma}_j(B_r^N)_p, \quad 0 \leqslant j \leqslant N < \infty \ \ (K_3 > 1),$$

where

$$C = \max\left\{ (2^{\alpha+1} K_1)^{\frac{p}{p-r}} K_3^{\left(\frac{p}{p-r}\right)^2}, K_2 \right\}$$

and we define

$$\frac{p(q-r)}{q(p-r)} = \begin{cases} \dfrac{p}{p-r} & \textit{if } p < q = \infty, \\ 1 & \textit{if } p = q = \infty. \end{cases}$$

Remarks. 1. As an immediate application of Theorem 1.1, let us consider the system $\{\phi_k\}_{k=0}^{\infty} = \{e^{ikx}\}_{k=0}^{\infty}$ of exponential functions on the unit circle $\mathbb{T}$. For this system, it had been known for a long time (see [10]) that for $2 \leqslant p < \infty$ and $1 \leqslant m \leqslant n$,

$$\sigma_m(B_2^n)_p \leqslant C\sqrt{p}\left(\frac{n}{m}\right)^{\frac{1}{2}},$$

with $C > 0$ an absolute constant. Using this estimate, and invoking Theorem 1.1(ii) with $r = 2$, $p = 3 + \log \frac{n}{m}$ and $q = \infty$, we obtain

$$\sigma_m(B_2^n)_\infty \leqslant C' \left(C\sqrt{p}\left(\frac{n}{m}\right)^{\frac{1}{2}}\right)^{\frac{p}{p-2}} \leqslant C'' \left(\frac{n}{m}\right)^{\frac{1}{2}} \left(1 + \log\frac{n}{m}\right)^{\frac{1}{2}}, \tag{1.1}$$

with $C'' > 0$ an absolute constant.

2. The inequality (1.1) is a consequence of a much stronger result obtained by DeVore and Temlyakov [5] in 1995 (see also Remark 3). Let Ω denote a topological space equipped with a finite measure $d\gamma$ and let $\{\phi_1, \ldots, \phi_N\}$ be a set of continuous functions on Ω satisfying the following two conditions:

(i) $\max_{1 \leqslant j \leqslant N} \|\phi_j\|_{L_\infty(d\gamma)} \leqslant K_1$.

(ii) There exist a constant K_2 and a set of points $x_j \in \Omega$, $j = 1, \ldots, M$, such that for each function $P \in \operatorname{span}\{\phi_j : 1 \leqslant j \leqslant N\}$, we have

$$\|P\|_{L_\infty(d\gamma)} \leqslant K_2 \max_{1 \leqslant j \leqslant M} |P(x_j)|.$$

Under the above assumptions (i) and (ii), the following remarkable inequality was proved by DeVore and Temlyakov [5, Theorem 3.1] in 1995:

$$\sigma_m(\mathcal{A}_N)_\infty \leqslant C K_1 K_2 m^{-1/2} \log^{1/2}(1 + M/m), \tag{1.2}$$

where

$$\mathcal{A}_N = \left\{ \sum_{j=1}^{N} c(j)\phi_j, \sum_{j=1}^{N} |c(j)| \leqslant 1 \right\}$$

and

$$\sigma_m(\mathcal{A}_N)_\infty = \sup_{f \in \mathcal{A}_N} \inf_{\substack{c_{j_1}, \ldots, c_{j_m} \in \mathbb{C} \\ 1 \leqslant j_1 < \cdots < j_m \leqslant N}} \left\| f - \sum_{k=1}^{m} c_{j_k}\phi_{j_k} \right\|_{L_\infty(d\gamma)}.$$

Note that condition (A) is not assumed for the validity of (1.2). Using inequality (1.2), DeVore and Temlyakov [5, Corollary 5.1] further proved the following general estimates:

$$\sigma_m(\mathcal{A}_\theta(\mathcal{T}_n))_\infty \leqslant \begin{cases} C m^{\frac{1}{2} - \frac{1}{\theta}} \log^{\frac{1}{2}}(1 + n^d/m) & \text{if } 0 < \theta \leqslant 1, \\ C n^{d - \frac{d}{\theta}} m^{-\frac{1}{2}} \log^{\frac{1}{2}}(1 + n^d/m) & \text{if } 1 < \theta \leqslant \infty, \end{cases} \tag{1.3}$$

where $1 \leqslant m \leqslant (2n+1)^d$,

$$\mathcal{A}_\theta(\mathcal{T}_n) := \left\{ \sum_{k\in\mathbb{Z}^d, |k|_\infty \leqslant n} c_k e^{ik\cdot x} : \|(c_k)\|_{\ell_\theta} \leqslant 1 \right\},$$

and $|k|_\infty = \max\{|k_1|, \ldots, |k_d|\}$ for $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. Though our proof of the inequality (1.1) is more direct, we are unable to apply Theorem 1.1 to deduce the estimates (1.3) for $\theta < 2$. Another interesting proof of inequality (1.1) was given by Belinskii [1] in 1998.

3. The following interesting result was proved by Temlyakov [14, Theorem 4.2] in 1998: for all $1 \leqslant p < \infty$ and $1 \leqslant m \leqslant n$,

$$C_1 n^{1/p} \max\left\{m^{-\frac{1}{p}}, m^{-\frac{1}{2}}\right\} \leqslant \sigma_m(B_p^n)_\infty \leqslant C_2 n^{1/p} \max\left\{m^{-\frac{1}{p}}, m^{-\frac{1}{2}}\right\} \ln \frac{3n}{m}, \tag{1.4}$$

where C_1, C_2 are two absolute positive constants, B_p^n, $1 \leqslant p \leqslant \infty$ denotes the unit L_p-ball in the space of trigonometric polynomials of degree at most n on the circle $\mathbb{T}$, and σ_m is defined with respect to the system $\{e^{ikx}\}_{k=0}^\infty$ of exponential functions on $\mathbb{T}$. We are unable to invoke Theorem 1.1 to deduce the upper estimates in (1.4) for $1 \leqslant p < 2$. Moreover, our method in this paper, in general, does not yield desired lower estimates of σ_m.

A more general application of Theorem 1.1 will yield the following:

Theorem 1.2. *Let $\{\phi_j\}_{j=0}^\infty$ be a sequence of orthonormal functions on* $(\Omega, \mathcal{F}, dm)$ *satisfying* (A) *and the following condition: for some $2 < p_0 < \infty$ and $\beta, \gamma \geqslant 0$,*

$$\|\phi_j\|_p \leqslant \begin{cases} K_4\left(1 + (p_0 - p)^{-\beta}\right) & \text{if } 2 \leqslant p < p_0, \\ K_5 (j+1)^\gamma & \text{if } p = \infty, \end{cases} \qquad j \geqslant 0, \tag{1.5}$$

where K_4, K_5 are independent of j, p, p_0. Then for $2 \leqslant p < q \leqslant \infty$ and $1 \leqslant m \leqslant n$,

$$\sigma_m(B_p^n)_q \leqslant \begin{cases} C_1\left(\sqrt{q}(1 + (p_0 - q)^{-\beta})\right)^{\frac{2(q-p)}{(q-2)p}} \left(\frac{n}{m}\right)^{\frac{q-p}{(q-2)p}} & \text{if } 2 < q < p_0, \\ C_2 p_0^{\frac{1}{p}} \left(\frac{n}{m}\log^{2\beta}\left(1 + \frac{n}{m}\right)\right)^{\frac{p_0(q-p)}{pq(p_0-2)}} & \text{if } p_0 \leqslant q \leqslant \infty \text{ and } p_0 < 3 + \log\frac{n}{m}, \\ C_3\left(\frac{n}{m}\log\left(1 + \frac{n}{m}\right)\right)^{\frac{1}{p}} & \text{if } p_0 \leqslant q \leqslant \infty \text{ and } p_0 \geqslant 3 + \log\frac{n}{m}, \end{cases}$$

where $\frac{q-p}{q-2} = \frac{q-p}{q} = 1$ for $q = \infty$, C_1, C_3 are independent of m, n, p, q, p_0, and C_2 is a constant which is independent of m, n, p, q and which is uniformly bounded as p_0 is bounded away from 2.

The point in Theorem 1.2 is that we do not need to assume $\sup_j \|\phi_j\|_\infty < \infty$. One typical example of orthonormal functions satisfying all the conditions in Theorem 1.2 is the system of normalized ultraspherical polynomials $P_k^\mu(t)\|P_k^\mu\|_2^{-1}$, $\mu > 0$, $k \in \mathbb{Z}_+$ (see [12] for precise definition), where p_0, β, γ can be taken to be $2 + \frac{1}{\mu}$, $\frac{\mu}{2\mu+1}$, μ, respectively (for the proof of this fact, see Section 4.2 of this paper).

We organize the paper as follows. Section 2 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is given in Section 3. In Section 4, the final section, we apply Theorem 1.2 in spherical m-widths and m-term approximation of the weighted Besov classes by the system of ultraspherical polynomials.

2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we need the following lemmas.

Lemma 2.1. *For $u > 0$, $1 \leqslant r \leqslant p \leqslant q \leqslant \infty$ and $f \in L_p(\Omega)$, there is a decomposition $f = f_1 + f_2$, such that*

$$\|f_1\|_q \leqslant u^{\frac{1}{p}-\frac{1}{q}} \|f\|_p,$$
$$\|f_2\|_r \leqslant u^{\frac{1}{p}-\frac{1}{r}} \|f\|_p,$$

where we define $\frac{1}{\infty} = 0$.

Lemma 2.1 can be easily obtained by setting

$$f_1(t) = \begin{cases} f(t) & \text{if } |f(t)| \leqslant u^{\frac{1}{p}} \|f\|_p, \\ \dfrac{u^{\frac{1}{p}} \|f\|_p f(t)}{|f(t)|} & \text{if } |f(t)| > u^{\frac{1}{p}} \|f\|_p, \end{cases}$$

and $f_2 = f - f_1$. We omit the details.

Lemma 2.2. *For $1 \leqslant m \leqslant n$ and $1 \leqslant r < p \leqslant q \leqslant \infty$,*

$$\sigma_m(B_r^n)_q \leqslant 2K_1 \left(\sigma_{[m/2]}(B_r^{2n})_q\right)^t \sigma_{[m/2]}(B_r^n)_p,$$

where

$$t = \left(\frac{1}{p} - \frac{1}{q}\right) \Big/ \left(\frac{1}{r} - \frac{1}{q}\right).$$

Proof. For $f \in B_r^n$, there exists a $T_1 \in \Sigma_{[m/2]}$ such that

$$\|f - T_1\|_p \leqslant (1+\varepsilon)\sigma_{[m/2]}(B_r^n)_p,$$

where $\varepsilon > 0$ is a sufficiently small number. Using Lemma 2.1, we have, for $u > 0$,

$$f - T_1 = f_1 + f_2, \tag{2.1}$$

where

$$\|f_1\|_q \leqslant u^{\frac{1}{p}-\frac{1}{q}} \|f - T_1\|_p \leqslant (1+\varepsilon) u^{\frac{1}{p}-\frac{1}{q}} \sigma_{[m/2]}(B_r^n)_p,$$
$$\|f_2\|_r \leqslant u^{\frac{1}{p}-\frac{1}{r}} \|f - T_1\|_p \leqslant (1+\varepsilon) u^{\frac{1}{p}-\frac{1}{r}} \sigma_{[m/2]}(B_r^n)_p. \tag{2.2}$$

For the function $V_n(f_2)$, there exists a $T_2 \in \Sigma_{[m/2]}$ such that

$$\begin{aligned} \|V_n(f_2) - T_2\|_q &\leqslant (1+\varepsilon)\sigma_{[m/2]}(B_r^{2n})_q \|V_n(f_2)\|_r \\ &\leqslant K_1(1+\varepsilon)^2 u^{\frac{1}{p}-\frac{1}{r}} \sigma_{[m/2]}(B_r^{2n})_q \sigma_{[m/2]}(B_r^n)_p. \end{aligned} \tag{2.3}$$

Now let $T = V_n(T_1) + T_2$. Then $T \in \Sigma_m$ and by (2.1)–(2.3), it follows that

$$\begin{aligned} \sigma_m(f)_q &\leqslant \|f - T\|_q = \|V_n(f - T_1) - T_2\|_q \leqslant \|V_n(f_1)\|_q + \|V_n(f_2) - T_2\|_q \\ &\leqslant K_1 \left((1+\varepsilon)u^{\frac{1}{p}-\frac{1}{q}} + (1+\varepsilon)^2 u^{\frac{1}{p}-\frac{1}{r}} \sigma_{[m/2]}(B_r^{2n})_q\right) \sigma_{[m/2]}(B_r^n)_p. \end{aligned}$$

Setting

$$u=\left(\sigma_{[m/2]}(B_r^{2n})_q\right)^{\frac{1}{\frac{1}{r}-\frac{1}{q}}},$$

and letting $\varepsilon \to 0$, we obtain

$$\sigma_m(f)_q \leqslant 2K_1\left(\sigma_{[m/2]}(B_r^{2n})_q\right)^t \sigma_{[m/2]}(B_r^n)_p$$

with $t=(\frac{1}{p}-\frac{1}{q})/(\frac{1}{r}-\frac{1}{q})$. Lemma 2.2 then follows by taking supremum over $f \in B_r^n$ on both sides of this last inequality. $\square$

Now we return to the proof of Theorem 1.1. We start with the proof of (i), which is simpler. For $f \in B_{r_2}^n$ and $u>0$, by Lemma 2.1, there is a decomposition

$$f=f_1+f_2 \tag{2.4}$$

such that

$$\|f_1\|_{r_1} \leqslant u^{\frac{1}{r_2}-\frac{1}{r_1}}, \quad \|f_2\|_p \leqslant u^{\frac{1}{r_2}-\frac{1}{p}}.$$

For $V_n(f_1)$, there exists a $T_m \in \Sigma_m$ such that

$$\|V_n(f_1)-T_m\|_p \leqslant (1+\varepsilon)\|V_n(f_1)\|_{r_1}\sigma_m(B_{r_1}^{2n})_p, \tag{2.5}$$

where $\varepsilon>0$ is sufficiently small. Now combining (2.4) with (2.5), we obtain

$$\begin{aligned}\sigma_m(f)_p &\leqslant \|f-T_m\|_p=\|V_n(f_1)+V_n(f_2)-T_m\|_p\\ &\leqslant \|V_n(f_1)-T_m\|_p+\|V_n(f_2)\|_p\\ &\leqslant K_1\left((1+\varepsilon)u^{\frac{1}{r_2}-\frac{1}{r_1}}\sigma_m(B_{r_1}^{2n})_p+u^{\frac{1}{r_2}-\frac{1}{p}}\right).\end{aligned}$$

Setting

$$u=\left(\sigma_m(B_{r_1}^{2n})_p\right)^{\frac{1}{\frac{1}{r_1}-\frac{1}{p}}}$$

and letting $\varepsilon \to 0$, we get

$$\sigma_m(f)_p \leqslant 2K_1\left(\sigma_m(B_{r_1}^{2n})_p\right)^{(\frac{1}{r_2}-\frac{1}{p})/(\frac{1}{r_1}-\frac{1}{p})}.$$

Since f is an arbitrary element from $B_{r_2}^n$, the conclusion (i) then follows.

To show the conclusion (ii), we let $t=(\frac{1}{p}-\frac{1}{q})/(\frac{1}{r}-\frac{1}{q})$ and

$$C=\max\left\{(2^{\alpha+1}K_1)^{\frac{p}{p-r}}K_3^{(\frac{p}{p-r})^2}, K_2\right\}.$$

We then claim that for any integer $k \geqslant 0$, the inequality

$$\sigma_m(B_r^n)_q \leqslant C\left(\widetilde{\sigma}_m(B_r^n)_p\right)^{\frac{1-t^k}{1-t}}(n+1)^{\alpha t^k}, \quad 1 \leqslant m \leqslant n<\infty, \tag{2.6}$$

holds, from which Theorem 1.1(ii) will follow by letting $k \to \infty$.

We prove the claim by induction on k. When $k = 0$, (2.6) is obvious since

$$\sigma_m(B_r^n)_q \leqslant \sigma_0(B_r^n)_q \leqslant K_2(n+1)^{\alpha}.$$

Next, assume the claim is true for some integer $k \geqslant 0$. Then by Lemma 2.2 and this assumption, it follows that

$$\begin{aligned}\sigma_m(B_r^n)_q &\leqslant 2K_1\left(\sigma_{[m/2]}(B_r^{2n})_q\right)^t \sigma_{[m/2]}(B_r^n)_p\\ &\leqslant 2K_1\left(C\left(\widetilde{\sigma}_{[m/2]}(B_r^{2n})_p\right)^{\frac{1-t^k}{1-t}}(n+1)^{\alpha t^k}\right)^t \widetilde{\sigma}_{[m/2]}(B_r^{2n})_p,\\ &\leqslant C\frac{2^{\alpha+1}K_1K_3^{\frac{1}{1-t}}}{C^{1-t}}(n+1)^{\alpha t^{k+1}}\left(\widetilde{\sigma}_m(B_r^n)_p\right)^{\frac{1-t^{k+1}}{1-t}}\\ &\leqslant C(n+1)^{\alpha t^{k+1}}\left(\widetilde{\sigma}_m(B_r^n)_p\right)^{\frac{1-t^{k+1}}{1-t}},\end{aligned}$$

proving the claim for $k+1$. This completes the proof.

3. Proof of Theorem 1.2

The proof relies on Theorem 1.1 and the following lemma.

Lemma 3.1. *For $2 \leqslant q < p_0$, $\Lambda \subset \mathbb{Z}_+$ with cardinality n, $1 \leqslant m \leqslant n$ and $f \in \operatorname{span}\{\phi_k : k \in \Lambda\}$, we have*

$$\inf_{\substack{c_{j_1},\ldots,c_{j_m}\in\mathbb{C}\\ \{j_1,\ldots,j_m\}\subset\Lambda}}\left\|f-\sum_{k=1}^m c_{j_k}\phi_{j_k}\right\|_q \leqslant C\sqrt{q}K_4\left(1+(p_0-q)^{-\beta}\right)\left(\frac{n}{m}\right)^{1/2}\|f\|_2, \tag{3.1}$$

where $C > 0$ is an absolute constant.

For the moment, we take this lemma for granted and proceed with the proof.

Lemma 3.1 implies that for $2 \leqslant q < p_0$,

$$\sigma_m(B_2^n)_q \leqslant C\sqrt{q}K_4\left(1+(p_0-q)^{-\beta}\right)\left(\frac{n}{m}\right)^{\frac{1}{2}}. \tag{3.2}$$

Therefore, according to Theorem 1.1(i), it will suffice to prove

$$\sigma_m(B_2^n)_q \leqslant \begin{cases} C_2 p_0^{\frac{1}{2}}\left(\frac{n}{m}\log^{2\beta}\left(1+\frac{n}{m}\right)\right)^{\frac{p_0(q-2)}{2q(p_0-2)}} & \text{if } p_0 \leqslant q \leqslant \infty \text{ and } p_0 < 3+\log\frac{n}{m},\\ C_3\left(\frac{n}{m}\log\left(1+\frac{n}{m}\right)\right)^{\frac{1}{2}} & \text{if } p_0 \leqslant q \leqslant \infty \text{ and } p_0 \geqslant 3+\log\frac{n}{m},\end{cases} \tag{3.3}$$

where C_3 is independent of m, n, p_0, q, and C_2 is a constant which is independent of m, n, q and which is uniformly bounded as p_0 is bounded away from 2.

To show (3.3), we take $2 < p_1 < p_0$ and use (3.2) and Theorem 1.1(ii) to obtain that for $p_0 \leqslant q \leqslant \infty$,

$$\sigma_m(B_2^n)_q \leqslant C'\left(\sqrt{p_1}\left(1+(p_0-p_1)^{-\beta}\right)\left(\frac{n}{m}\right)^{\frac{1}{2}}\right)^{\frac{p_1(q-2)}{q(p_1-2)}}.$$

Setting

$$p_1 = \begin{cases} p_0 - \min\left\{(p_0-2)/2, \left(1+\log\frac{n}{m}\right)^{-1}\right\} & \text{if } p_0 < 3+\log\frac{n}{m}, \\ 2.5+\log\frac{n}{m} & \text{if } p_0 \geqslant 3+\log\frac{n}{m}, \end{cases}$$

we deduce (3.3) by straightforward computation. The proof is then complete by assuming Lemma 3.1.

Now we return to the proof of Lemma 3.1. For simplicity, in the proof below, we shall use the notation $|A|$ to denote the cardinality of a finite set A.

Proof of Lemma 3.1. Let $\{r_j\}_{j=0}^{\infty}$ be independent ± 1-valued random variables with mean 0 on some probability space (X, P). We assume $\frac{n}{m} \sim 2^l$ and $f = \sum_{k\in\Lambda} c_k\phi_k$. We then rewrite f as

$$f(t) = \Phi(t, x) + T_{\Lambda(x)}(t), \tag{3.4}$$

where $t \in \Omega$, $x = (x_1, \ldots, x_l) \in X^l$ and

$$\Phi(t, x) := \sum_{j=1}^{l}\sum_{k\in\Lambda}(1-r_k(x_1))\ldots(1-r_k(x_{j-1}))r_k(x_j)c_k\phi_k(t), \tag{3.5}$$

$$T_{\Lambda(x)}(t) := \sum_{k\in\Lambda(x)}(1-r_k(x_1))\ldots(1-r_k(x_l))c_k\phi_k(t), \tag{3.6}$$

$$\Lambda(x) := \{k \in \Lambda : (1-r_k(x_1))\ldots(1-r_k(x_l)) \neq 0\}. \tag{3.7}$$

Here and below we will employ the slight abuse of notation that $(1-r_k(x_1))\ldots(1-r_k(x_{j-1})) = 1$ for $j = 1$.

It will be shown that there is a vector $x^* = (x_1^*, \ldots, x_l^*) \in X^l$ such that

$$|\Lambda(x^*)| \sim m \tag{3.8}$$

and

$$\|\Phi(\cdot, x^*)\|_q \leqslant CK_4\sqrt{q}\left(1+(p_0-q)^{-\beta}\right)2^{l/2}\|f\|_2, \tag{3.9}$$

which combined with (3.4) and (3.6) will give (3.1).

To see this, first, by (3.7), it follows that

$$|\Lambda(x)| = \frac{1}{2^l}\sum_{k\in\Lambda}(1-r_k(x_1))\ldots(1-r_k(x_l)),$$

and hence

$$\int_{X^l}|\Lambda(x)|P(dx_1)\ldots P(dx_l) = \frac{n}{2^l} \sim m. \tag{3.10}$$

Second, from (3.5), we have

$$\int_{X^l}\|\Phi(\cdot, x)\|_q P(dx_1)\ldots P(dx_l)$$
$$\leqslant \sum_{j=1}^{l}\int_{X^j}\left\|\sum_{k\in\Lambda}(1-r_k(x_1))\ldots(1-r_k(x_{j-1}))r_k(x_j)c_k\phi_k\right\|_q P(dx_1)\ldots P(dx_j)$$

$$
\begin{aligned}
&\leqslant \sum_{j=1}^{l} \int_{X^{j-1}} \left\| \left(\int_X \left| \sum_{k\in\Lambda} (1-r_k(x_1))\dots(1-r_k(x_{j-1}))r_k(x_j)c_k\phi_k \right|^q P(dx_j) \right)^{\frac{1}{q}} \right\|_q \\
&\quad \times P(dx_1)\dots P(dx_{j-1}) \\
&\leqslant C\sqrt{q} \sum_{j=1}^{l} \int_{X^{j-1}} \left\| \left(\sum_{k\in\Lambda} (1-r_k(x_1))^2\dots(1-r_k(x_{j-1}))^2 |c_k|^2 |\phi_k|^2 \right)^{\frac{1}{2}} \right\|_q \\
&\quad \times P(dx_1)\dots P(dx_{j-1}) \\
&\leqslant C\sqrt{q} \sum_{j=1}^{l} \int_{X^{j-1}} \left(\sum_{k\in\Lambda} (1-r_k(x_1))^2\dots(1-r_k(x_{j-1}))^2 |c_k|^2 \|\phi_k\|_q^2 \right)^{\frac{1}{2}} \\
&\quad \times P(dx_1)\dots P(dx_{j-1}),
\end{aligned}
$$

where the second inequality follows by Fubini's theorem and Hölder's inequality, the third by Khinchine's inequality, and the last by Minkowski's inequality. Hence, using (1.5) and integrating with respect to $x_1, \dots, x_{l-1}$, we obtain

$$
\begin{aligned}
&\int_{X^l} \|\Phi(\cdot, x)\|_q P(dx_1)\dots P(dx_l) \\
&\quad \leqslant CK_4\sqrt{q}\left(1 + (p_0 - q)^{-\beta}\right) \\
&\quad\quad \times \sum_{j=1}^{l} \int_{X^{j-1}} \left(\sum_{k\in\Lambda} |c_k|^2 (1-r_k(x_1))^2\dots(1-r_k(x_{j-1}))^2 \right)^{\frac{1}{2}} P(dx_1)\dots P(dx_{j-1}) \\
&\quad \leqslant CK_4\sqrt{q}\left(1 + (p_0 - q)^{-\beta}\right) 2^{l/2} \|f\|_2.
\end{aligned} \tag{3.11}
$$

Now combining (3.11) with (3.10), we conclude that there must be a $x^* = (x_1^*, \dots, x_l^*) \in X^l$ such that both (3.8) and (3.9) hold. The proof is therefore complete. □

4. Applications

4.1. Spherical m-widths for the Sobolev classes on the unit sphere

Let $\mathbb{S}^{d-1}$ denote the unit sphere of the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the usual rotation invariant measure $d\sigma(x)$ and let $\mathcal{H}_k$, $k \in \mathbb{Z}_+$, be the space of spherical harmonics of degree k on $\mathbb{S}^{d-1}$. Given $r > 0$, the Sobolev class W_p^r, $1 \leqslant p \leqslant \infty$, is defined to be the class of all functions f on $\mathbb{S}^{d-1}$ of the form

$$
f(x) = \int_{\mathbb{S}^{d-1}} g(y) F_r(x\cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \ \ \|g\|_{L_p(\mathbb{S}^{d-1})} \leqslant 1,
$$

where

$$
F_r(t) = \sum_{k=1}^{\infty} (k(k+d-2))^{-\frac{r}{2}} \left(k + \frac{d-2}{2}\right) P_k^{\frac{d-2}{2}}(t),
$$

and $P_k^{\frac{d-2}{2}}(t)$ denotes the usual ultraspherical polynomial of degree k normalized by $P_k^{\frac{d-2}{2}}(1) = \frac{\Gamma(k+d-2)}{\Gamma(d-2)\Gamma(k+1)}$.

For $1 \leqslant q \leqslant \infty$ and a function class $\mathcal{B} \subset L_q(\mathbb{S}^{d-1})$, we define the spherical m-width $d_m^{\mathrm{S}}(\mathcal{B}, L_q)$ by

$$d_m^{\mathrm{S}}(\mathcal{B}, L_q) := \inf_{\Lambda \in G_m} \sup_{f \in \mathcal{B}} \inf_{f_k \in \mathcal{H}_k, k \in \Lambda} \left\| f - \sum_{k \in \Lambda} f_k \right\|_q ,$$

where

$$G_m := \left\{ \Lambda \subset \mathbb{Z}_+ : \sum_{k \in \Lambda} k^{d-2} \leqslant m \right\}.$$

(k^{d-2} appears in the definition of G_m because of the fact that $\dim \mathcal{H}_k \sim k^{d-2}$.)

We point out that in the special case $d = 2$, d_m^{S} is the well-known trigonometric m-width, for which the orders of the Sobolev classes are completely known (see [10,9]). However, in the higher-dimensional case, it seems that so far very few investigations on d_m^{S} have been done. Our result in this subsection is the following:

Theorem 4.1. *For $r > \frac{d(d-1)}{2}$ and $2 \leqslant q \leqslant \infty$,*

$$d_m^{\mathrm{S}}(W_1^r, L^q) \asymp m^{-\frac{r}{d-1} + \frac{1}{2}}.$$

The proof of Theorem 4.1 is based on Theorem 1.2 with $\phi_k(t) = \| P_k^{\frac{d-2}{2}} \|_2^{-1} P_k^{\frac{d-2}{2}}(t)$, and follows the standard method (see [10,9]). The proof of the fact that the ultraspherical polynomials satisfy the hypothesis of Theorem 1.2 will be given in Section 4.2.

4.2. Approximation of weighted Besov classes by ultraspherical polynomials

First, we state the definition of ultraspherical polynomials and show that they satisfy the hypothesis of Theorem 1.2. For $\mu > 0$ which will be fixed throughout this subsection, we denote by $L_{p,\mu}$, $1 \leqslant p \leqslant \infty$, the space of all functions f on $[-1, 1]$ with

$$\infty > \|f\|_{p,\mu} := \begin{cases} \left(\int_{-1}^1 |f(t)|^p (1-t^2)^{\mu - \frac{1}{2}} \, dt \right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{t \in [-1,1]} |f(t)| & \text{if } p = \infty. \end{cases}$$

The ultraspherical polynomials $P_k^\mu(t)$, $k = 0, 1, \ldots,$ $t \in [-1, 1]$, are defined as usual via the generating function

$$(1 - 2tz + z^2)^{-\mu} = \sum_{k=0}^\infty P_k^\mu(t) z^k,$$

where $|z| < 1$, $|t| < 1$. For simplicity, for the rest of the paper, we set

$$\varphi_k^\mu(t) := \frac{P_k^\mu(t)}{\|P_k^\mu\|_{2,\mu}}.$$

It is known that φ_k^μ is an algebraic polynomial of degree k and $\{\varphi_k^\mu\}_{k=0}^\infty$ forms a complete orthonormal system for $L_{2,\mu}$. Moreover, it follows from [12, p. 80, (4.7.1); p. 81, (4.7.15); p. 169, (7.32.5)] that

$$|\varphi_k^\mu(t)| \leqslant C(\mu) \min\left\{(k+1)^\mu, (1-t^2)^{-\frac{\mu}{2}}\right\}. \tag{4.1}$$

By (4.1) and a straightforward calculation, we deduce

$$\|\varphi_k^\mu\|_{p,\mu} \leqslant \begin{cases} C(\mu)(p_0-p)^{-\frac{1}{p_0}} & \text{if } 2 \leqslant p < p_0 := 2 + \dfrac{1}{\mu}, \\ C(\mu)(k+1)^\mu & \text{if } p = \infty. \end{cases} \tag{4.2}$$

This means that condition (1.5) with $p_0 = 2 + \frac{1}{\mu}$, $\beta = \frac{\mu}{2\mu+1}$ and $\gamma = \mu$ is satisfied for φ_j^μ, $j = 0, 1, \ldots$. [1]

In order to show that condition (A) in Section 1 is satisfied, we have to state some known results on Cesàro summability of the ultraspherical expansions. For $f \in L_{1,\mu}$, the Cesàro means $C_N^\delta(f)$ of f of order $\delta > -1$ are defined as usual by

$$C_N^\delta(f)(x) = \sum_{k=0}^N \frac{A_{N-k}^\delta}{A_N^\delta} \langle f, \varphi_k^\mu \rangle \varphi_k^\mu(x), \quad x \in [-1, 1], \quad N = 1, 2, \ldots,$$

where

$$A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$$

and throughout this subsection,

$$\langle f, \varphi_k^\mu \rangle := \int_{-1}^1 f(t)\varphi_k^\mu(t)(1-t^2)^{\mu-\frac{1}{2}}\, dt.$$

It is well known that (see [12, p. 273]) if $\delta > \mu$ then for all $1 \leqslant p \leqslant \infty$ and all $f \in L_{p,\mu}$,

$$\sup_{N\in\mathbb{N}} \|C_N^\delta(f)\|_p \leqslant C(\mu, \delta)\|f\|_{p,\mu}. \tag{4.3}$$

Now we are in a position to show that condition (A) is satisfied for φ_j^μ, $j = 0, 1, \ldots$. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 1$ for $|t| \leqslant 1$, $0 < \eta(t) < 1$ for $1 < |t| < 2$, and $\eta(t) = 0$ for $|t| \geqslant 2$. Given $f \in L_{1,\mu}$, define

$$V_n(f) := \sum_{k=0}^\infty \eta\left(\frac{k}{n}\right) \langle f, \varphi_k^\mu \rangle \varphi_k^\mu, \quad n = 1, 2, \ldots . \tag{4.4}$$

---

[1] For $p \geqslant p_0 = 2 + \frac{1}{\mu}$, by [12, p. 391, Exercise 91], we have

$$\|\varphi_k^\mu\|_{p,\mu}^p \asymp \begin{cases} \log(k+1) & \text{if } p = p_0, \\ (k+1)^{\mu(p-p_0)} & \text{if } p > p_0. \end{cases}$$

Thus, $\sup_{k\in\mathbb{N}} \|\varphi_k^\mu\|_{p,\mu} < \infty$ if and only if $0 < p < p_0 = 2 + \frac{1}{\mu}$.

Let Π_n and Σ_n be as defined in Section 1 with $\{\phi_k\} = \{\varphi_k^\mu\}_{k=0}^\infty$. By (4.3) and a summation by parts, we obtain

$$\sup_{n\geqslant 1} \|V_n(f)\|_{p,\mu} \leqslant C\|f\|_{p,\mu}, \quad 1\leqslant p\leqslant\infty, \quad f\in L_{p,\mu},$$

with $C > 0$ independent of f. On the other hand, by the definition (4.4), it is obvious that $V_n(f) = f$ for $f \in \Pi_n$, and $V_n(\Sigma_v) \subset \Sigma_v$ for $v = 1, 2, \ldots$. Thus condition (A) in Section 1 is satisfied. We will keep the notations V_n and η for the rest of this subsection.

In summary, we have shown that the hypothesis of Theorem 1.2 is satisfied for the normalized ultraspherical polynomials φ_j^μ, $j = 0, 1, \ldots$.

Next, we give the definition of Besov classes. For $\varphi(x) = \sqrt{1-x^2}$ and an integer $r > 0$, the Ditzian–Totik K-functional $K_{r,\varphi}(f, t^r)_{p,\mu}$ is defined by

$$K_{r,\varphi}(f, t^r)_{p,\mu} := \inf\left\{\|f-g\|_{p,\mu} + t^r\|\varphi^r g^{(r)}\|_{p,\mu} : g^{(r-1)} \in A.C._{\mathrm{loc}}\right\},$$

where $g^{(r-1)} \in A.C._{\mathrm{loc}}$ means that g is $r-1$ times differentiable and $g^{(r-1)}$ is absolutely continuous in every $[c, d] \subset (-1, 1)$. As is well known (see [7, Section 6.1]), for $\mu \geqslant \frac{1}{2}$, $K_{r,\varphi}(f, t^r)_{p,\mu}$ are equivalent to the computable weighted Ditzian–Totik moduli $\omega_\varphi^r(f, t)_{w,p}$ with $w = (1-x^2)^{(\mu-\frac{1}{2})/p}$. For $\alpha > 0$, $1\leqslant\tau\leqslant\infty$ and $0 < s \leqslant\infty$, we define the Besov class $B_s^\alpha(L_{\tau,\mu})$ to be the class of all functions f on $[-1, 1]$ such that

$$|f|_{B_s^\alpha(L_{\tau,\mu})} := \left\|t^{-\alpha}K_{r,\varphi}(f, t^r)_{\tau,\mu}\right\|_{L_s([0,1],\frac{dt}{t})} \leqslant 1,$$

where $r = [\alpha] + 1$ and $\varphi(x) = \sqrt{1-x^2}$.

The Besov classes $B_s^\alpha(L_{\tau,\mu})$ can be characterized in terms of ultraspherical expansions. In fact, following the standard method, we have the following equivalence:

$$|f|_{B_s^\alpha(L_{\tau,\mu})} \approx \left\|\left\{2^{j\alpha}\|f - V_{2^j}(f)\|_{\tau,\mu}\right\}_{2^j\geqslant r}\right\|_{\ell_s(\mathbb{Z}_+)}.$$

Our purpose in this subsection is to consider the asymptotic orders of the m-term approximation of $B_s^\alpha(L_{\tau,\mu})$ by ultraspherical polynomials. We define $\sigma_m(f)_{p,\mu}$ and $\sigma_m(\mathcal{B})_{p,\mu}$ as in Section 1 with $\{\phi_k\}_{k=0}^\infty = \{\varphi_k^\mu\}_{k=0}^\infty$ and $L_p(\Omega) = L_{p,\mu}$. Our main result can be stated as follows.

Theorem 4.2. *Let* $0 < s\leqslant\infty$, $1\leqslant p, \tau\leqslant\infty$ *and let*

$$\alpha(p,\tau) := \begin{cases} (2\mu+1)(1/\tau - 1/p)_+ & \text{if } 1\leqslant\tau\leqslant p\leqslant 2 \text{ or } 1\leqslant p\leqslant\tau\leqslant\infty, \\ \max\left\{2\mu\left(\frac{1}{\tau}-\frac{1}{p}\right)+\frac{1}{2}, \frac{2\mu+1}{\tau}-\frac{2\mu}{p}\right\} & \text{otherwise.}\end{cases}$$

Then for $\alpha > \alpha(p,\tau)$, *we have*

$$\sigma_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu} \asymp \begin{cases} m^{-\alpha+(2\mu+1)(\frac{1}{\tau}-\frac{1}{p})} & \text{if } 1\leqslant\tau\leqslant p\leqslant 2, \\ m^{-\alpha+(2\mu+1)(\frac{1}{\tau}-\frac{1}{2})} & \text{if } 1\leqslant\tau\leqslant 2\leqslant p\leqslant\infty, \\ m^{-\alpha} & \text{if } s=\infty \text{ and } 2\leqslant p,\tau\leqslant\infty \end{cases}$$

and

$$\sigma_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu} \leqslant Cm^{-\alpha} \quad \text{if } 1\leqslant p\leqslant\tau\leqslant\infty \text{ and } p < 2,$$

where C and the constants of equivalency are dependent only on α, p, τ, s *and* μ.

It is interesting to compare the orders of $\sigma_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu}$ with the corresponding orders of Kolmogorov m-widths $d_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu}$ (for precise definition of Kolmogorov widths, we refer to [11,13]). Indeed, it was shown in [3] that for $\alpha > \alpha(p,\tau)$,

$$d_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu} \asymp \begin{cases} m^{-\alpha+(\frac{1}{\tau}-\frac{1}{p})_+} & \text{if } 1\leqslant p\leqslant \tau<\infty \text{ or } 1<\tau\leqslant p\leqslant 2,\\ m^{-\alpha+(\frac{1}{\tau}-\frac{1}{2})_+} & \text{if } 1\leqslant \tau\leqslant p\leqslant\infty \text{ and } p\geqslant 2. \end{cases}$$

Therefore, in most cases, the orders of $d_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu}$ are significantly less than those of $\sigma_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu}$. This is somewhat surprising since in the periodic case, for the usual Besov classes and the system $\{e^{ijx}\}_{j=0}^\infty$ of exponential functions, it was shown by DeVore and Temlyakov [5] that d_m and σ_m have the same orders as $m\to\infty$.

Now we return to the proof of Theorem 4.2.

Proof of Theorem 4.2. The proof of upper estimates is based on Theorem 1.2 with $\{\phi_k\}=\{\varphi_k^\mu\}$, and in fact, runs along the same lines as that of Theorem 6.4 of [5]. We omit the details.

For the proof of the lower estimates, we first consider the case $1\leqslant\tau\leqslant p\leqslant 2$, from which the case when $1\leqslant\tau\leqslant 2\leqslant p\leqslant\infty$ will follow by the inequality $\|\cdot\|_{p,\mu}\geqslant C\|\cdot\|_{2,\mu}$. Let

$$K_{2m,\eta}(t)=\sum_{k=0}^{4m}\eta\left(\frac{k}{2m}\right)\varphi_k^\mu(1)\varphi_k^\mu(t).$$

By the known estimates for the Cesàro kernels (see [2, Theorem 2.1]), it follows that

$$\|K_{2m,\eta}\|_{\tau,\mu}\leqslant C(\mu,\tau)m^{(2\mu+1)(1-\frac{1}{\tau})}.$$

So

$$\left(C'(\mu,\tau)\right)^{-1}m^{-\alpha+(2\mu+1)(\frac{1}{\tau}-1)}K_{2m,\eta}\in B_s^\alpha(L_{\tau,\mu}). \tag{4.5}$$

On the other hand, for any $t_m\in\Sigma_m$, by Nikolskii's inequality for ultraspherical expansions, we have

$$\begin{aligned}\|K_{2m,\eta}-t_m\|_{p,\mu} &\geqslant C\|K_{2m,\eta}-V_{4m}(t_m)\|_{p,\mu}\\ &\geqslant Cm^{(2\mu+1)(\frac{1}{2}-\frac{1}{p})}\|K_{2m,\eta}-V_{4m}(t_m)\|_{2,\mu}\\ &\geqslant Cm^{(2\mu+1)(\frac{1}{2}-\frac{1}{p})}\inf_{\substack{\Lambda\subset[0,8m]\\ |\Lambda|\leqslant m}}\left(\sum_{\substack{k\notin\Lambda\\ 0\leqslant k\leqslant 2m}}k^{2\mu}\right)^{\frac{1}{2}}\\ &\geqslant Cm^{(2\mu+1)(1-\frac{1}{p})}.\end{aligned} \tag{4.6}$$

The third inequality follows from the fact that $\varphi_k^\mu(1)\sim k^\mu$. Now a combination of (4.5) and (4.6) gives

$$\sigma_m(B_s^\alpha(L_{\tau,\mu}))_{p,\mu}\geqslant Cm^{(2\mu+1)(1-\frac{1}{p})}m^{-\alpha+(2\mu+1)(\frac{1}{\tau}-1)}=Cm^{-\alpha+(2\mu+1)(\frac{1}{\tau}-\frac{1}{p})},$$

the same as desired in this case.

Next, we consider the case when $2\leqslant p\leqslant\tau\leqslant\infty$ and $s=\infty$. By the inequality $\|\cdot\|_{p,\mu}\geqslant C\|\cdot\|_{2,\mu}$ and the embedding $B_\infty^\alpha(L_\infty)\subset B_\infty^\alpha(L_{\tau,\mu})$, it will suffice to prove the lower estimate for $\tau=\infty$,

$p = 2$ and $s = \infty$. To this end, we let $\psi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \psi(x) \leqslant 1$ for $x \in \mathbb{R}$, $\psi(x) = 1$ for $\frac{1}{4} \leqslant x \leqslant \frac{1}{2}$ and $\psi(x) = 0$ for $x \notin (0, 1)$. We take an integer N so that $m \leqslant c_0 N \leqslant m + 1$ with c_0 a sufficiently small absolute constant. We define

$$\psi_j(x) = \frac{\psi(2Nx + j)}{\|\psi(2N \cdot + j)\|_{2,\mu}}, \quad j = -N + 1, -N + 2, \ldots, N,$$

and

$$\mathcal{B}_N := \left\{ \sum_{j=-N+1}^{N} a_j \psi_j(x) : \max_{-N+1 \leqslant j \leqslant N} |a_j| \leqslant 1 \right\}.$$

Then $\{\psi_j\}_{j=-N+1}^{N}$ is an orthonormal system on $([-1, 1], (1 - x^2)^{\mu - \frac{1}{2}} dx)$ and therefore, by a general result of Kashin [8, Corollary 2], it follows that

$$\sigma_m(\mathcal{B}_N)_{2,\mu} \geqslant C N^{\frac{1}{2}}. \tag{4.7}$$

It will be shown that

$$C N^{-\alpha - \frac{1}{2}} \mathcal{B}_N \subset B^\alpha_\infty(L_\infty) \tag{4.8}$$

with $C > 0$ an absolute constant, which combined with (4.7) will give the desired lower estimate:

$$\sigma_m(B^\alpha_\infty(L_\infty))_{2,\mu} \geqslant C N^{-\alpha - \frac{1}{2}} N^{\frac{1}{2}} \sim m^{-\alpha}.$$

To show (4.8), we define

$$D_\mu := (1 - t^2) \frac{d^2}{dt^2} - (2\mu + 1) t \frac{d}{dt}. \tag{4.9}$$

As is well known,

$$D_\mu(\varphi_k^\mu) = -k(k + 2\mu) \varphi_k^\mu, \quad k = 0, 1, \ldots .$$

In view of this fact, we also define $(-D_\mu)^\beta$ $(\beta > 0)$ in a distributional sense by

$$\langle (-D_\mu)^\beta (f), \varphi_k^\mu \rangle = (k(k + 2\mu))^\beta \langle f, \varphi_k^\mu \rangle, \quad k = 0, 1, \ldots, \text{ for a distribution } f.$$

Now by the definition, and in view of (4.9), one can easily verify that for $f \in \mathcal{B}_N$ and an integer $\ell_0 > 0$,

$$\|D_\mu^{\ell_0}(f)\|_\infty \leqslant C N^{2\ell_0 + \frac{1}{2}}$$

and

$$\|f\|_\infty \leqslant C N^{\frac{1}{2}}.$$

It then follows by Kolmogorov type inequality (see [6, Theorem 8.1]) that for $f \in \mathcal{B}_N$ and an integer $\ell_0 > \frac{\alpha}{2}$,

$$\|D_\mu^{\frac{\alpha}{2}}(f)\|_\infty \leqslant C \|f\|_\infty^{\frac{2\ell_0 - \alpha}{2\ell_0}} \|D_\mu^{\ell_0}(f)\|^{\frac{\alpha}{2\ell_0}} \leqslant C N^{\alpha + \frac{1}{2}}.$$

Therefore, by Jackson type inequality (see [6, Theorem 7.2]), we obtain, for $f \in \mathcal{B}_N$,

$$N^{-\alpha-\frac{1}{2}}\|f - V_{2^j}(f)\|_\infty \leqslant C2^{-j\alpha}N^{-\alpha-\frac{1}{2}}\|D_\mu^{\frac{\alpha}{2}}(f)\|_\infty \leqslant C2^{-j\alpha}, \quad j = 0, 1, 2 \ldots,$$

and (4.8) then follows.

Finally, the case $2 \leqslant \tau \leqslant p \leqslant \infty$ follows from the inequality $\|\cdot\|_{p,\mu} \geqslant C\|\cdot\|_{\tau,\mu}$ and what we have just proved for $2 \leqslant p = \tau \leqslant \infty$.

This completes the proof of lower estimates. □

Acknowledgments

The author would like to express his sincere gratitude to the anonymous referees for many helpful comments. He also would like to thank Professor V.N. Temlyakov for kindly pointing out an error in the first version of this paper.

References

[1] E.S. Belinskii, Decomposition theorems and approximation by a "floating" system of exponentials, Trans. Amer. Math. Soc. 350 (1) (1998) 43–53.
[2] A. Bonami, J.L. Clerc, Sommes de Cesàro et multiplicateurs des dèveloppments en harmonique sphériques, Trans. Amer. Math. Soc. 183 (1973) 223–263.
[3] F. Dai, On n-widths of weighted Besov classes, J. Math. Anal. Appl. 315(2) (2006) 711–724.
[4] R.A. DeVore, Nonlinear approximation, Acta Numer. 7 (1998) 51–150.
[5] R.A. DeVore, V.N. Temlyakov, Nonlinear approximation by trigonometric sums, J. Fourier Anal. Appl. 2 (1) (1995) 29–48.
[6] Z. Ditzian, Fractional derivatives and best approximation, Acta Math. Hungar. 81 (1998) 323–348.
[7] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer, Berlin, 1987.
[8] B.S. Kashin, Approximation properties of complete orthonormal systems, Proc. Steklov Inst. Math. 3 (1987) 207–211 (English transl.).
[9] V.E. Maiorov, Trigonometric widths of Sobolev classes W_p^r in the space L_q, Mat. Zametki 40 (2) (1986) 161–173.
[10] Y. Makovoz, On trigonometric n-widths and their generalizations, J. Approx. Theory 41 (1984) 361–366.
[11] A. Pinkus, N-widths in Approximation Theory, Springer, New York, 1985.
[12] G. Szegö, Orthogonal Polynomials, American Mathematical Society, New York, 1975.
[13] V.N. Temlyakov, Approximation of Periodic Functions, in: Computational Mathematics and Analysis Series, Nova Science Publishers, Inc., Commack, NY, 1993.
[14] V.N. Temlyakov, Greedy algorithm and m-term trigonometric approximation, Constr. Approx. 14 (4) (1998) 569–587.
[15] V.N. Temlyakov, Nonlinear methods of approximation, Found. Comput. Math. 3 (1) (2003) 33–107.
[16] V.N. Temlyakov, Greedy-type approximation in Banach spaces and applications, Constr. Approx. 21 (2) (2005) 257–292.