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On the scientific work of Kunyang Wang

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Dedicated to Professor Kunyang Wang on the occasion of his 70th birthday.

This paper surveys some of the scientific work on positive polynomial sums, Fourier analysis and spherical approximation on the sphere that Kunyang Wang did in the past 20 years.

Keywords: Positive polynomial sums; spherical harmonics; unit sphere.

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1. Positivity of Certain Basic Sums of Ultraspherical Polynomials

1.1. *A problem of Szegö*

The ultraspherical polynomials $C_n^\lambda(x)$ are defined by the generating function

$$\frac{1}{(1-2xz+z^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) z^n, \quad |z|<1, \ \ x\in[-1,1].$$

The following important inequality is shown by Féjer[34] for the case of $\nu = \frac{1}{2}$:

$$\sum_{k=0}^{n} \frac{C_k^\nu(x)}{C_k^\nu(1)} > 0, \quad -1<x<1. \tag{1.1}$$

Feldheim[36] observes the following remarkable formula for the ultraspherical polynomials[a]: for $\nu > \lambda > -\frac{1}{2}$ and $0 < \theta < \frac{1}{2}\pi$,

$$C_n^\nu(\cos\theta) = \frac{2\Gamma\left(\nu+\dfrac{1}{2}\right)(2\nu)_n \cos^{n+2\lambda+1}\theta}{\Gamma\left(\lambda+\dfrac{1}{2}\right)\Gamma(\nu-\lambda)(2\lambda)_n\cos^{2\nu-1}\theta} \times \int_0^\theta \frac{\sin^{2\lambda}\psi[\cos^2\psi-\cos^2\theta]^{\nu-\lambda-1}C_n^\lambda(\cos\psi)}{\cos^{n+2\nu}\psi}d\psi. \tag{1.2}$$

This formula is used in Ref. 36 to deduce from the result of Féjer that (1.1) holds for all $\nu \geq \frac{1}{2}$. Indeed, as can be easily seen, the formula (1.2) of Feldheim implies that the inequality (1.1) holds for all $\mu > \nu$ if it holds for an index ν. Since (1.1) fails for $\nu = 0$, this implies that there must exist an optimal index ν' such that (1.1) holds for $\nu > \nu'$, but fails for some n for each $\nu < \nu'$. The editor Szegö, asks on p. 280 of his paper[36] what is ν' (see also pp. 821–830 of Ref. 54)?

We can also formulate this problem of Szegö as follows.

Problem 1. Characterize all values of ν such that (1.1) holds for all positive integers n.

This longstanding problem of Szegö was solved by Brown, Koumandos and Wang in 1998 in Ref. 17.

Theorem 1.1 (Brown, Koumandos and Wang[17]). *Let*

$$S_n^\lambda(x) =: \sum_{k=0}^n \frac{C_k^\lambda(x)}{C_k^\lambda(1)}$$

and define $\alpha' := -0.2693885\ldots$ to be the solution α of the transcendental equation $\int_0^{j_{\alpha,2}} t^{-\alpha}J_\alpha(t)dt = 0$, where $j_{\alpha,2}$ is the second positive zero of the Bessel function $J_\alpha(t)$. Let $\lambda' = \alpha' + 1/2$. Then for $\lambda \geq \lambda'$,

$$S_n^\lambda(x) > 0, \quad -1 < x < 1, \;\; n = 1, 2, \ldots,$$

while for $\lambda < \lambda'$,

$$\inf\{S_n^\lambda(x) \colon -1 < x < 1, n \in \mathbb{N}\} = -\infty.$$

Thus, this result of Brown, Koumandos and Wang shows that (1.1) holds for all $n \in \mathbb{N}$ if and only if $\nu \geq \lambda'$.

[a]The main part of Ref. 36 is a slightly modified version of a letter of the young and able Hungarian mathematician Ervin Feldheim, dated 12 March 1944, a few months before he became the victim of the terror of the Nazis.

Problem 1 is closely related to a second problem raised by Szegö in Ref. 36:

Problem 2. Characterize all values of α such that

$$\int_0^z t^{-\alpha} J_\alpha(t)dt > 0 \tag{1.3}$$

holds for all $z > 0$.

Since

$$\lim_{n\to\infty} \frac{C_n^\nu\left(\cos\frac{z}{n}\right)}{C_n^\nu(1)} = 2^\alpha\Gamma(\alpha+1)z^{-\alpha}J_\alpha(z), \quad \alpha = \nu - \frac{1}{2},$$

uniformly in every fixed disk $|z| \le R$, it follows that for all $z > 0$

$$\lim_{n\to\infty} \frac{z}{n}\sum_{k=0}^{n} \frac{C_k^\nu\left(\cos\frac{z}{n}\right)}{C_k^\nu(1)} = 2^\alpha\Gamma(\alpha+1)\int_0^z t^{-\alpha}J_\alpha(t)dt, \quad a = \nu - \frac{1}{2}.$$

Thus, the result of Brown, Koumandos and Wang[17] implies that $\int_0^z t^{-\alpha}J_\alpha(t)dt \ge 0$ for all $z > 0$ if $\alpha \ge \alpha'$.

Problem 2 was solved by Szegö,[36] who showed that $\int_0^z t^{-\alpha}J_\alpha(t)dt > 0$ for all $z > 0$ if $\alpha > \alpha'$ and this inequality fails for some $z > 0$ for each $\alpha < \alpha'$.

1.2. *Positivity of some basic Legendre polynomial sums*

A classical result of Fejér[34] states that

$$\sum_{k=0}^{n} P_k(\cos\theta) \ge 0, \quad 0 \le \theta \le \pi, \tag{1.4}$$

where $P_k(x) = C_k^{\frac{1}{2}}(x)$ is the Legendre polynomial of degree k. Equation (1.4) is a special case of the more general inequality (1.1) on ultraspherical polynomials.

The following new inequalities concerning positivity of basic sums of Legendre polynomials are proved by Brown, Koumandos and Wang.[18]

Theorem 1.2 (Brown, Koumandos and Wang[18]). *For* $n = 0, 1, \ldots,$

$$\frac{1}{2} + \sum_{k=1}^{n} P_{2k}(\cos\theta) \ge 0, \quad 0 \le \theta \le \pi, \tag{1.5}$$

$$\frac{\sqrt{15}}{45} + \sum_{k=1}^{n} P_{2k-1}(\cos\theta) \ge 0, \quad 0 \le \theta \le \frac{\pi}{2}, \tag{1.6}$$

$$\frac{1}{2} + \sum_{k=1}^{n} P_{4k}(\cos\theta) > 0, \quad 0 \le \theta \le \pi. \tag{1.7}$$

2. Positivity of Certain Sums of Jacobi Polynomials

2.1. *A conjecture of Gasper*

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{k!(\alpha+1)_k}\left(\frac{1-x}{2}\right)^k, \quad x\in[-1,1].$$

Over the years, some special cases of the inequality

$$\sum_{k=0}^{n}\frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geq 0, \quad -1\leq x\leq 1, \tag{2.1}$$

have been proved by several authors (see Ref. 15 and the references therein). By the application of Bateman's integral, it is true that if (2.1) holds for some (α,β) then it also holds for $(\alpha-\mu,\beta+\mu)$ with $\mu>0$.

Gasper[39] proves (2.1) under the more general assumptions $\alpha+\beta\geq 0$, $\beta\geq -\frac{1}{2}$ or $\alpha+\beta\geq -2$, $\beta\geq 0$. He also conjectured

Conjecture. Equation (2.1) is true for $-1<\alpha<\frac{1}{2}$, $\alpha'\leq\beta<0$, with $\alpha'=-0.2693885\ldots$ being the same as in Theorem 7.1.

Gasper[39] pointed out that any case of his conjecture should be treated by completely different methods than his own. Askey[7] draws attention to this conjecture, mentioning that it would be interesting to prove it for $-1<\alpha<0$ and some $\beta<0$, predicting, however, that this is probably quite hard.

The result of Brown, Koumandos and Wang,[17] Theorem 7.1, establishes (2.1) for $\alpha=\beta\geq\alpha'$, which partially confirms Gasper's conjecture.

In Ref. 15, Brown, Koumandos and Wang extend the progress on Gasper's conjecture for $-1<\alpha\leq -\frac{1}{2}$ and negative β. They prove the following result in 1996.

Theorem 2.1 (Brown, Koumandos and Wang[15]). *The inequality*

$$\sum_{k=0}^{n}(-1)^k\frac{C_{2k}^{\lambda}(\cos\theta)}{C_{2k}^{\lambda}(1)} \geq 0, \quad n=1,2,\ldots,\ 0<\theta\leq\pi, \tag{2.2}$$

holds for all $\lambda\geq\lambda_0$, where $\lambda_0=0.308443\ldots$ is the Littlewood–Salem–Izumi number, which is the unique root $\lambda\in(0,1)$ of the equation $\int_0^{\frac{3\pi}{2}}\frac{\cos t}{t^\lambda}dt=0$. Furthermore, the only cases of equality in (2.2) are when $\theta=0$ and n is odd.

Applying the quadratic transformation

$$C_{2k}^{\lambda}(x) = \frac{(\lambda)_k}{\left(\mu+\frac{1}{2}\right)_k}P_k^{(-\frac{1}{2},\lambda-\frac{1}{2})}(2x^2-1),$$

we deduce from (2.2) that (2.1) holds for $\alpha=-\frac{1}{2}$ and $\beta=\lambda-\frac{1}{2}\geq\lambda_0-\frac{1}{2}=\alpha'$. Thus, Theorem 2.1 settles the above conjecture of Gasper for the case of $\alpha=-\frac{1}{2}$.

2.2. Positive Jacobi polynomial sums from quadrature

Let

$$(1-x)^{-\gamma}(1+x)^{-\delta} \sim \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x) \tag{2.3}$$

be the Jacobi orthogonal polynomial expansion of the function $f(x) := (1-x)^{-\gamma}(1+x)^{-\delta}$ on $[-1,1]$, where

$$a_k = \frac{\int_{-1}^{1} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx}{\int_{-1}^{1} |P_k^{(\alpha,\beta)}(x)|^2(1-x)^{\alpha}(1+x)^{\beta}dx}, \quad k = 0, 1, \ldots. \tag{2.4}$$

In his SIAM conference lectures, Askey[5] draws attention to the following problem.

Problem. Find conditions on $\alpha, \beta, \delta, \gamma$ such that

$$\sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(x) \geq 0, \quad -1 \leq x \leq 1, \;\; n = 0, 1, \ldots, \tag{2.5}$$

where the coefficients a_k are given in (2.4).

The background of this problem can be described as follows. Denote the zeros of $P_n^{(\alpha,\beta)}(x)$, arranged in decreasing order, by $x_{n,k}$, $k = 1, 2, \ldots, n$. The Cotes numbers $\lambda_{n,k}$ for integration with respect to $(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx$ on $[-1,1]$ are defined by

$$\sum_{k=1}^{n} \lambda_{n,k} x_{n,k}^r = \int_{-1}^{1} x^r (1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx, \quad r = 0, 1, \ldots, n.$$

Define the quadrature rule Q_n by

$$Q_n(f) = \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k}),$$

so that

$$Q_n f = \int_{-1}^{1} f(x)(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx$$

holds for all algebraic polynomials f on $[-1,1]$ of degree at most n. It is highly desirable in practice that all the Cotes numbers $\lambda_{n,k}$ in the quadrature rule are positive. It follows from (7.3.9) of Ref. 1 that

$$\lambda_{n,k} = c_{n,k} \sum_{k=0}^{n} \frac{\int_{-1}^{1} P_k^{(\alpha,\beta)}(x)(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx}{\int_{-1}^{1} |P_k^{(\alpha,\beta)}(x)|^2(1-x)^{\alpha}(1+x)^{\beta}dx} P_k^{(\alpha,\beta)}(x_{n,k})$$

for some positive constant $c_{n,k}$. Thus, (2.5) is a sufficient condition for the positivity of the cotes numbers $\lambda_{n,k}$.

In Ref. 17, Brown, Koumandos and Wang consider the above problem for the special case $\beta = -\frac{1}{2}$, which was noted by Askey[3] to be of special interest. Previous to the work,[17] the following cases of (2.5) for $\beta = -\frac{1}{2}$ were known:

(1) $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = 1$, $\delta = 0$ (positivity of the Fejér kernel);
(2) $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{4}$, $\delta = -\frac{1}{4}$ (a cosine sum of Vietoris);
(3) $\alpha = \frac{1}{2} = \gamma$, $\beta = \delta = -\frac{1}{2}$ (contained in results of Fejer[35] and Szegö[53]);
(4) $\alpha > \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\delta = 0$, $\gamma = \frac{1}{2}\alpha + \frac{3}{4}$ (which follows from work of Gasper[38]).

The following results were proven in Ref. 17.

Theorem 2.2 (Brown, Koumandos and Wang[17]). *The strict inequality in* (2.5) *holds if one of the following conditions is satisfied*:

(i) $\alpha \geq 0$, $\beta = -\frac{1}{2}$, $\delta = 0$, $0 < \gamma \leq -\frac{1}{2}$.
(ii) $\alpha = 0$, $\beta = -\frac{1}{2}$, $\delta = \frac{1}{4}$, $\gamma = \frac{1}{2}$.
(iii) $\delta = 0$, $\alpha + \beta + 1 > 0$, $\alpha \geq \beta$, $-1 < \gamma < 0$.
(iv) $\delta = 0$, $-\beta - 1 < \alpha < \beta$, $-1 < \gamma \leq \frac{1}{2}(\alpha - \beta - 1)$.

3. Positive Trigonometric Sums

3.1. *Extensions of Vietoris' inequality on cosine sums*

In 1958, Vietoris[56] proves that if $a_0 \geq a_1 \geq \cdots \geq a_n > 0$ and $a_{2k} \leq (1 - \frac{1}{2k})a_{2k-1}, k \geq 1$, then for all $n \geq 1$ and $\theta \in (0, \pi)$,

$$\sum_{k=1}^{n} a_k \sin k\theta > 0 \quad \text{and} \quad \sum_{k=0}^{n} a_k \cos k\theta > 0, \tag{3.1}$$

which extend both the Fejér–Jackson–Gronwall inequality,

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} > 0, \quad 0 < \theta < \pi, \tag{3.2}$$

and the W. H. Young inequality,

$$1 + \sum_{k=1}^{n} \frac{\cos k\theta}{k} > 0, \quad 0 < \theta < \pi. \tag{3.3}$$

These inequalities of Vietoris have turned out to be very useful. In fact, Askey and Steinig[4] prove that they can be applied to yield various new results, including improved estimates for the localization of zeros of a class of trigonometric polynomials and new positive sums of ultraspherical polynomials, while Askey[8] shows that one of the problems suggested by these inequalities leads to the derivation of the hypergeometric summation formula and to other summation formulas.

The Vietoris inequality (3.1) on cosine sums has been extended in two different directions by Brown, Dai and Wang in Refs. 13 and 12 respectively, both being best possible in certain sense. In one direction, they prove the following result in Ref. 13.

Theorem 3.1 (Brown, Dai and Wang[13]). *Let* $\lambda_0 = 0.308443\ldots$ *denote the Littlewood–Salem–Izumi number, and assume that* $a_0 \geq a_1 \geq \cdots \geq a_n > 0$ *and* $a_{2k} \leq (1 - \frac{\lambda_0}{k})a_{2k-1}, k \geq 1$. *Then*

$$\sum_{k=0}^{n} a_k \cos k\theta > 0, \quad \theta \in (0, \pi).$$

The number λ_0 *is optimal in the sense that for any* $\lambda \in (0, \lambda_0)$ $\lim_{n\to\infty} \min_{\theta\in(0,\pi)} \sum_{k=0}^{n} u_k(\lambda) \cos k\theta = -\infty$, *where the coefficients* $u_k(\lambda)$ *are defined as* $u_0(\lambda) = u_1(\lambda) = 1, u_{2k}(\lambda) = u_{2k+1}(\lambda) = (1 - \frac{\lambda}{k})u_{2k-1}(\lambda), k \geq 1$.

Clearly, Theorem 3.1 with $\lambda_0 = \frac{1}{2}$ corresponds to the original inequality of Vietoris (3.1) on cosine sums.

For the extension in another direction, we note that by summation by parts, the Vietoris inequality (3.1) is in fact equivalent to

$$\sum_{k=1}^{n} c_k \sin k\theta > 0 \quad \text{and} \quad \sum_{k=0}^{n} c_k \cos k\theta > 0,$$

where $\{c_k\}_{k=0}^{\infty}$ is a sequence of positive numbers given by

$$c_0 = c_1 = 1, \quad c_{2k} = c_{2k+1} = \frac{2k-1}{2k} c_{2k-1}, \quad k = 1, 2, \ldots. \tag{3.4}$$

This motivates us to define, for a general parameter $\alpha > -1$,

$$c_0(\alpha) = c_1(\alpha) = 1, \quad c_{2k}(\alpha) = c_{2k+1}(\alpha) = \frac{2k-1+\alpha}{2k+\alpha} c_{2k-1}(\alpha), \quad k \geq 1,$$

and set

$$C_n^{\alpha}(\theta) = \sum_{k=0}^{n} c_k(\alpha) \cos(k\theta).$$

Thus, the Vietoris inequality (3.1) asserts that $C_n^0(\theta) > 0$ for all $n \in \mathbb{N}$ and $\theta \in (0, \pi)$. Next, we define $\alpha_0 := 2.3308\ldots$ to be the unique solution $\alpha \in (-1, \infty)$ of the equation $\min_{\theta\in(0,\pi)} C_6^{\alpha}(\theta) = 0$.

With the above notation, the following result was proved in Ref. 12.

Theorem 3.2 (Brown, Dai and Wang[12]). *If* $\{a_k\}_{k=0}^{\infty}$ *is a decreasing sequence of positive real numbers satisfying* $(2k + \alpha_0)a_{2k} \leq (2k - 1 + \alpha_0)a_{2k-1}$, *then*

$$\sum_{k=0}^{n} a_k \cos(k\theta) \geq 0, \quad x \in (0, \pi), \ \ n \in \mathbb{N}, \tag{3.5}$$

with equality for some $\theta \in (0, \pi)$ *only when* $n = 6$ *and* $a_k = c_k(\alpha_0)$. *The number* $\alpha_0 = 2.3308\ldots$ *is best possible in the sense that if* $\alpha > \alpha_0$ *then the inequality* (3.5) *with* $a_k = c_k(\alpha)$ *is not true for some* n *and* $\theta \in (0, \pi)$.

Clearly, the Vietoris inequality (3.1) on cosine sums corresponds to the above theorem with $\alpha_0 = 0$.

3.2. *An extension of the Fejér–Jackson inequality*

The classical positive trigonometric sums are those of Young (3.3) and Fejér–Jackson (3.2). Rogosinki and Szegö[47] extended Young's inequality (3.3) to

$$\frac{1}{1+\alpha} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+\alpha} \geq 0, \quad 0 \leq \theta \leq \pi, \ \ -1 < \alpha \leq 1. \tag{3.6}$$

The case $\alpha = 1$ is interesting, but Gasper[37] showed that the result admits considerable improvement. In fact, he showed that (3.6) holds for $-1 < \alpha \leq \widetilde{\alpha} := 4.567\ldots$ and $\widetilde{\alpha}$ is best possible.

In Ref. 19, Brown and Wang extended the Fejér–Jackson inequality (3.2) in a similar way, considering the partial sums

$$T_n^\alpha(\theta) = \sum_{k=1}^{n} \frac{\sin k\theta}{k+\alpha}, \quad \alpha > -1, \ \ n \in \mathbb{N}.$$

Theorem 3.3 (Brown and Wang[19]). *If* $-1 < \alpha < \alpha_0$, *then*

$$T_{2n-1}^\alpha(\theta) > 0, \quad 0 < \theta < \pi, \ \ n \in \mathbb{N},$$

and

$$T_{2n}^\alpha(\theta) > 0, \quad 0 < \theta \leq \pi - \frac{\mu_0 \pi}{2n+0.5},$$

where $\alpha_0 = 2.1102\ldots$ *and* $\mu_0 = 0.8128252\ldots$ *are both best possible.*

The precise definition of the constants α_0 and μ_0 in the above theorem is as follows. Let $\widehat{\lambda} := 0.4302967\ldots$ be the unique solution $\lambda \in (0, 1/2)$ of the equation $(1+\lambda)\pi = \tan(\lambda\pi)$. We then define μ_0 to be the solution $\mu > 0$ of the equation

$$\frac{\sin \mu\pi}{\mu\pi} = \frac{\sin \widehat{\lambda}\pi}{(1+\widehat{\lambda})\pi},$$

and α_0 to be the solution α of the equation

$$\sum_{k=1}^{\infty} \frac{2k}{(2k-1+\alpha)(2k+\alpha)(2k+1+\alpha)} = \frac{\sin \widehat{\lambda}\pi}{2(1+\widehat{\lambda})\pi}.$$

3.3. *Positivity of some basic cosine sums*

Brown, Wang and Wilson[22] studied the positivity of the cosine sums

$$L_n^\alpha(\theta) = 1 + \sum_{k=1}^{n} k^{-\alpha} \cos k\theta, \quad n \in \mathbb{N}, \ \ \alpha > 0.$$

The classical result (3.3) of Young is that $L_n^\alpha(\theta) \geq 0$ for $\alpha = 1$ and all $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Littlewood and Salem proved that there is $\alpha_0 \in (0, 1)$ such that for $\alpha \geq \alpha_0$, the L_n^α are uniformly bounded below, while for $\alpha < \lambda_0$, they are not.

Izumi (see V. 2.29 of Ref. 61) showed that $\alpha_0 = \lambda_0$ is the unique root of the equation $\int_0^{3\pi/2} t^{-\alpha} \cos t dt = 0$.

The following remarkable result, which quantifies a boundedness result of Littlewood and Salem, was proved in Ref. 22.

Theorem 3.4 (Brown, Wang and Wilson[22]). *If $\alpha \geq \lambda_0$, then $L_n^\alpha(\theta) > 0.0376908$ for all $\theta \in \mathbb{R}$ and $n > 1$, while for $\alpha < \lambda_0$, $L_n^\alpha(\theta) < 0$ for some $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$.*

Using summation by parts, one can deduce from Theorem 3.4 the following corollary.

Corollary 3.5. *Let $\alpha \geq \lambda_0$ and suppose $\{a_k\}_{k=0}^\infty$ is a non-increasing sequence of non-negative real numbers satisfying $k^\alpha a_k \geq (k+1)^\alpha a_{k+1}$. Then*

$$\sum_{k=0}^n a_k \cos k\theta \geq 0$$

for all $\theta \in \mathbb{R}$ and $n \geq 0$.

These results were further extended by Brown, Dai and Wang in Ref. 14. Note that for $0 < \alpha < 1$ and $k \geq 1$,

$$\frac{(k+1)^{-\alpha}}{k^{-\alpha}} = \left(1 - \frac{1}{k+1}\right)^\alpha \leq 1 - \frac{\alpha}{k+1} = \frac{a_{k+1}(\alpha)}{a_k(\alpha)}, \tag{3.7}$$

where

$$a_k(\alpha) = \frac{\Gamma(k+1-\alpha)}{\Gamma(2-\alpha)k!}, \quad k = 1, 2, \ldots. \tag{3.8}$$

Define

$$S_n(x, \alpha) := 1 + \sum_{k=1}^n a_k(\alpha) \cos kx. \tag{3.9}$$

Using (3.7) and summation by parts, we see that if $S_n(x, \alpha) \geq 0$ for all $x \in \mathbb{R}$ and some number $\alpha \in (0, 1)$, then $1 + \sum_{k=1}^n k^{-\alpha} \cos k\theta \geq 0$ for all $x \in \mathbb{R}$.

Positivity of the sums (3.9) was investigated in Ref. 14. To describe the results, we define α^* to be the unique solution $\alpha \in (0, 1)$ of the equation

$$\min_{x \in [0,\pi]} S_7(x, \alpha) = 0. \tag{3.10}$$

Numerical evaluation shows that $\alpha^* = 0.33542\ldots$. The following results were proved in Ref. 14.

Theorem 3.6 (Brown, Dai and Wang[14]). (i) *If $\alpha^* \leq \alpha < 1$ then*

$$S_n(x, \alpha) \geq 0, \quad x \in [0, \pi], \; n = 1, 2, \ldots, \tag{3.11}$$

with equality being true for some $x \in [0, \pi]$ if and only if $\alpha = \alpha^*$ and $n = 7$.

(ii) *If* $0.308443\ldots = \lambda_0 < \alpha < \alpha^* = 0.33542\ldots$, *then* (3.11) *fails for some* $n \leq 7$, *but there exists an integer* $N(\alpha)$ *that depends only on* α *such that* $S_n(x,\alpha) \geq 0$ *for all* $x \in \mathbb{R}$ *whenever* $n \geq N(\alpha)$.

(iii) *If* $\alpha = \lambda_0$ *then*

$$\limsup_{n\to\infty}(\min\{S_n(x,\lambda_0) : x \in [0,\pi]\}) = -\frac{\alpha_0}{1-\alpha_0} = -0.446014\ldots. \qquad (3.12)$$

(iv) *If* $0 < \alpha < \lambda_0$ *then*

$$\liminf_{n\to\infty}(\min\{S_n(x,\alpha) : x \in [0,\pi]\}) = -\infty.$$

Theorem 3.6 and summation by parts imply the following corollary.

Corollary 3.7. (i) *If* $\{c_k\}_{k=0}^{\infty}$ *is a sequence of positive numbers satisfying* $c_0 \geq c_1$ *and*

$$\left(1 - \frac{\alpha^*}{k+1}\right) c_k \geq c_{k+1}, \quad k = 1, 2, \ldots,$$

then

$$\sum_{k=0}^{n} c_k \cos kx \geq 0, \quad x \in [0,\pi], \ \ n = 1, 2, \ldots,$$

with equality being true for some $x \in [0,\pi]$ *if and only if* $n = 7$ *and* $c_k = c_0 a_k(\alpha^*)$ *for* $0 \leq k \leq 7$ *and some positive number* c_0.

(ii) *If* $\lambda_0 < \alpha < \alpha^* = 0.33542\ldots$, *and* $\{c_k\}_{k=0}^{\infty}$ *is a sequence of positive numbers satisfying* $c_0 \geq c_1$,

$$\left(1 - \frac{\alpha}{k+1}\right) c_k \geq c_{k+1}, \quad k = 1, 2, \ldots,$$

then we have

$$\sum_{k=0}^{n} c_k \cos kx \geq \min_{0\leq j<N(\alpha)} \left(\sum_{k=0}^{j} c_k \cos kx\right), \quad x \in [0,\pi], \ \ n = 0, 1, 2, \ldots,$$

where $N(\alpha)$ *is the same as in the second assertion of Theorem* 3.6.

The proof in Ref. 14 also yields the following useful asymptotic estimate of $S_n(x,\alpha)$: for $\alpha \in (0,1)$, we have

$$S_n(x,\alpha) = -\frac{\alpha}{1-\alpha} + \frac{\sin\left(\frac{\pi\alpha}{2} + \frac{(1-\alpha)x}{2}\right)}{(1-\alpha)\left(2\sin\frac{x}{2}\right)^{1-\alpha}} - \frac{x^\alpha}{2\Gamma(2-\alpha)\sin\frac{x}{2}} \int_{(n+\frac{1}{2})x}^{\infty} \frac{\cos t}{t^\alpha} dt + O(n^{-\alpha}), \qquad (3.13)$$

uniformly for $x \in (0,\pi]$, as $n \to \infty$. For $n \geq 101$, the remainder term $O(n^{-\alpha})$ in (3.13) can be controlled by a very small, computable constant.

4. Integrability of Double Lacunary Sine Series

In Ref. 21, Brown and Wang settled a problem of Móricz[44] on integrability of double lacunary sine series. To describe the results, we need a few notations. Let $a_{ij}, i, j \in \mathbb{N}$, be real numbers satisfying the condition

$$\sigma = \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 \right)^{1/2} < \infty. \tag{4.1}$$

Suppose $q > 1$ and m_i, n_j are positive numbers satisfying

$$\frac{m_{i+1}}{m_i} \geq q, \quad \frac{n_{j+1}}{n_j} \geq q, \quad m_1 = n_1 = 1, \ \ i, j \in \mathbb{N}. \tag{4.2}$$

Define

$$f(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} (\sin m_i x)(\sin n_j y),$$

$$g_j(x) = \sum_{i=1}^{\infty} a_{ij} \sin m_i x, \quad h_j(y) = \sum_{j=1}^{\infty} a_{ij} \sin n_j y,$$

where the infinite sums are understood in the sense of L^2-convergence.

In the special case when $m_i = n_i = 2^{i-1}$, $i \in \mathbb{N}$, Móricz[44] proved that the condition

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{k=i}^{\infty} \sum_{l=j}^{\infty} a_{kl}^2 \right)^{1/2} < \infty \tag{4.3}$$

is equivalent to

$$\frac{f(x, y)}{xy} \in L^1([0, 1]^2), \quad \frac{g_i(x)}{x} \in L^1[0, 1], \quad \frac{h_i(y)}{y} \in L[0, 1], \quad i \in \mathbb{N}. \tag{4.4}$$

He further conjectured in the general case that (4.4) is satisfied if and only if

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{m_{i+1}}{m_i} \log \frac{n_{j+1}}{n_j} \left(\sum_{k=i}^{\infty} \sum_{l=j}^{\infty} a_{kl}^2 \right)^{1/2} < \infty. \tag{4.5}$$

Brown and Wang[21] disproved this conjecture and established the following modified result.

Theorem 4.1 (Brown and Wang[21]). *Let a_{ij}, m_i, n_j satisfy* (4.1) *and* (4.2). *Let f, g_j, h_i be as above. Define*

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|, \quad T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{m_{i+1}}{m_i} \left(\sum_{k=i+1}^{\infty} a_{kj}^2 \right)^{1/2},$$

$$U = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \log \frac{n_{j+1}}{n_j} \left(\sum_{l=j+1}^{\infty} a_{il}^2 \right)^{1/2},$$

$$V = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\frac{m_{i+1}}{m_i}\log\frac{n_{j+1}}{n_j}\left(\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty} a_{kl}^2\right)^{1/2}.$$

Then the condition (4.4) *is equivalent to the condition*

$$S + T + U + V < \infty. \tag{4.6}$$

It is worthwhile to point out that in the above theorem, m_i, m_j need not be integers. If $m_i = n_i = 2^{i-1}$, then (4.5) is equivalent to (4.6). But in general, an example of Ref. 21 shows that (4.5) is stronger than (4.6) and they are not equivalent.

5. Equiconvergent Operators of Cesàro Means for Spherical Harmonic Expansions

In 1993, Wang[57,58] introduced and studied a family of linear operators, called the equiconvergent operators (EOs), in the investigation of Cesàro summability of the spherical harmonic expansions. These operators have properties very similar to those of the Cesàro operators, but are much easier to deal with in applications. They have become a very powerful tool for the study of the Cesàro means of spherical harmonic expansions. The works of Wang[57,58] opened a new line of research for spherical harmonic analysis on the sphere, and have significantly influenced the further research of his students and participants in his seminar.

To describe the works of Wang on EOs, we need to introduce some necessary notations first. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual Lebesgue measure $d\sigma(x)$ normalized by $\sigma(\mathbb{S}^{d-1}) = 1$. In sequel, all functions and sets are assumed to be Lebesgue measurable. Let $L^p(\mathbb{S}^{d-1})$ denote the Lebesgue L^p-space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^{d-1}$, and $C(\mathbb{S}^{d-1})$ the space of all continuous functions on $\mathbb{S}^{d-1}$. We denote by $\mathcal{H}_n \equiv \mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^{d-1}$, and $\Pi_n \equiv \Pi_n^d$ the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$. Let Y_k denote the orthogonal projection from $L^2(\mathbb{S}^{d-1})$ onto $\mathcal{H}_k^d$. *The Cesàro means of $f \in L(\mathbb{S}^{d-1})$ of order* $\delta > -1$ are defined by

$$\sigma_n^\delta(f) := \sum_{k=0}^{n} \frac{A_{n-k}^\delta}{A_n^\delta} Y_k(f), \quad n = 0, 1, \dots, \tag{5.1}$$

where $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$. As is well known (see Ref. 11), $\lambda := \frac{d-2}{2}$ is the critical index for the Cesàro summability in $L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, in the sense that if $\delta > \lambda$, then

$$\lim_{n\to\infty} \|\sigma_n^\delta f - f\|_p = 0$$

for all $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, while this is no longer true when $\delta < \lambda$. In general, the σ_n^λ behave like the partial sums of the Fourier series in one variable.

Definition 5.1 (Wang[57]). The EOs of the Cesàro means of order $\delta > -1$ are defined by

$$E_N^\delta(f)(x) := \gamma_N^\delta \int_{\mathbb{S}^{d-1}} f(y) P_N^{(\frac{d-1}{2}+\delta, \frac{d-3}{2})}(x \cdot y) d\sigma(y), \tag{5.2}$$

for $x \in \mathbb{S}^{d-1}$ and $f \in L^1(\mathbb{S}^{d-1})$, where

$$\gamma_N^\delta = \frac{\Gamma(\delta+1)\Gamma(N+1)\Gamma(N+d-1)}{(4\pi)^{\frac{d-1}{2}}\Gamma(N+\delta+1)\Gamma\left(N+\dfrac{d-1}{2}\right)} \sim N^{\frac{d-1}{2}-\delta}, \tag{5.3}$$

and this constant is chosen so that $E_N^\delta(\mathbb{1}) = 1$.

The connection between the EOs and the Cesàro means can be seen from the following equation (see Ref. 57):

$$E_N^\delta(f) = \alpha_N^\delta \sigma_N^\delta(f) + \sum_{v=1}^{\infty} O(1) v^{-d-1-\delta} \sigma_N^{\delta+v}(f), \tag{5.4}$$

where $\alpha_N^\delta = 1 + O(N^{-1})$, and the $O(1)$ denote general constants whose absolute values are bounded above by a constant depending only on d and δ. Explicit expressions of the coefficients α_N^δ and $O(1)$ can be found in Ref. 57 (see also pp. 85–87 of Ref. 59).

The following proposition collects some useful properties of the EOs, which also justify the name of these operators.

Proposition 5.2 (Wang[58]). *Let $\delta > -1$.*

(i) *Let $f \in L^1(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$. Then*

$$\lim_{N\to\infty} \sigma_N^\delta(f)(x) = f(x)$$

if and only if

$$\lim_{N\to\infty} E_N^\delta(f)(x) = f(x).$$

(ii) *Let D be a subset of $\mathbb{S}^{d-1}$, and assume that $f \in L^p(D)$ with $1 \le p < \infty$, or $f \in C(D)$ when $p = \infty$. Then*

$$\lim_{N\to\infty} \|\sigma_N^\delta(f) - f\|_{L^p(D)} = 0$$

if and only if

$$\lim_{N\to\infty} \|E_N^\delta(f) - f\|_{L^p(D)} = 0.$$

(iii) $E_N^\delta f = \sum_{k=0}^N b_{N,k}^\delta Y_k f$ *for each $f \in L^1(\mathbb{S}^{d-1})$, where*

$$b_{N,k}^\delta := \frac{\Gamma(N+d-1)\Gamma(N+\delta+k+d-1)}{\Gamma(N+k+d-1)\Gamma(N+\delta+d-1)} \frac{A_{N-k}^\delta}{A_N^\delta}, \quad 0 \le k \le N.$$

In the next few subsections, we shall list several interesting applications of the EOs.

5.1. *Application I: Tests for the pointwise convergence of Cesàro means*

Various tests for the pointwise convergence of Cesàro means were obtained by the second author in Refs. 57 and 58, using the EOs. Here, we only describe two very interesting results from Ref. 58, the first of which is known as the Dini type test.

The translation operator S_θ on $\mathbb{S}^{d-1}$ with step $\theta \in [0, \pi]$ is defined by

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{d-2}| \sin^{d-2}\theta} \int_{\{y \in \mathbb{S}^{d-1} : xy = \cos\theta\}} f(y) d\ell_{x,\theta}(y),$$

where $f \in L^1(\mathbb{S}^{d-1}), x \in \mathbb{S}^{d-1}$ and $d\ell_{x,\theta}(y)$ denotes the Lebesgue measure element on the set $\{y \in \mathbb{S}^{d-1} : y \cdot x = \cos\theta\}$.

Theorem 5.3 (Wang[58]). *Let $f : \mathbb{S}^{d-1} \to \mathbb{R}$ be an integrable function on $\mathbb{S}^{d-1}$, and x_0 be a fixed point on $\mathbb{S}^{d-1}$. If the function*

$$\theta \mapsto \theta^{-1}(\pi - \theta)^{\frac{d-2}{2}} |S_\theta(f)(x_0) - f(x_0)|$$

is integrable on $(0, \pi)$, then $\lim_{N\to\infty} \sigma_N^\lambda(f)(x_0) = f(x_0)$. Furthermore, if

$$\int_0^{\frac{\pi}{2}} \frac{\|f - S_\theta f\|_1}{\theta} d\theta < \infty,$$

then $\lim_{N\to\infty} \sigma_N^\lambda(f)(x) = f(x)$ for a.e. $x \in \mathbb{S}^{d-1}$.

The second result is the Salem type theorem.

Theorem 5.4 (Wang[58]). *Let $f : \mathbb{S}^{d-1} \to \mathbb{R}$ be an integrable function on $\mathbb{S}^{d-1}$ and let $\xi \in \mathbb{S}^{d-1}$ be such that both ξ and $-\xi$ are the Lebesgue points of f. Define a function $\varphi_\xi : [0, \pi] \to \mathbb{R}$ by*

$$\varphi_\xi(\theta) := \sin^{d-2}\theta[S_\theta f(\xi) - f(\xi)].$$

If

$$\sup_{\theta \in [2h, \frac{\pi}{2}]} \frac{1}{\theta^{d-2}} \int_0^h [\varphi_\xi(\theta + t) - \varphi_\xi(\theta - t)] dt = o\left(\frac{h}{|\log h|}\right), \quad \text{as } h \to 0+,$$

then

$$\lim_{N\to\infty} \sigma_N^\delta(f)(\xi) = f(\xi).$$

5.2. *Application II: Localization*

Theorem 5.5 (Wang[58]). *Assume that $f \in L^1(\mathbb{S}^{d-1})$ varnishes on a spherical cap $B(\xi_0, r) := \{x \in \mathbb{S}^{d-1} : \arccos(x \cdot \xi) \le r\}$ centered at $\xi_0 \in \mathbb{S}^{d-1}$ having radius $r \in (0, \frac{\pi}{2})$.*

(i) *If $\delta \ge d - 2$ then $\sigma_N^\delta(f)$ converges uniformly to 0 on any closed subset of $B(\xi_0, r)$ as $N \to \infty$.*

(ii) *If $\lambda < \delta < d-2$, and if, in addition, the following anti-pole condition is satisfied uniformly on some nonempty subset D of $B(\xi_0, r)$:*

$$\lim_{\theta \to 0+} \frac{1}{\theta^{d-2-\delta}} \int_0^\theta |S_t f(\xi)| t^{d-2} dt = 0, \quad \xi \in D, \tag{5.5}$$

then $\sigma_N^\delta(f)$ converges uniformly to 0 on D as $N \to \infty$.

(iii) *The antipole condition (5.5) is necessary for the convergence of $\sigma_N^\delta(f)$ on $B(\xi_0, r)$ when $-1 < \delta < d-2$. Indeed, if $-1 < \delta < d-2$ and $\xi_0 \in \mathbb{S}^{d-1}$, then there exists a function $f \in L^1(\mathbb{S}^{d-1})$ varnishing on the spherical cap $B(\xi_0, \frac{\pi}{2})$ but satisfying $\lim_{N\to\infty} |\sigma_N^\delta(f)(\xi_0)| = \infty$.*

The second statement of Theorem 5.5 improves an earlier result of Bonami and Clerc[11] in 1973.

5.3. *Application III: A.e. convergence of Cesàro means at the critical index*

We start with some background information. In the case of $d = 2$, the Cesàro mean σ_N^λ at the critical index $\lambda = \frac{d-2}{2}$ becomes the Nth partial sum $S_N f$ of the Fourier series of $f \in L(\mathbb{S}^1) \equiv L^1[-\pi, \pi]$. The celebrated Carleson–Hunt theorem asserts that if $f \in L^p[-\pi, \pi]$ with $1 < p < \infty$ then

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \quad \text{for a.e. } x \in [-\pi, \pi]. \tag{5.6}$$

Sjölin[51] sharpens this result by showing that (5.6) remains true for $f \in L(\log^+ L)(\log^+ \log^+ L)$. Antonov[2] further improves Sjölin's result for all $f \in L(\log^+ L)(\log^+ \log^+ \log^+ L)$.

When the dimension $d-1$ of the sphere $\mathbb{S}^{d-1}$ is bigger than one, summability at the "critical index" $\lambda := \frac{d-2}{2}$ was the correct analogue of the convergence, for phenomena near L^1. In this sense, many classical results for the partial sums S_k of the Fourier series were extended to the higher-dimensional sphere $\mathbb{S}^{d-1}$ for the Cesàro means σ_N^λ at the critical index. For instance, as is well known, (see Refs. 11 and 55), if $f \in L^p(\mathbb{S}^{d-1})$ and $1 < p < \infty$, then

$$\lim_{N\to\infty} \sigma_N^\lambda(f)(x) = f(x) \quad \text{for a.e. } x \in \mathbb{S}^{d-1} \tag{5.7}$$

whereas if $p = 1$, there exists a function $f \in H^1(\mathbb{S}^{d-1}) \subset L^1(\mathbb{S}^{d-1})$ for which $\limsup_{N\to\infty} |\sigma_N^\lambda(f)(x)| = \infty$ for a.e. $x \in \mathbb{S}^{d-1}$, where $H^p(\mathbb{S}^{d-1})$ denotes the Hardy space on $\mathbb{S}^{d-1}$. It is natural, therefore, to ask whether (5.7) holds for every function f "near" $L^1(\mathbb{S}^{d-1})$.

In the case of more variables, for the Bochner–Riesz means $S_R^{\frac{d-1}{2}} f$ of integrable functions on the d-torus T^d at the critical index $\frac{d-1}{2}$, a celebrated result of Stein[52] states that

$$\lim_{R\to\infty} S_R^{\frac{d-1}{2}} f(x) = f(x), \quad \text{a.e. } x \in T^d, \; f \in L^1(\log^+ L)^2(T^d).$$

In Ref. 20, Brown and Wang extended this classical result of Stein to the case of spherical harmonic expansions.

Theorem 5.6 (Brown and Wang[20]). *If* $f \in L^1(\log^+ L)^2(\mathbb{S}^{d-1})$ *then*

$$\lim_{N\to\infty} \sigma_N^\lambda(f)(x) = f(x)$$

holds a.e. on $\mathbb{S}^{d-1}$. *Moreover,*

$$\|E_*^\lambda(f)\|_1 \le c \int_{\mathbb{S}^{d-1}} |f(x)| \log^2(1+|f(x)|) d\sigma(x), \tag{5.8}$$

where $E_*^\lambda f(x) = \sup_N |E_N^\lambda(f)(x)|$.

The EOs E_n^λ play a vital role in the proof of Theorem 5.6 in Ref. 20. It should be pointed out that without using the EOs, it seems very difficult to deduce (5.7) for all $f \in L(\log^+ L)^2$ directly from known estimates on Cesàro means, as observed by Brown and Wang.[20] To be more precise, define the maximal Cesàro operator σ_*^δ by $\sigma_*^\delta f(x) := \sup_N |\sigma_N^\delta f(x)|$ for a complex δ with $\mathrm{Re}\delta > -1$. Using summation by parts and known estimates for Cesàro means of real orders, Bonami and Clerc[11] obtained the following two estimates on σ_*^δ: for $\varepsilon \in (0,1)$ and $\tau \in \mathbb{R}$,

$$\|\sigma_*^{\lambda+\varepsilon+i\tau} f\|_p \le c_d \varepsilon^{-2} e^{c\tau^2} \frac{1}{p-1} \|f\|_p, \quad \text{for all } 1 < p \le 2 \tag{5.9}$$

and

$$\|\sigma_*^{\varepsilon+i\tau} f\|_2 \le c_d \varepsilon^{-1} e^{3|\tau|} \|f\|_2. \tag{5.10}$$

Using (5.9) and (5.10), and applying Stein's interpolation theorem for a family of analytic operators, one can deduce

$$\|\sigma_*^\lambda f\|_p \le c_d (p-1)^{-3} \|f\|_p, \quad 1 < p \le 2,$$

which, using the standard extrapolation technique,[52] would yield (5.7) for all functions in the space $L(\log^+ L)^3$ rather than the slightly larger space $L(\log^+ L)^2$. On the other hand, for the EOs, using some more delicate new estimates, Brown and Wang[20] was able to prove a better estimate

$$\|E_*^\lambda f\|_p \le c_d (p-1)^{-2} \|f\|_p, \quad 1 < p \le 2,$$

which, using extrapolation, will imply (5.8).

It is worthwhile to mention that an important ingredient in the proof of Brown and Wang[20] is the following useful estimates for Jacobi polynomials with complex indices.[b]

[b]Additional factors ε^{-1} and $(\pi-\theta)^{-\varepsilon}$ appear in the original estimates of Ref. 20, but can be easily removed using the analyticity of the indices.

Proposition 5.7 (Brown and Wang[20]). *Let $A \geq 1$ be a given number. Then for $\alpha, \beta \in [0, A]$ and $\tau \in \mathbb{R}$,*

$$|P_k^{(\alpha+i\tau,\beta)}(\cos\theta)| \leq \begin{cases} c_A e^{3|\tau|} k^{\alpha}, & \text{if } 0 \leq \theta \leq k^{-1}, \\ c_A e^{3|\tau|} k^{-\frac{1}{2}} \theta^{-\alpha-\frac{1}{2}} (\pi-\theta)^{-\beta-\frac{1}{2}}, & \text{if } k^{-1} \leq \theta \leq \pi - k^{-1}, \\ c_A e^{3|\tau|} k^{\beta}, & \text{if } \pi - k^{-1} \leq \theta \leq \pi. \end{cases} \tag{5.11}$$

As a very interesting application of (5.11), Wang and Ma[60] obtained the following result on the boundedness of the maximal translation operator.

Theorem 5.8 (Wang and Ma[60]). *For $d \geq 4$ and $p \geq \frac{d-1}{d-2}$,*

$$\|S_* f\|_p \leq c_p \|f\|_p,$$

whereas for $d \geq 3$ and $1 < p < \infty$,

$$\|S_{**} f\|_p \leq C_p \|f\|_p,$$

where $S_* f(x) := \sup_{\theta\in\mathbb{R}} |S_\theta f(x)|$, and $S_{**} f(x) = \sup_{k\in\mathbb{N}} |S_{2^{-k}} f(x)|$.

5.4. *Strong summability of Cesàro means at critical index on Hardy spaces*

According to Definition 5.1, kernels of the EOs E_n^δ are simply constant multiples of the Jacobi polynomials $P_n^{(\frac{d-1}{2}+\delta, \frac{d-3}{2})}(t)$, which are orthogonal with respect to the weight $(1-t)^{\frac{d-1}{2}+\delta}(1+t)^{\frac{d-3}{2}}$ on $[-1, 1]$. The orthogonality of the kernels of the EOs makes them extremely useful in the study of strong approximation by spherical polynomials.

Given $0 < p \leq 1$, we denote by $H^p(\mathbb{S}^{d-1})$ the Hardy spaces on $\mathbb{S}^{d-1}$. For $0 < p \leq 1$, $\delta(p) := \frac{d-1}{p} - \frac{d}{2}$ is known as the critical index for the summability of the Cesàro means σ_N^δ in the Hardy space $H^p(\mathbb{S}^{d-1})$ (see Ref. 24), in the sense that

$$\|\sigma_k^\delta(f)\|_{H^p} \leq C_{\delta,p} \|f\|_{H^p}, \quad k = 0, 1, \ldots$$

holds if and only if $\delta > \delta(p) := \frac{d-1}{p} - \frac{d}{2}$.

Using the EOs, Dai and Wang[25,27] proved the following result.

Theorem 5.9 (Dai and Wang[25,27]). *Let $0 < p \leq 1$ and $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. Then for $f \in H^p(\mathbb{S}^{d-1})$,*

$$\sum_{j=1}^{N} \frac{1}{j} \|\sigma_j^\delta(f) - f\|_{H^p}^p \approx \sum_{j=1}^{N} \frac{1}{j} E_j^p(f, H^p),$$

where

$$E_j(f, H^p) \equiv E_j(f)_{H^p} := \inf_{g\in\Pi_j^d} \|f - g\|_{H^p}.$$

6. The Jackson Inequality on the Sphere

A central topic in polynomial approximation is to connect the rate of approximation to smoothness properties of functions. The core of this theory lies in Jackson's theorem and its Stechkin-type converses. Although for the trigonometric case the direct and inverse results had been proven to be matching pairs long time ago, for spherical polynomial approximation on the sphere the correct formulation and proof of the Jackson inequality and its converse were done only fairly recently. With Riemenschneider, Wang[46] finalized the proof of a Jackson inequality on the sphere.

To formulate the results we need several notations. Let Δ_0 denote the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. As is well known, $f \in \mathcal{H}_n^d$ if and only if $f \in C^2(\mathbb{S}^{d-1})$ and $\Delta_0 f = -n(n+d-2)f$. The fractional *rth order Laplace–Beltrami operator* $(-\Delta_0)^r$, $r > 0$ is defined in a distribution sense by

$$Y_k((-\Delta_0)^r f) = (-k(k+d-2))^r Y_k(f), \quad k = 0, 1, \dots. \tag{6.1}$$

Accordingly, we define the *rth order K-functional* associated with the differential operator $(-\Delta_0)^r$ by

$$K_r(f,t)_p := \inf\{\|f-g\|_p + t^r \|(-\Delta_0)^{\frac{r}{2}} g\|_p : (-\Delta_0)^{r/2} g \in L^p(\mathbb{S}^{d-1})\}. \tag{6.2}$$

Given $1 \le p \le \infty$, we denote by $E_n(f)_p$ the best approximation of a function $f : \mathbb{S}^{d-1} \to \mathbb{R}$ by spherical harmonics of degree $\le n$ in the $L^p(\mathbb{S}^{d-1})$- metric; namely,

$$E_n(f)_p = \inf\{\|f-g\|_p : g \in \Pi_n^d\}, \quad n = 0, 1, \dots.$$

The *rth order difference operator* Δ_t^r with step $t > 0$ is defined by

$$\Delta_t^r := (I - S_t)^{\frac{r}{2}} \equiv \sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} (S_t)^k, \tag{6.3}$$

where $\psi(x) = (1-x)^{\frac{r}{2}}$, while the *rth order modulus of smoothness* of order r in the L^p-metric is given by

$$\omega^r(f,t)_p = \sup_{0<\theta\le t} \|\Delta_\theta^r f\|_p.$$

One of the fundamental results in polynomial approximation on the sphere is the following Jackson type inequality

$$E_n(f)_p \le C_{p,r} \omega^r(f, n^{-1})_p, \quad n \in \mathbb{N}, \ r > 0, \tag{6.4}$$

and its Stechkin type inverse

$$\omega^r(f, n^{-1})_p \le c_{p,r} n^{-r} \sum_{0\le k\le n} (k+1)^{r-1} E_k(f)_p, \tag{6.5}$$

where we assume $f \in L^p(\mathbb{S}^{d-1})$ when $1 \le p < \infty$, and $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$.

The Stechkin type inverse (6.5) is relatively easier to prove. In fact, it is a simple consequence of the sub-linearity of $\omega^r(\cdot,t)_p$ and the fact that for $g \in \Pi_k$ with $0 \le k \le n$,

$$\omega^r(g, n^{-1})_p \le c n^{-r} \|(-\Delta_0)^{\frac{r}{2}} g\|_p \le c \left(\frac{k}{n}\right)^r \|g\|_p. \tag{6.6}$$

The proof of (6.6) is based on summation by parts and can be found in Theorem 4.6.3 of Ref. 59.

On the other hand, however, the proof of the Jackson inequality (6.4) has a very long history, and many researchers have made contributions to this problem. Indeed, (6.4) was proved by Kušnirenko[42] in 1958 for $d = 3$ (the two sphere $\mathbb{S}^2$), $r = 2$ and $p = \infty$, by Butzer and Jansche[23] in 1971 for $r = 2$ and $1 \le p \le \infty$, by Pawelke[45] in 1972 for $r = 2$ and $1 \le p \le \infty$, by Lizorkin and Nikolskii[43] in 1988 for $r > 0$ and $p = 2$ and by Kalyabin[40] in 1987 for $r > 0, 1 < p < \infty$. In 1992–1993, Rustamov[50,49] announced a proof of (6.4) for all $r > 0$ and all $1 \le p \le \infty$. However, it was soon observed by Riemenschneider and Wang[46] that his proof contains a serious flaw at the endpoints $p = 1$ and $p = \infty$. Indeed, the proof of Rustamov[50,49] relies on two crucial lemmas (see Lemma 3.9 of Ref. 50 and Lemma 3 of Ref. 48), which are incorrect at the endpoints $p = 1, \infty$, claiming that the unit ball of the Sobolev space $W_p^r := \{f \in L^p(\mathbb{S}^{d-1}) : (-\Delta_0)^{r/2} f \in L^p(\mathbb{S}^{d-1})\}$ is weakly compact in the space $L^p(\mathbb{S}^{d-1})$.

In 1995, Riemenschneider and Wang finalized the proof of (6.4) for all the remaining cases. Their proof was first announced in Ref. 46, and then written in detail in Chap. 5 of Ref. 59. The following theorem summarizes the main results of Ref. 46.

Theorem 6.1 (Riemenschneider and Wang[46]). *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then the Jackson inequality (6.4) holds for all $r > 0$ and $1 \le p \le \infty$, and furthermore, for $r > 0$, $1 \le p \le \infty$ and $t \in (0, \pi)$,*

$$\omega_r(f,t)_p \approx K_r(f,t)_p \approx \|f - V_t f\|_p + t^r \|(-\Delta)^{r/2} V_t f\|_p, \tag{6.7}$$

where $V_t f = \sum_{0 \le k \le 2/t} \eta(kt) Y_k f$, and η is a $C^{[d/2]+1}(\mathbb{R})$-function be such that $\eta(t) = 1$ for $|t| \le 1$ and $\eta(t) = 0$ for $|t| \ge 2$.

The significance in the equivalence (6.7) lies in the fact that the K-functional is defined in terms of the fractional Laplace–Beltrami operator, and hence has a close connection with spherical harmonic expansions on the sphere, where some powerful techniques in harmonic analysis and orthogonal polynomials are applicable.

It has turned out that techniques developed in Ref. 46 are very useful in treating some other problems in this area. We conclude this section with a brief description of a related result proved by Wang and his graduate students. To this end, we need the following definition: for $\theta \in (0,\pi)$, the average operator B_θ is defined by

$$B_\theta f(x) := \frac{1}{\displaystyle\int_{B(x,\theta)} 1 d\sigma(y)} \int_{B(x,\theta)} f(y) d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

where $B(x,\theta) := \{y \in \mathbb{S}^{d-1} : \arccos x \cdot y \le \theta\}$.

The following result, which confirms a conjecture of Ditzian and Runovskii,[31] was proved by Wang and his two graduate students.

Theorem 6.2 (Dai, Wang and Yu[28]). *For $1 \le p \le \infty$,*

$$\lim_{m\to\infty} \sup_{\theta\in[0,\pi]} \|\theta^2 \Delta_0 B_\theta^m\|_{(p,p)} = 0,$$

where $\|T\|_{(p,p)} = \sup_{\|f\|_p=1} \|Tf\|_p$ for a linear operator defined on $L^p(\mathbb{S}^{d-1})$.

7. Approximation by Bernstein–Durrmeyer Operator on a Simplex

With his two graduate students, Wang[26] confirms a conjecture of Berens, Schmid, and Xu,[9] as well as a related conjecture of Ditzian[30] on equivalence of L^p-norms of different differential operators on the simplex.

We start with some notation and background information. Given $\kappa = (\kappa_1, \ldots, \kappa_{d+1}) \in \mathbb{R}^{d+1}_{\ge 0}$, consider the Jacobi weight function $W_\kappa(x) := \prod_{i=1}^d x_i^{\kappa_i - 1/2}(1 - |x|)^{\kappa_{d+1}-1/2}$ on the simplex

$$S_d := \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \min_{1\le j\le d} x_j \ge 0, 1 - |x| \ge 0 \right\},$$

where $|x| = \sum_{j=1}^d |x_j|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $L^p(W_\kappa; S_d)$ denote the usual weighted Lebesgue space endowed with the norm

$$\|f\|_{\kappa,p} := \left(\int_{S_d} |f(y)|^p W_\kappa(y) dy \right)^{1/p}, \quad 1 \le p < \infty.$$

The weighted Ditzian–Totik K-functional of order $r \in \mathbb{N}$ of a function $f \in L^p(W_\kappa; S_d)$ is defined by

$$K_{r,\Phi}(f,t)_{\kappa,p} := \inf_{g\in C^r(S_d)} \left(\|f-g\|_{\kappa,p} + t \max_{1\le i\le j\le d} \|\varphi_{ij}^r D_{ij}^r g\|_{\kappa,p} \right),$$

where $\varphi_{ii}(x) = \sqrt{x_i(1-|x|)}$, $\varphi_{ij}(x) = \sqrt{x_i x_j}$, $i \ne j$, and

$$D_{ii} = \frac{\partial}{\partial x_i}, \quad D_{ij} = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}, \quad i \ne j.$$

Such a K-functional is equivalent to a computable, weighted Ditzian–Totik modulus of smoothness on the simplex (see, for instance, Refs. 9 and 32).

For each $n \in \mathbb{N}$, the weighted Bernstein–Durrmeyer operator $M_{n,\kappa}$ on S_d is given by

$$M_{n,\kappa}(f)(x) = \sum_{|\mathbf{k}|\le n} p_{\mathbf{k}n}(x) \left(\int_{S_d} p_{\mathbf{k}n}(y) W_\kappa(y)dy\right)^{-1} \int_{S_d} f(y) p_{\mathbf{k}n}(y) W_\kappa(y)dy,$$

where $x = (x_1, \ldots, x_d) \in S_d$, $\mathbf{k} = (k_1, \ldots, k_d)$, $|\mathbf{k}| = |k_1| + |k_2| + \cdots + |k_d|$, and

$$p_{\mathbf{k}n}(x) = \frac{n!}{(n-|\mathbf{k}|)! \prod_{j=1}^{d} k_j!} x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} (1-|x|)^{n-|\mathbf{k}|}.$$

In the unweighted case, (where $\kappa = (\frac{1}{2}, \cdots, \frac{1}{2})$), Berens, Schmid and Xu[9] proved in 1992 that the direct estimate

$$\|f - M_{n,\kappa} f\|_{\kappa,p} \le c K_{2,\Phi}(f, n^{-1})_{\kappa,p} + c n^{-1} \|f\|_{\kappa,p} \tag{7.1}$$

for $1 \le p < \infty$, as well as the reverse inequality

$$K_{2,\Phi}(f, n^{-1})_{\kappa,p} \le C \|f - M_{n,\kappa} f\|_{\kappa,p} \tag{7.2}$$

for $p = 2$. They also conjectured in the unweighted case that (7.2) holds for all $1 < p < \infty$.

This conjecture of Berens, Schmid and Xu[9] was confirmed recently by Dai, Huang and Wang.[26] Indeed, both (7.1) and (7.2) hold for all $1 < p < \infty$ and $\kappa \in \mathbb{R}^{d+1}_{\ge 0}$.

Theorem 7.1 (Dai, Huang and Wang[26]). *If $1 < p < \infty$ and $f \in L^p(W_k; S_d)$, then*

$$c^{-1} K_{2,\Phi}(f, n^{-1})_{\kappa,p} \le \|f - M_{n,\kappa} f\|_{\kappa,p} \le c K_{2,\Phi}(f, n^{-1})_{\kappa,p} + c n^{-1} \|f\|_{\kappa,p}, \tag{7.3}$$

where the constant c is independent of n and f.

The problem of Berens, Schmid and Xu[9] is closely related to a problem of Ditzian,[30] which was formulated with respect to a general Jacobi weight. The background for the problem of Ditzian is as follows. It was shown by Knoop and Zhou[41] in the unweighted case and by Ditzian[29] in the weighted case that for $1 \le p \le \infty$,

$$\|f - M_{n,\kappa} f\|_{\kappa,p} \sim \inf_{g \in C^2(S_d)} \{\|f - g\|_{\kappa,p} + n^{-1} \|P(D)_\kappa g\|_{\kappa,p}\} \equiv: \widetilde{K}_2(f, n^{-1})_{\kappa,p}. \tag{7.4}$$

Here, $P(D)_\kappa$ is a self-adjoint, second-order differential operator in $L^2(W_\kappa; S_d)$ whose eigenvalue expansion in $L^2(W_\kappa; S_d)$ corresponds to the weighted orthogonal polynomial expansion with respect to the weight W_κ on S_d. More precisely, if we denote by $\mathcal{V}_n^d(W_\kappa)$ the space of polynomials of degree n on S_d that are orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(W_\kappa; S_d)$, then a $C^2(S_d)$-function P belongs to the space $\mathcal{V}_n^d(W_\kappa)$ if and only if (see (2.3.11) on p. 46 of Ref. 33)

$$P(D)_\kappa P = -n\left(n + |\kappa| + \frac{d-1}{2}\right) P. \tag{7.5}$$

The differential operator $P(D)_\kappa$ enjoys the following elegant decomposition on S_d:

$$P(D)_\kappa = \sum_{\xi \in E} P(D)_{\kappa,\xi}, \tag{7.6}$$

where

$$P(D)_{\kappa,\xi} := W_\kappa(x)^{-1} \frac{\partial}{\partial \xi}(\widetilde{d}(\xi, x) W_\kappa(x)) \frac{\partial}{\partial \xi}, \tag{7.7}$$

$\widetilde{d}(\xi, x) := \sup\{ts : t, s \geq 0, x + t\xi, x - s\xi \in S_d\}$, and E is the set of all unit directions parallel to the edges of S_d in which we do not distinguish ξ and $-\xi$. Equation (7.6) was proved by Berens, Schmid and Xu[9] in the unweighted case, and by Ditzian[30] in the weighted case. The main point in the decomposition (7.7) is that the differential operators $P(D)_{\kappa,\xi}$ are much simpler than $P(D)_\kappa$, and have several remarkable properties which make them much easier to deal with. In fact, each of the operators $P(D)_{\kappa,\xi}$ is self-adjoint on $L^2(W_\kappa; S_d)$ and invariant on the spaces $\mathcal{V}_n^d(W_\kappa)$ of weighted orthogonal polynomials. As a matter of fact, it is easy to show by integration by parts that for $1 < p < \infty$, (see Refs. 26 and 9),

$$\begin{aligned} C^{-1} \max_{1 \leq i \leq j \leq d} \|\varphi_{ij}^2 D_{ij}^2 g\|_{\kappa,p} &\leq \max_{\xi \in S} \|P(D)_{\kappa,\xi} g\|_{\kappa,p} \\ &\leq C \max_{1 \leq i \leq j \leq d} \|\varphi_{ij}^2 D_{ij}^2 g\|_{\kappa,p} + C\|g\|_{\kappa,p}. \end{aligned} \tag{7.8}$$

In 1995, Ditzian[30] proved the following equivalence for $p = 2$:

$$\|P(D)_\kappa f\|_{\kappa,p} \sim \max_{\xi \in E} \|P(D)_{\kappa,\xi} f\|_{\kappa,p}, \quad 1 < p < \infty, \tag{7.9}$$

which combined with (7.4) and (7.8) will imply both the direct estimate (7.1) and the inverse estimate (7.2). He further conjectured that (7.2) holds for all $1 < p < \infty$. This was also confirmed in the paper of Dai, Huang and Wang[26]:

Theorem 7.2 (Dai, Huang and Wang[26]). *The equivalence holds for all* $f \in C^2(S_d)$ *and* $1 < p < \infty$.

Combining (7.4) with (7.2), one can also establish the following relationship between the two different K-functionals $K_{2,\Phi}(f, n^{-1})_{\kappa,p}$ and $\widetilde{K}_2(f, n^{-1})_{\kappa,p}$.

Corollary 7.3 (Dai, Huang and Wang[26]). *For* $1 < p < \infty$,

$$K_{2,\Phi}(f, n^{-1})_{\kappa,p} \leq c\widetilde{K}_2(f, n^{-1})_{\kappa,p} \leq c^2 K_{2,\Phi}(f, n^{-1})_{\kappa,p} + c^2 n^{-1}\|f\|_{\kappa,p}.$$

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