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JACKSON INEQUALITY FOR BANACH SPACES ON THE SPHERE

F. DAI and Z. DITZIAN

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
e-mail: dfeng@math.ualberta.ca, zditzian@math.ualberta.ca

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Abstract. The best rate of approximation of functions on the sphere by spherical polynomials is majorized by recently introduced moduli of smoothness. The treatment applies to a wide class of Banach spaces of functions.

1. Introduction

For B, a Banach space of functions on the sphere

$$S^{d-1} = \left\{ x = (x_1, \dots, x_d) : \, x_1^2 + \dots + x_d^2 = 1 \right\},$$

new moduli of smoothness $\omega^r(f,t)_B$ were recently introduced in [8]. $\omega^r(f,t)_B$ is given by

$$\omega^r(f,t)_B = \sup \left\{ \left\| \Delta_\rho^r f \right\|_B : \, \rho \in O_t \right\}, \quad t \geqq 0 \tag{1.1}$$

where $\Delta_\rho f(x) = f(\rho x) - f(x)$, $\Delta_\rho^r f(x) = \Delta_\rho \big(\Delta_\rho^{r-1} f(x) \big)$,

$$O_t = \left\{ \rho \in SO(d) : \, \max_{x \in S^{d-1}} \rho x \cdot x \geqq \cos t \right\} \tag{1.2}$$

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and $SO(d)$ is the collection of $d\times d$ orthonormal real matrices with determinants equal to 1.

Some results were proved about $\omega^r(f,t)_B$ in [8] under the condition on B

$$\| f(\rho\,\cdot)\|_B = \| f(\cdot)\|_B, \quad \forall\, \rho \in SO(d) \tag{1.3}$$

i.e. an operation by an element of $SO(d)$ is an isometry, and in most situations under the condition

$$\| f(\rho\,\cdot) - f(\cdot)\|_B \to 0 \quad \text{as} \quad |\rho - I| \to 0 \tag{1.4}$$

where $|\rho-\eta|^2 = \max_{x\in S^{d-1}} \big((\rho x - \eta x)\cdot(\rho x - \eta x)\big)$. (Note that $\max(\rho x\cdot x) \geqq \cos t$ is equivalent to $|\rho - I| \leqq 2\big|\sin\frac{t}{2}\big|$.) Of course when (1.4) fails, we may consider B_0, the subspace of B for which (1.4) is satisfied, and majorizing by $\omega^r(f,t)_B$, the interesting situation is when $f\in B_0$, since otherwise $\omega^r(f,t)_B$ is not $o(1)$ as $t\to 0$.

The space H_k of spherical harmonic polynomials of degree k, is defined by

$$H_k \equiv \big\{\varphi:\ \widetilde{\Delta}\varphi = -k(k+d-2)\varphi\big\} \tag{1.5}$$

where $\widetilde{\Delta}$ is the Laplace–Beltrami differential operator given, using the Laplacian Δ $\Big(\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}\Big)$, by

$$\widetilde{\Delta} f(x) = \Delta F(x), \quad \text{for} \quad x\in S^{d-1} \quad \text{where} \quad F(x) = f\left(\frac{x}{|x|}\right). \tag{1.6}$$

The Laplace–Beltrami operator is the tangential component of the Laplacian on S^{d-1}. We denote by

$$E_n(f)_B \equiv \inf\left\{\|f-\psi\|_B:\ \psi\in B\cap \operatorname{span}\left(\bigcup_{k<n} H_k\right)\right\} \tag{1.7}$$

the rate of best n-th degree spherical harmonic approximation to f in the Banach space B.

We do not assume $H_k\subset B$ for all k, in spite of the fact that in many familiar cases this is so, since we do not use this fact. A simple example that such an assumption is not always satisfied is

$$B = \left\{ f\in L_2(S^{d-1}):\ f = \sum_{k=\ell}^{\infty}\sum_{j=1}^{d_k} a_{k,j}Y_{k,j},\ \sum_{k=\ell}^{\infty}\sum_{j=1}^{d_k} a_{k,j}^2 < \infty\right\}$$

where $Y_{k,j}$ is an orthonormal basis of H_k.

A Jackson-type estimate (or inequality) is

$$E_n(f)_B \leqq C\omega^r\left(f, \frac{1}{n}\right)_B. \tag{1.8}$$

We also assume $B \subset L_1(S^{d-1})$ with

(1.9)
$$\|f\|_{L_1(S^{d-1})} \leqq \|f\|_B \text{ for } f \in B, \text{ and } g = 0 \text{ a.e. in } S^{d-1} \text{ implies } \|g\|_B = 0.$$

Traditionally, the second part of (1.9) is not stated but is implicit (see [10] on S^{d-1} and [11, p. 15] on T). As it is used in our paper, perhaps it is better to state it explicitly.

For $B = L_p(S^{d-1})$, $1 \leqq p < \infty$, the Jackson inequality (1.8), using the moduli given in (1.1) was proved in [9] (and in case $p = \infty$ for $B_0 = C(S^{d-1})$). Here we use the recent innovations in [6] and [10] as well as some new ideas to give a simpler proof of the Jackson-type inequality which is at the same time applicable to a much wider class of spaces than just $L_p(S^{d-1})$. A method using a Marchaud-type inequality to convert a Jackson-type inequality for $r = 1$ or $r = 2$ to a Jackson-type inequality for higher r given here may also be useful for other situations.

We find it surprising that the proof for even d (of S^{d-1}) is simpler and yields a more general result than that for odd d. We deal with the case of odd d as well, as we feel it is important. (After all we live on a dilation of S^{d-1} for $d = 3$.)

It is clear that L_p and Orlicz spaces on S^{d-1} satisfy the conditions on B given in this paper. For other examples of interest we refer the reader to [10, Section 7].

2. Result for even dimensions

Let M_θ be the $d \times d$ (d even) matrix given by

$$M_\theta = \begin{pmatrix} \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} \end{pmatrix}. \tag{2.1}$$

Obviously $M_0 = I$, $(M_\theta)^j = M_{j\theta}$ and $(M_\theta)^{-1} = M_{-\theta}$. The average operator $S_\theta : L_1(S^{d-1}) \to L_1(S^{d-1})$ is given (as usual) by

$$S_\theta f = S_\theta f(x) = \frac{1}{m_\theta} \int_{xy=\cos\theta} f(y) d\gamma(y), \quad S_\theta 1 = 1 \tag{2.2}$$

where $d\gamma$ is the measure on the set $\{y \in S^{d-1} : x \cdot y = \cos\theta\}$ induced by the Lebesgue measure on S^{d-2} $\big(\{y : x \cdot y = \cos\theta\}$ is an isorphic isometric map of dilation on $S^{d-2}\big)$, and m_θ is given by $S_\theta 1 = 1$.

We now have the following result.

THEOREM 2.1. *Suppose $f \in L_1(S^{d-1})$, and d is even. Then we have*

$$S_\theta f(x) = \int_{SO(d)} f(Q^{-1} M_\theta Q x)\, dQ \tag{2.3}$$

where dQ is the Haar measure on $SO(d)$ normalized by $\int_{SO(d)} dQ = 1$.

PROOF. For a fixed $x \in S^{d-1}$ the group

$$SO(d-1, x) = \big\{ \rho \in SO(d) : \rho x = x \big\} \tag{2.4}$$

is an exact copy of $SO(d-2)$. Denote by $d\mu(\rho) = d\mu_x(\rho)$ the Haar measure on $SO(d-1, x)$ normalized to satisfy $\int_{SO(d-1,x)} d\mu(\rho) = 1$. Clearly we can express $S_\theta f(x)$ by

$$S_\theta f(x) = \frac{1}{|S^{d-2}|} \int_{S_x^{d-2}} f(x\cos\theta + y\sin\theta)\, dy \tag{2.5}$$

where $S^{d-2} \approx S_x^{d-2} = \{y \in S^{d-1} : x \cdot y = 0\}$ and $|S^{d-2}|$ is the measure of S^{d-2}. Using now the well-known fact

$$\frac{1}{|S^{\ell-2}|} \int_{S^{\ell-2}} f(y)\, dy = \int_{SO(\ell-1)} f(Qx)\, dQ = \int_{SO(\ell-1)} f(Q^{-1}x)\, dQ, \tag{2.6}$$

we have

$$S_\theta f(x) = \int_{SO(d-1,x)} f(\rho^{-1} z)\, d\mu(\rho) \tag{2.7}$$

for any $z \in S(x, \theta) = \{y \in S^{d-1} : x \cdot y = \cos\theta\}$. (Note that up to this point of the proof we did not use the fact that d is even and hence that part of the proof will be applicable to the proof of the forthcoming Theorem 4.1.)

We now observe that $Q^{-1}M_\theta Qx \in S(x,\theta)$ for all $Q \in SO(d)$ (d even) since $(M_\theta v)\cdot v = \cos\theta$ for all $v \in S^{d-1}$ and

$$(Q^{-1}M_\theta Qx)\cdot x = (M_\theta Qx)\cdot(Qx) = (M_\theta v)\cdot v = \cos\theta.$$

We now use (2.7) (valid for any $z \in S(x,\theta)$) with $z = Q^{-1}M_\theta Qx$ for any $Q \in SO(d)$, and integrate on $Q \in SO(d)$ to obtain

$$S_\theta f(x) = \int_{SO(d-1,x)} f(\rho^{-1}Q^{-1}M_\theta Qx)\,d\mu(\rho)$$

$$= \int_{SO(d)}\int_{SO(d-1,x)} f(\rho^{-1}Q^{-1}M_\theta Qx)\,d\mu(\rho)\,dQ$$

$$= \int_{SO(d-1,x)}\int_{SO(d)} f(\rho^{-1}Q^{-1}M_\theta Q\rho x)\,dQ\,d\mu(\rho)$$

$$= \int_{SO(d-1,x)}\int_{SO(d)} f(\widetilde{Q}^{-1}M_\theta\widetilde{Q}x)\,d\widetilde{Q}\,d\mu(\rho) = \int_{SO(d)} f(\widetilde{Q}^{-1}M_\theta\widetilde{Q}x)\,d\widetilde{Q}$$

where we used Fubini's theorem and $\rho x = x$ for the third equality, and the change of variable $\widetilde{Q} = Q\rho$ together with the invariance under ρ of the Haar measure for the fourth equality. $\square$

COROLLARY 2.2. *Suppose B is a Banach space of functions satisfying* (1.3), (1.4) *and* (1.9). *Then $S_\theta f \in B$,*

$$\|S_\theta f\|_B \leqq \|f\|_B \tag{2.8}$$

and

$$\|S_\theta f - f\|_B \leqq \frac{1}{2}\omega^2(f,\theta)_B \leqq \omega(f,\theta)_B. \tag{2.9}$$

PROOF. Since both $L_1(S^{d-1})$ and B satisfy (1.3) and (1.4), we can consider the integral

$$\int_{SO(d)} f(Q^{-1}M_\theta Qx)\,dQ$$

as a Riemann vector valued integral of a continuous B or L_1 valued function $f(Q^{-1}M_\theta Qx)$ of Q. Note that $|Q_1 - Q_2| < \delta$ implies $\left|Q_1^{-1}M_\theta Q_1 - Q_2^{-1}M_\theta Q_2\right| < 2\delta$. As the limits introduced by the Riemann integration are the same for $L_1(S^{d-1})$ and B (and both exist), they are equal. Using (2.3) for L_1 and hence a.e., (1.9) implies $S_\theta f \in B$. We now use (1.3), and hence $\left\| f(Q^{-1}M_\theta Q\,\cdot)\right\|_B = \left\| f(\cdot)\right\|_B$, to obtain

$$\|S_\theta f\|_B \leqq \int_{SO(d)} \left\| f(Q^{-1}M_\theta Q\,\cdot)\right\|_B dQ \leqq \|f\|_B.$$

The inequality $\|S_\theta f - f\|_B \leqq \omega(f,\theta)_B$ follows from the above and (1.1) for $r=1$. To show the remainder of (2.9) we note that

$$\int_{SO(d)} f(Q^{-1}M_\theta Qx)\,dQ = \int_{SO(d)} f(Q^{-1}M_{-\theta}Qx)\,dQ$$
$$= \int_{SO(d)} f(QM_{-\theta}Q^{-1}x)\,dQ,$$

and use

$$\|\Delta_\rho^2 f\|_B = \left\|\left(T_{\rho^{-1}} - 2I + T_\rho\right) f\right\|_B \quad \text{for} \quad T_\rho f(x) = f(\rho x),$$
$$\rho = Q^{-1}M_\theta Q \quad \text{and} \quad \rho^{-1} = QM_{-\theta}Q^{-1}. \quad \square$$

Theorem 2.1 and Corollary 2.2 imply the boundedness of the Cesàro summability of f, of some order, which in turn is crucial for many results (see [4] and [7]).

For $f \in L_1(S^{d-1})$ where $d \geqq 3$ and

$$P_k f(x) = \int_{S^{d-1}} \sum_{i=1}^{d_k} Y_{k,i}(x)Y_{k,i}(y)f(y)\,dy \tag{2.10}$$

where $\{Y_{k,i}\}_{i=1}^{d_k}$ is (any) orthonormal basis of H_k given in (1.5), the Cesàro summability of order δ is given by

$$C_N^\delta f(x) = \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta P_k f(x) \tag{2.11}$$

where $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(\delta+1)\Gamma(k+1)}$. For $\delta = \ell$ with $\ell \in \mathbf{N}$,

$$\frac{A_{N-k}^\ell}{A_N^\ell} = \left(1 - \frac{k}{N+1}\right) \cdots \left(1 - \frac{k+\ell}{N+\ell+1}\right).$$

For the Jackson inequality it is sufficient to deal with $\delta = \ell$ with some integer ℓ. However, the boundedness of the Cesàro summability for the wider class of spaces B and $\delta > \frac{d-2}{2}$ may be useful in the future and adds no additional difficulty here.

THEOREM 2.3. *For* $\delta > \frac{d-2}{2}$, *d even and* $B \subset L_1(S^{d-1})$ *satisfying* (1.3) *and* (1.4) *we have*

$$C_n^\delta(f,x) = \int_0^\pi \mu_n^\delta(\theta) S_\theta(f,x)\, d\theta; \quad \left\| C_n^\delta(f,\cdot) \right\|_B \leqq C\|f\|_B \tag{2.12}$$

with $C = 1$ *if* $\delta > d-1$.

REMARK 2.4. For $B = L_p$ (2.8) and (2.12) are well-known. For a somewhat less general space B, but for all $d \geqq 3$, (2.8) and (2.12) were proved in [10].

PROOF. In fact (2.12) is known for $f \in L_1(S^{d-1})$ with

$$\mu_n^\delta(\theta) \equiv m(S^{d-2}) K_n^\delta(\cos\theta) \sin^{d-2}\theta \tag{2.13}$$

and

$$\int_0^\pi \left|\mu_n^\delta(\theta)\right| d\theta \leqq C \quad \text{for} \quad \delta > \frac{d-2}{2} \quad \text{with} \quad C = 1 \quad \text{for} \quad \delta > d-1. \tag{2.14}$$

Using Corollary 2.2, $S_\theta f \in B$. Moreover, $S_\theta f$ is a continuous B valued function on θ since

$$\left|Q^{-1}M_{\theta_1}Q - Q^{-1}M_{\theta_2}Q\right| = |M_{\theta_1} - M_{\theta_2}| = \left|M_{\theta_2}^{-1}M_{\theta_1} - I\right| = 2\left|\sin\frac{\theta_1-\theta_2}{2}\right|,$$

and hence

$$\|S_{\theta_1}f - S_{\theta_2}f\|_B \leqq \int_{SO(d)} \left\| f(Q^{-1}M_{\theta_1}Q\,\cdot) - f(Q^{-}1M_{\theta_2}Q\,\cdot)\right\|_B dQ$$

is small when $|\theta_1 - \theta_2|$ is. Therefore, the integral in (2.12) can be construed as a Riemann B valued integral and the inequality $\left\|C_n^\delta(f,\cdot)\right\|_B \leqq C\|f\|_B$ follows from (2.13) and (2.14). □

The Jackson-type estimate for $r > 1$ will be given in Section 6. For $r = 1$ it is given in the following section together with other applications.

3. Applications for the case of even d

In earlier papers results were given which would imply the Jackson inequality here (for even d and $r = 1$) if we assume in addition that spherical harmonic polynomials are dense in B. For even d this is derived from (1.3) and (1.4) in the following theorem.

THEOREM 3.1. *Suppose that* $B \subset L_1(S^{d-1})$ *with even* $d > 3$ *and that* B *satisfies* (1.3) *and* (1.4). *Then* $\operatorname{span}\big(\bigcup_{k=0}^{\infty} H_k\big)$ *is dense in* B.

PROOF. Using (2.7) and recalling $\int_0^\pi \mu_n^\delta(\theta)\,d\theta = 1$, we have

$$C_n^\delta(f,x) - f(x) = \int_0^\pi \mu_n^\delta(\theta)\big(S_\theta(f,x) - f(x)\big)\,d\theta,$$

and hence

$$\big\| C_n^\delta(f,\cdot) - f(\cdot)\big\|_B \leqq \int_0^\pi \big|\mu_n^\delta(\theta)\big|\,\|S_\theta f - f\|_B\,d\theta$$

$$= \int_0^\eta \big|\mu_n^\delta(\theta)\big|\,\|S_\theta f - f\|_B\,d\theta + \int_\eta^\pi \big|\mu_n^\delta(\theta)\big|\,\|S_\theta f - f\|_B\,d\theta.$$

For appropriate $\delta\big(\delta > \frac{d-2}{2}\big)$, we have $\int_0^\eta \big|\mu_n^\delta(\theta)\big|\,d\theta \leqq \int_0^\pi \big|\mu_n^\delta(\theta)\big|\,d\theta \leqq M(\delta)$ and $\int_\eta^\pi \big|\mu_n^\delta(\theta)\big|\,d\theta \leqq \varepsilon$ for $n \geqq n_0(\delta,\eta)$.

Recalling (2.8) and (2.9), we now have

$$\big\| C_n^\delta(f,\cdot) - f(\cdot)\big\|_B \leqq M(\delta)\sup_{\theta \leqq \eta}\|S_\theta f - f\|_B + \varepsilon\cdot 2\|f\|_B$$

$$\leqq M(\delta)\omega(f,\eta)_B + 2\varepsilon\|f\|_B.$$

Using (1.4), we may choose η so that $\omega(f,\eta)_B \leqq \varepsilon$, and we then choose n_0 to complete the proof. $\square$

As a corollary of Theorems 2.1, 2.3 and 3.1 and Corollary 2.2, we may use the results in [4] and [7] to obtain the following result.

THEOREM 3.2. *Suppose* $B \subset L_1(S^{d-1})$ *with even* $d > 3$ *and* B *satisfies* (1.3) *and* (1.4). *Then for any* $f \in B$ *we have*

$$E_n(f)_B \leqq C_\alpha K_{2\alpha}\big(f,\widetilde{\Delta},n^{-2\alpha}\big)_B, \quad \alpha > 0 \tag{3.1}$$

where

$$K_{2\alpha}\big(f,\widetilde{\Delta},t^{2\alpha}\big)_B \equiv \inf\Big(\|f-g\|_B + t^{2\alpha}\big\|(-\widetilde{\Delta})^\alpha g\big\|_B : (-\widetilde{\Delta})^\alpha g \in B\Big) \tag{3.2}$$

and

$$(-\widetilde{\Delta})^\alpha g \sim \sum \big(k(k+d-2)\big)^\alpha P_k g. \tag{3.3}$$

PROOF. See Theorem 3.6 in [4] for integer α and Theorem 5.1 in [7] where (3.1) is proved for fractional α. (In both places the result is more general and proved in a more general setup.) □

We now prove the following strong converse inequality.

THEOREM 3.3. *Suppose $B \subset L_1(S^{d-1})$ with even $d > 3$ and B satisfies* (1.3) *and* (1.4). *Then for $f \in B$ and $|\theta| \leqq \frac{\pi}{2\ell}$*

$$\left\| f + \frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} S_{j\theta} f \right\|_B \approx K_{2\ell}\big(f, \widetilde{\Delta}, \theta^{2\ell}\big)_B. \tag{3.4}$$

PROOF. The proof is the same as that of Theorem 4.1 of [6] as only (2.8), (2.12) and the density of $\operatorname{span}\big(\bigcup_{k=1}^{\infty} H_k\big)$ in B were used. (For L_p, $p = \infty$, it was shown that the result is valid as well but not interesting unless $f \in C(S^{d-1})$, in which case both sides of (3.4) tend to zero as $\theta \to 0$.) □

We remark that the result (3.4) for $\ell = 1$ is sufficient for our purpose below and that was proved for $L_p(S^{d-1})$, $1 \leqq p \leqq \infty$, in [1].

For even $d > 3$ the Jackson-type estimate by the moduli of smoothness given in (1.1) now follows from the previous theorems.

THEOREM 3.4. *Suppose $B \subset L_1(S^{d-1})$ with even $d > 3$ and B satisfies* (1.3) *and* (1.4). *Then*

$$E_n(f)_B \leqq C\omega^2\left(f, \frac{1}{n}\right)_B \leqq 2C\omega\left(f, \frac{1}{n}\right)_B. \tag{3.5}$$

PROOF. Combining Theorems 3.2 and 3.3 for $\ell = 1$ with (2.9) of Corollary 2.2, we obtain our result. □

For $B \subset L_1(S^{d-1})$ satisfying (1.3) and (1.4) denote by B_T the space of functions $f \in B$ for which $Tf \in B$ where the multiplier operator T induced by the sequence $\{\nu_k\}$ is given by

$$Tf \sim \sum \nu_k P_k f \quad \text{for} \quad f \sim \sum P_k f. \tag{3.6}$$

Define $\|f\|_{B_T}$ by

$$\|f\|_{B_T} = \|f\|_B + \|Tf\|_B < \infty. \tag{3.7}$$

REMARK 3.5. We may replace the space B by B_T in Theorems 2.1, 2.3, 3.1, 3.2, 3.3 and 3.4, as B_T satisfies the exact same conditions. We note that for odd d we will impose a condition on B which is not satisfied by B_T.

REMARK 3.6. We could have proved directly the estimate $E_n(f)_B \leqq C\omega\big(f, \frac{1}{n}\big)_B$ by showing

$$\|J_n^\ell f - f\|_B \leqq C\omega\left(f, \frac{1}{n}\right)_B \tag{3.8}$$

where

$$J_n^\ell(f,x) = C_{n,\ell}\int_0^\pi K_n^\ell(\cos\theta)^2 \sin^{d-2}\theta S_\theta(f,x)\,d\theta, \quad J_n^\ell(1,x) = 1 \tag{3.9}$$

for some ℓ (say $\ell > d-1$) or

$$\|V_n f - f\|_B \leqq C\omega\left(f, \frac{1}{n}\right)_B \tag{3.10}$$

where

$$V_n(f,x) = \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) P_k f(x) \tag{3.11}$$

and $\eta(t) \in C^\infty[0,\infty)$, $\eta(t) = 1$ for $t \leqq 1$ and $\eta(t) = 0$ for $t \geqq 2$. Proving (3.8) or (3.10), we do not need to prove Theorems 3.1, 3.2 or 3.3, but we believe that Theorems 3.2 and 3.3 should be given in any case. If we deal with$C_n^\ell f$, we only obtain

$$\|C_n^\ell f - f\|_B \leqq C\log n \cdot \omega\left(f, \frac{1}{n}\right)_B, \tag{3.12}$$

which is optimal, no matter how large ℓ is.

Note that it is sufficient to confirm the optimality of (3.12) just for $\ell > d-1$ for which the kernel $\mu_n^\delta(\theta)$ (mentioned in the proof of Theorem 3.1) is positive. It is also sufficient to show it for some given space, and we choose $B = L_\infty(S^d)$ and for some given function f and we choose $f(x_1,\dots,x_{d-1},x_d) = \sqrt{1-x_d^2}$. For $f(x_1,\dots,x_d) = \sqrt{1-x_d^2}$, $\omega(f,t)_\infty \approx t$ and $\|S_t f - f\|_\infty \approx t$. Simple calculations using the behaviour of the kernel$\mu_n^\delta(\theta)$ (see Theorem 5.2) yield $\big|C_n^\ell f(0,\dots,0,1) - 0\big| \approx n^{-1}\log n$. The proof that (3.12) is valid for all B satisfying (1.3) and (1.4) is computational again using the behaviour of $\mu_n^\ell(\theta)$ for $\ell > d-1$.

4. Basic results for odd dimensions

For a Banach space of functions on S^{d-1} with odd dimension d we cannot prove the results (2.3), (2.8), (2.9) and (2.12) without extra conditions. In [10] (2.8) and (2.12) were proved for the class of functions $SHBS$ satisfying dual conditions to (1.3) and (1.4). Here we impose alongside (1.3) and (1.4) the condition that our Banach space is lattice compatible (B is a Banach lattice), that is

(4.1)
$$\left|f(x)\right| \leqq \left|g(x)\right|, \ g \in B \text{ and } f \in L_1(S^{d-1}) \text{ implies } f \in B \text{ and } \|f\|_B \leqq \|g\|_B.$$

In Section 7 we will compare the theorems resulting from assuming (4.1) with those assuming that B is a $SHBS$ space. It is an open question as to whether the assumption that $B \in SHBS$ or that B satisfies (4.1) (see also Remarks 4.5 and 5.5) are necessary for the proof of the Jackson inequality in case d is odd.

The matrix M_θ is a $d \times d$ orthogonal matrix having along the diagonal the matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $\left(\frac{d-1}{2} \text{ times}\right)$, with 1 the last entry on the diagonal and all other entries equal to zero, that is

$$M_\theta \equiv \begin{pmatrix} \cos\theta & \sin\theta & & & & \\ -\sin\theta & \cos\theta & & & 0 & \\ & & \ddots & & & \\ & & & \cos\theta & \sin\theta & \\ & 0 & & -\sin\theta & \cos\theta & \\ & & & & & 1 \end{pmatrix}. \tag{4.2}$$

Define $A_\theta(f,x)$ (which we denote by $A_\theta f$ when there is no danger of confusion) for $f \in B \subset L_1(S^{d-1})$ (d odd) with B satisfying (1.3) and (1.4) by

$$A_\theta(f,x) = \int_{SO(d)} f\left(Q^{-1}M_\theta Qx\right) dQ, \qquad \int_{SO(d)} dQ = 1. \tag{4.3}$$

The integral (4.3) is well-defined as a Riemann B valued integral since $|Q_1 - Q_2| < \delta$ implies $\left|Q_1^{-1}M_\theta Q_1 - Q_2^{-1}M_\theta Q_2\right| < 2\delta$. We can now give another useful description of $A_\theta(f,x)$.

THEOREM 4.1. *For* $f \in L_1(S^{d-1})$

$$A_\theta(f,x) = C\int_0^{\pi/2} \cos^{d-2}\varphi S_{\psi(\varphi,\theta)}(f,x)\, d\varphi \quad a.e. \tag{4.4}$$

where $C\int_0^{\pi/2}\cos^{d-2}\varphi\,d\varphi = 1$ *and* $\sin\frac{1}{2}\psi(\varphi,\theta) = \sin\frac{\theta}{2}\cos\varphi$. *Moreover, for* $f \in B \subset L_1(S^{d-1})$ *with* B *satisfying* (1.3), (1.4) *and* (1.9) *we have*

$$\|A_\theta f\|_B \leqq \|f\|_B \tag{4.5}$$

and

$$\|A_\theta f - f\|_B \leqq \omega(f,\theta)_B. \tag{4.6}$$

Note that $S_\theta f$ (and hence $S_{\psi(\varphi,\theta)}f$) is defined by (2.2) on $L_1(S^{d-1})$ and is a contraction in $L_1(S^{d-1})$ and that the right hand side of (4.4) is defined in L_1 and hence a.e. (for $x \in S^{d-1}$). However, we can not show that $f \in B$ implies $S_\theta f \in B$. Nevertheless, the integral on the right of (4.4) is in B if f is because the integral on the right of (4.3) is and they are equal.

Proof. We follow essentially the geometric ideas in [9, Lemma 3.1] and Theorem 2.1 here with different M_θ. As in the proof of Theorem 2.1, we endow the geometric ideas with analytic proof in the present more complicated situation. Recall that (2.4), (2.5), (2.6) and (2.7) in the proof of Theorem 2.1 were proved without the assumption that d is even. Replace (2.7) by

$$\int_{SO(d-1,x)} f(\rho^{-1}z)\,d\mu(\rho) = S_\psi f(x) \tag{4.7}$$

for fixed x, $f \in L_1(S^{d-1})$, $SO(d-1,x) = \{\rho \in SO(d) : \rho x = x\}$ and $z \in S(x,\psi) = \{y \in S^{d-1} : x\cdot y = \cos\psi\}$. We now write for any $\rho \in SO(d-1,x)$

$$I = \int_{SO(d)} f(Q^{-1}M_\theta Qx)\,dQ = \int_{SO(d)} f(\rho^{-1}Q^{-1}M_\theta Qx)\,dQ.$$

Therefore,

$$\begin{aligned} I &= \int_{SO(d-1,x)}\int_{SO(d)} f(\rho^{-1}Q^{-1}M_\theta Qx)\,dQ\,d\mu_x(\rho) \\ &= \int_{SO(d)}\int_{SO(d-1,x)} f(\rho^{-1}Q^{-1}M_\theta Qx)\,d\mu_x(\rho)\,dQ \end{aligned}$$

where $d\mu_x(\rho)$ is the Haar measure on $SO(d-1,x)$.

Using (4.7) with $Q^{-1}M_\theta Qx = z$, we have

$$I = \int_{SO(d)} S_{\psi(Q^{-1}M_\theta Qx\cdot x)} f(x)\,dQ$$

where $\psi(t)=\arccos t$. As $(Q^{-1}M_\theta Qx)\cdot x=(M_\theta Qx)\cdot(Qx)=M_\theta y\cdot y$, we may write

$$I=\int_{S^{d-1}} S_{\psi(M_\theta y\cdot y)}f(x)\,dy.$$

Recalling (4.2) and writing $y=(y_1,\dots,y_d)$, we have $(M_\theta y)\cdot y=(1-y_d^2)\cos\theta+y_d^2$, and hence for a given ψ, $\cos\psi=(1-y_d^2)\cos\theta+y_d^2$. If we set $y_d=\sin t$ with $-\frac{\pi}{2}\leqq t\leqq\frac{\pi}{2}$, the measure of all $y=(y_1,\dots,y_d)$ for which $y_d=\sin t$ is proportional to $\cos^{d-2}t$ (that is to the volume of $(y_1,\dots,y_{d-1})$ for which $y_1^2+\dots+y_{d-1}^2=\cos^2 t$). Therefore, for $\Phi(t)$ such that $\cos\Phi(t)=\cos^2 t\cos\theta+\sin^2 t$ for which $\Phi(t)=\psi(M_\theta y\cdot y)$ when $y_d=\sin t$ and $y\in S^{d-1}$, we have

$$I=C\int_{-\pi/2}^{\pi/2} S_{\Phi(t)}f(x)\cos^{d-2}t\,dt=2C\int_0^{\pi/2} S_{\Phi(t)}f(x)\cos^{d-2}t\,dt$$

and as $S_{\Phi(t)}1=1$ and $A_\theta 1=1$, the constant C satisfies $C\int_{-\pi/2}^{\pi/2}\cos^{d-2}t\,dt=1$.

This completes the proof of (4.4). The integral in (4.3) is defined as a Riemann vector valued integral with either L_1 or B values (using (1.4) for B and for $L_1(S^{d-1})$). The limit is the same and is also equal to the right hand side of (4.4) a.e. Using (1.3), this argument implies (4.5), and using (1.4) together with (1.1), it implies (4.6). $\square$

In the next section, we will prove the crucial boundedness of the Cesàro summability, the Jackson inequality for $r=1$ and other results following from them. For this we need the following lemmas.

LEMMA 4.2. *For $f\in L_1(S^{d-1})$, $t\in\left(0,\frac{\pi}{2}\right)$ and a measurable function $m(\theta)$ satisfying $\sup_{\theta\in[t/\sqrt{2},t]}\left|m(\theta)\right|\leqq M(t)$ we have*

$$\left|\frac{1}{t}\int_{t/\sqrt{2}}^{t} m(\theta)S_\theta(f,x)\,d\theta\right|\leqq CM(t)A_t\big(\left|f(\cdot)\right|,x\big)\equiv CM(t)A_t\big(|f|,x\big) \tag{4.8}$$

and

$$\left|\frac{1}{t}\int_{t/\sqrt{2}}^{t} m(\theta)\big(S_\theta(f,x)-f(x)\big)\,d\theta\right|\leqq CM(t)A_t\big(\left|f(\cdot)-f(x)\right|,x\big). \tag{4.9}$$

PROOF. For a fixed t, $t\in\left(0,\frac{\pi}{2}\right)$ set $\sin\frac{\psi(\varphi,t)}{2}\equiv\sin\frac{t}{2}\cos\varphi=\sin\frac{\theta}{2}$ with $\psi(\varphi,t)\in[0,\pi]$ and $t\in[0,\pi]$, which implies $0\leqq\theta\leqq t\leqq\frac{\pi}{2}$. We now have $\left|\frac{d\varphi}{d\theta}\right|$

$= \frac{1}{2}\frac{\cos\frac{\theta}{2}}{\sin\frac{t}{2}\sin\varphi}$ and therefore for $0<\varphi<\frac{\pi}{4}$, $\left|\frac{d\varphi}{d\theta}\right| \geqq \frac{1}{4}\frac{1}{\sin\frac{t}{2}} \geqq \frac{1}{2t}$. Using (4.4) and as $t=\theta$ when $\varphi=0$ and $\theta \leqq \frac{t}{\sqrt{2}}$ when $\varphi=\frac{\pi}{4}$, we write

$$A_t\big(\left|f(\cdot)\right|,x\big) \geqq C_1\int_0^{\pi/4}\cos^{d-2}\varphi S_{\psi(\varphi,t)}\big(\left|f(\cdot)\right|,x\big)\,d\varphi$$

$$\geqq C_2\int_0^{\pi/4} S_{\psi(\varphi,t)}\big(\left|f(\cdot)\right|,x\big)\,d\varphi \geqq \frac{C_2}{2t}\int_{t/\sqrt{2}}^{t} S_\theta\big(\left|f(\cdot)\right|,x\big)\,d\theta$$

$$\geqq \frac{C_2}{2t}\frac{1}{M(t)}\int_{t/\sqrt{2}}^{t}\left|m(\theta)\right| S_\theta\big(\left|f(\cdot)\right|,x\big)\,d\theta$$

$$\geqq \frac{C_2}{2t}\frac{1}{M(t)}\left|\int_{t/\sqrt{2}}^{t} m(\theta)S_\theta\big(f(\cdot),x\big)\,d\theta\right|.$$

To obtain (4.9), replace $f(y)$ by $f(y)-f(x)$. □

LEMMA 4.3. *Suppose $f\in L_1(S^{d-1})$ and $m(\theta)$ is a measurable function satisfying $\left|m(\theta)\right| \leqq M$ for $0<\theta\leqq t$. Then*

(4.10)
$$\left|\int_0^t m(\theta)S_\theta(f,x)\sin^{d-2}\theta\,d\theta\right| \leqq CMt^{d-1}\sum_{j=0}^{\infty}2^{-j(d-1)/2}A_{2^{-j/2}t}\big(|f|,x\big).$$

PROOF. Recalling (4.7), we may write

$$\left|\int_0^t m(\theta)S_\theta(f,x)\sin^{d-2}\theta\,d\theta\right| \leqq M\int_0^t S_\theta\big(|f|,x\big)\,\sin^{d-2}\theta\,d\theta$$

$$\leqq M\sum_{j=0}^{\infty}\int_{t2^{-(j+1)/2}}^{t2^{-j/2}} S_\theta\big(|f|,x\big)\,\big(2^{-j/2}t\big)^{d-2}\,d\theta$$

$$\leqq MCt^{d-1}\sum_{j=0}^{\infty}A_{2^{-j/2}t}\big(|f|,x\big)\,2^{-j(d-1)/2}. \quad \square$$

Using (4.5) and (4.6), we can deduce from Lemma 4.3 the following corollary about $B_t(f,x)$, the average on a cap of the sphere, i.e.

(4.11)
$$\begin{cases} B_t(f,x) = \dfrac{1}{m_1(t)}\displaystyle\int_{x\cdot y\geqq\cos t} f(y)d\sigma(y) = \dfrac{1}{m(t)}\displaystyle\int_0^t S_\theta(f,x)\sin^{d-2}\theta\,d\theta, \\ B_t(1,x)=1. \end{cases}$$

COROLLARY 4.4. *For* $f \in B \subset L_1(S^{d-1})$ *with* B *satisfying* (1.3), (1.4) *and* (4.1) *we have*

$$\|B_t f\|_B \leqq C\|f\|_B \tag{4.12}$$

and

$$\|B_t f - f\|_B \leqq C\omega(f,t)_B. \tag{4.13}$$

PROOF. Note that $m(t) = Cm_1(t) \approx t^{d-1}$ and that by (4.5) $A_{2^{-j/2}t}\big(|f|,x\big) \in B$ if $|f| \in B$, and using (4.1), $|f| \in B$ if $f \in B$. Moreover, $\big\| |f| \big\|_B = \|f\|_B$ and $\big\| A_{2^{-j/2}t} f \big\|_B \leqq \|f\|_B$. Therefore, using (4.10), $\frac{1}{m(t)} \int_0^t S_\theta(f,x) \sin^{d-2}\theta \, d\theta \in B$ and it satisfies (4.12). Following the same argument but using (4.9) instead of (4.8), we have

$$\|B_t f - f\|_B \leqq C \max_{u \leqq t} \big\| A_u\big(\big| f(\cdot) - f(x) \big|, x\big) \big\|_B.$$

We can now write

$$\big\| A_u\big(\big| f(\cdot) - f(x) \big|, x\big) \big\|_B = \bigg\| \int_{SO(d)} \big| f(Q^{-1}M_u Q x) - f(x) \big| \, dQ \bigg\|_B$$

$$\leqq \int_{SO(d)} \big\| f(Q^{-1}M_u Q x) - f(x) \big\|_B \, dQ \leqq \omega(f,u)_B. \quad \square$$

Remark that for odd d we were not able to prove that $S_\theta f$ is bounded in B using only (1.3), (1.4) and (4.1). The second author believes that under these conditions S_θ is a contraction on B.

REMARK 4.5. Corollary 4.4 is valid for B_T given in (3.6) and (3.7) though B_T does not satisfy (4.1). To justify this, we give the estimate on $f \in B$ and on $Tf \in B$ separately. As T is a multiplier operator, it commutes with A_θ, S_θ and B_θ, which allows us to deal with f and Tf separately. We will use this method repeatedly in the next few sections.

5. Boundedness of Cesàro summability and its applications

In this section we prove the boundedness of the Cesàro summability in B, the Jackson inequality for $r = 1$ and some other applications.

THEOREM 5.1. *Suppose* $B \subset L_1(S^{d-1})$ *with* $d \geqq 3$ *and* B *satisfies* (1.3), (1.4) *and* (4.1). *Then for* $\delta > \frac{d-2}{2}$

$$\|C_n^\delta f\|_B \leqq C\|f\|_B \tag{5.1}$$

where $C_n^\delta(f,x)$ *is given by* (2.11).

Recall that for even d, (5.1) was already proved (even without assuming (4.1)).

Note that for the Jackson type result we need only to prove Theorem 5.1 for large enough δ. We prove (5.1) for the optimal δ i.e. $\delta > \frac{d-2}{2}$ for completeness. We need an estimate of the kernel of $C_n^\delta(f,x)$ proved by Bonami and Clerc (based on the text by Szegö [12]). In [2, Corollaire (2.5), p. 234] $n, L, d(x,1)$ and x there is $d-1$, n, θ and $\cos\theta$ here. See also [13, Theorem 2.3.8, p. 53].

THEOREM 5.2 (Bonami-Clerc). *For $K_n^\delta(\cos\theta)$ given by*

$$C_n^\delta(f,x) = \int_0^\pi K_n^\delta(\cos\theta)\sin^{d-2}\theta S_\theta(f,x)\,d\theta \tag{5.2}$$

(for $f \in L_1(S^{d-1})$) the following estimates hold.

I. *For all θ*

$$\left|K_n^\delta(\cos\theta)\right| \leqq Cn^{d-1}.$$

II. *For $0 < \theta \leqq \frac{\pi}{2}$*

$$\left|K_n^\delta(\cos\theta)\right| \leqq \begin{cases} Cn^{\frac{d-2}{2}-\delta}\theta^{-\frac{d-2}{2}-\delta-1}, & \delta \leqq \dfrac{d}{2} \\ Cn^{-1}\theta^{-d}, & \delta \geqq \dfrac{d}{2}. \end{cases}$$

III. *For $\frac{\pi}{2} \leqq \theta < \pi - n^{-1}$*

$$\left|K_n^\delta(\cos\theta)\right| \leqq \begin{cases} Cn^{(d-2)/2-\delta}(\pi-\theta)^{-(d-2)/2}, & \delta \leqq \dfrac{d}{2} \\ Cn^{-1}(\pi-\theta)^{-d+1+\delta}, & \dfrac{d}{2} \leqq \delta \leqq d-1 \\ Cn^{-1}, & \delta > d-1. \end{cases}$$

IV. *For $\pi - n^{-1} \leqq \theta \leqq \pi$*

$$\left|K_n^\delta(\cos\theta)\right| \leqq \begin{cases} Cn^{d-2-\delta}, & 0 < \delta \leqq d-1 \\ Cn^{-1}, & \delta \geqq d-1. \end{cases}$$

PROOF OF THEOREM 5.1. Write for $f \in L_1(S^{d-1})$

$$\left|C_n^\delta(f,x)\right| = \left|\left\{\int_0^{1/n} + \int_{1/n}^{\pi/2} + \int_{\pi/2}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right\} K_n^\delta(\cos\theta)\sin^{d-2}\theta S_\theta(f,x)\,d\theta\right|$$

$$\leqq \left\{ \int_0^{1/n} + \int_{1/n}^{\pi/2} + \int_{\pi/2}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi} \right\} \left| K_n^\delta(\cos\theta) \right| \sin^{d-2}\theta S_\theta\big(|f|,x\big)\,d\theta$$

$$\equiv I_1(x) + I_2(x) + I_3(x) + I_4(x).$$

Clearly, $I_j(x) \in L_1(S^{d-1})$. If we show that $I_j(x) \in B$ and $\left\| I_j(\cdot) \right\|_B \leqq C\|f\|_B$, (4.1) will imply $C_n^\delta(f,x) \in B$ and (5.1). Using (I) of Theorem 5.2, we may apply Lemma 4.3 with $t = \frac{1}{n}$ and $m(\theta) = \left| K_n^\delta(\cos\theta) \right| \leqq Cn^{d-1}$ to obtain $I_1(x) \in B$ and $\left\| I_1(x) \right\|_B \leqq C\|f\|_B$. We now use II of Theorem 5.2 with $\frac{d-2}{2} < \delta \leqq \frac{d}{2}$, $\sin\theta \leqq \theta$ for $\theta < \frac{\pi}{2}$ and $j_0 = \max\big(j : n^{-1}2^{(j+1)/2}\big) \leqq \frac{\pi}{2}$, to obtain

$$\left| I_2(x) \right| \leqq C \left\{ \sum_{j=0}^{j_0} \int_{2^{j/2}n^{-1}}^{2^{(j+1)/2}n^{-1}} n^{\frac{d-2}{2}-\delta} \theta^{-\frac{d-2}{2}-\delta-1} \sin^{d-2}\theta S_\theta\big(|f|,x\big)\,d\theta \right.$$

$$\left. + \int_{\pi/2\sqrt{2}}^{\pi/2} S_\theta\big(|f|,x\big)\,d\theta \right\}$$

$$\leqq C_1 \left\{ \sum_{j=0}^{j_0} n^{\frac{d-2}{2}-\delta} n^{-\frac{d-2}{2}+\delta+1} 2^{\frac{j}{2}[(\frac{d-2}{2})-\delta-1]} \int_{2^{j/2}n^{-1}}^{2^{(j+1)/2}n^{-1}} S_\theta\big(|f|,x\big)\,d\theta \right.$$

$$\left. + A_{\frac{\pi}{2}}\big(|f|,x\big) \right\}$$

$$\leqq C_2 \left\{ \sum_{j=0}^{j_0} 2^{\frac{j}{2}(\frac{d-2}{2}-\delta)} A_{2^{(j+1)/2}n^{-1}}\big(|f|,x\big) + A_{\pi/2}\big(|f|,x\big) \right\},$$

and as all terms are in B, we have $\left\| I_2(x) \right\|_B \leqq C_3\|f\|_B$. For $\frac{d}{2} \leqq \delta$ the result is simpler and in any case the result (5.1) being valid for $\frac{d-2}{2} < \delta \leqq \frac{d}{2}$ implies (5.1) for $\delta \geqq \frac{d}{2}$ as it is a finite average of elements in B.

To estimate $I_3(x)$ and $I_4(x)$ note that

$$\sin^{d-2}\theta = \sin^{d-2}(\pi-\theta), \quad S_{\pi-\theta}\big(|f|,x\big) = S_\theta\big(|f|,-x\big)$$

and that $A_t\big(|f|,-x\big) \in B$ satisfies $\left\| A_t\big(|f|,-x\big) \right\|_B \leqq \|f\|_B$ for $t \leqq \frac{\pi}{2}$. Similarly, we now use III and IV of Theorem 5.2 to obtain

$$\left\| I_3(x) \right\|_B \leqq A_3\|f\|_B \quad \text{and} \quad \left\| I_4(x) \right\|_B \leqq A_4\|f\|_B.$$

(In fact, $A_3 = o(1)$ and $A_4 = o(1)$ as $n \to \infty$.) □

To prove the Jackson inequality for $r = 1$ recall the operator $V_n f$ given by (3.11) which satisfies

$$\|V_n f\|_B \leqq C\|f\|_B \tag{5.3}$$

for B satisfying (1.3), (1.4) and (4.1),

$$V_n f \in \operatorname{span}\left\{\bigcup_{0\leqq k<2n} H_k\right\} \tag{5.4}$$

and

$$V_n\varphi = \varphi \quad \text{for} \quad \varphi \in \operatorname{span}\left\{\bigcup_{k=0}^{n} H_k\right\}. \tag{5.5}$$

While (5.4) and (5.5) are immediate and $V_n f \in L_1(S^{d-1})$ when $f \in L_1(S^{d-1})$, one also has $V_n f \in B$ whenever $f \in B$ and (5.3) because of Theorem 5.1. (It can also be proved directly.) Therefore,

$$\|V_n f - f\|_B \leqq (C+1)E_n(f)_B \quad \text{and} \quad E_n(f)_B \leqq \left\|V_{[n/2]} f - f\right\|_B. \tag{5.6}$$

We can now prove the Jackson inequality for $r = 1$.

THEOREM 5.3. *For $B \subset L_1(S^{d-1})$ satisfying* (1.3), (1.4) *and* (4.1)

$$E_n(f)_B \leqq C\omega\left(f, \frac{1}{n}\right)_B. \tag{5.7}$$

PROOF. Using (5.6), it is sufficient to show that

$$\|V_n f - f\|_B \leqq C\omega\left(f, \frac{1}{n}\right)_B. \tag{5.8}$$

We further note that in (5.8) we may replace f by $f_1 = f - A$ with any constant A and choose $A = \int_{Q\in SO(d)} f(Qx)\,dQ$, and hence

$$\|f_1\|_B = \left\|f(x) - A\right\|_B \leqq \omega(f, \pi)_B. \tag{5.9}$$

In what follows we assume that (5.9) is satisfied by f, that is f is the f_1 described above.

For $K_{n,V}(t)$ given by

(5.10)
$$V_n(f,x) \equiv \int_{S^{d-1}} K_{n,V}(x\cdot y)f(y)\,dy = C\int_0^\pi K_{n,V}(\cos\theta)\sin^{d-2}\theta S_\theta(f,x)\,d\theta,$$

recall that (see [3, Lemma 3.3])

$$\big|K_{n,V}(\cos\theta)\big| \leqq J(\ell)n^{d-1}(1+n\theta)^{-\ell} \tag{5.11}$$

for any integers ℓ (and we assume ℓ is large enough).

We now follow (4.9) of Lemma 4.2, (4.13) of Corollary 4.4 and the proof of Theorem 5.1 to obtain

$$\begin{aligned}\big|V_n(f,x)-f(x)\big| &\leqq C\int_0^\pi \big|K_{n,V}(\cos\theta)\big|\sin^{d-2}\theta S_\theta\big(\big|f(\cdot)-f(x)\big|,x\big)\,d\theta\\ &\leqq CJ(\ell)\bigg\{\int_0^{1/n}+\int_{1/n}^{\pi/2}+\int_{\pi/2}^{\pi}\bigg\}n^{d-1}(1+n\theta)^{-\ell}\\ &\quad\times\sin^{d-2}\theta S_\theta\big(\big|f(\cdot)-f(x)\big|,x\big)\,d\theta = I_1(x)+I_2(x)+I_3(x).\end{aligned}$$

When we show $I_j(x)\in B$ and $\big\|I_j(x)\big\|_B \leqq C_j\omega\big(f,\frac{1}{n}\big)_B$, we will complete the proof. Using (4.9) and (4.13), we have

$$\big\|I_1(x)\big\|_B \leqq C\cdot J(\ell)\bigg\|\int_0^{1/n}(n\theta)^{d-2}nS_\theta\big(\big|f(\cdot)-f(x)\big|,x\big)\,d\theta\bigg\|_B \leqq C_1\omega(f,t)_B.$$

We now estimate $I_2(x)$:

$$\begin{aligned}I_2(x) \leqq C(1)\bigg\{&\sum_{j=0}^{j_0}\int_{\frac{1}{n}2^{j/2}}^{\frac{1}{n}2^{(j+1)/2}} n^{d-1}\frac{1}{(n\theta)^\ell}\theta^{d-2}S_\theta\big(\big|f(\cdot)-f(x)\big|,x\big)\,d\theta\\ &+\int_{\frac{1}{n}2^{(j_0+1)/2}}^{\pi/2} n^{d-\ell-1}S_\theta\big(\big|f(\cdot)-f(x)\big|,x\big)\,d\theta\bigg\}\end{aligned}$$

with $j_0=\max\big(j : 2^{(j_0+1)/2}\leqq\frac{\pi n}{2}\big\}$. Using (4.9),

$$\begin{aligned}I_2(x) \leqq C(2)\bigg\{&\sum_{j=0}^{j_0} n^{d-\ell-1}n^{\ell-d+1}2^{-\frac{j}{2}(\ell-d+1)}A_{\frac{1}{n}2^{(j+1)/2}}\big(\big|f(\cdot)-f(x)\big|,x\big)\\ &+n^{d-\ell-1}A_{\frac{\pi}{2}}\big(\big|f(\cdot)-f(x)\big|,x\big)\bigg\}.\end{aligned}$$

Using $\big\|A_u\big(\big|f(\cdot)-f(x)\big|,x\big)\big\|_B \leqq C\omega(f,u)_B$, we obtain

$$\big\| I_2(x)\big\|_B \leqq C(3)\sum_{j=0}^{\infty} 2^{-\frac{j}{2}(\ell-d+1)}\omega\left(f,\frac{1}{n}2^{(j+1)/2}\right)_B,$$

which, as $\omega(f,2u)_B \leqq 2\omega(f,u)_B$ (see [8]), implies $\big\| I_2(x)\big\|_B \leqq C_2\omega\big(f,\frac{1}{n}\big)_B$ for $\ell+1>d$.

We now estimate $I_3(x)$:

$$I_3(x)=\int_{\pi/2}^{\pi}\big|K_{n,V}(\cos\theta)\big|\sin^{d-2}\theta\big(S_\theta\big(|f|,x\big)+\big|f(x)\big|\big)\,d\theta$$

$$\leqq C\left\{\int_0^{\pi/2} n^{d-1-\ell}S_\theta\big(|f|,-x\big)\,d\theta+\frac{1}{n}\big|f(x)\big|\right\},$$

and following the estimates above, we have

$$\big\| I_3(x)\big\|_B \leqq C_1\frac{1}{n}\|f\|_B.$$

Using (5.9) and (2.3) of [8], we now have

$$\big\| I_3(x)\big\|_B \leqq C_1\frac{1}{n}\omega(f,\pi)_B \leqq C_2\omega\left(f,\frac{1}{n}\right)_B. \quad \square$$

REMARK 5.4. Theorem 5.3 and condition (1.4) establish the density of the spherical polynomials in B and hence, using Theorem 5.1, all the theorems in [4] and [7] are applicable. However, the K-functionals representing the moduli of smoothness in those papers are different from $\omega^r(f,t)_B$, and we still have to prove the general Jackson inequality for $r>1$.

REMARK 5.5. Theorems 5.1 and 5.3 are valid for the space B_T following the procedure mentioned in Remark 4.5.

6. Marchaud inequality and Jackson inequality

The Marchaud inequality on S^{d-1} is given by the following theorem.

THEOREM 6.1. *Suppose $f\in B$, B is a Banach space of functions on S^{d-1} satisfying* (1.3), *and $\omega^r(f,t)_B$ is given by* (1.1) *and* (1.2). *Then*

$$\omega^r(f,t)_B \leqq Ct^r\int_t^A \frac{\omega^{r+1}(f,u)_B}{u^{r+1}}\,du \tag{6.1}$$

for any fixed A and $t<A$.

PROOF. (This was essentially proved in [8].) We use Theorem 3.1 of [8], proved on page 195 there with $q = 1$ for which (3.1) of [8] is simply the triangle inequality. The proof in [8, p. 195] yields

$$\omega^r(f,t)_B \leqq Ct^r \int_t^\infty \frac{\omega^{r+1}(f,t)_B}{t^{r+1}}\,dt,$$

and as $\omega^{r+1}(f,t)_B \leqq \omega^{r+1}(f,\pi)_B$ for $t \geqq \pi$, (6.1) follows with $A = \pi$. Validity for any fixed A follows from (2.3) of [8]. $\square$

REMARK 6.2. The above minor modification to Theorem 3.1 of [8] is valid for any q there. Hence, in (3.2) of Theorem 3.1 of [8] the second term on the right is redundant.

THEOREM 6.3. *Suppose B is a Banach space of functions on S^{d-1} satisfying* (1.3) *and for ℓ large enough*

$$\|C_n^\ell f\|_B \leqq C\|f\|_B. \tag{6.2}$$

Then

$$E_n(f)_B \leqq C_1\omega\left(f,\frac{1}{n}\right)_B \equiv C_1\omega^1\left(f,\frac{1}{n}\right)_B \tag{6.3}$$

implies

$$E_n(f)_B \leqq C_r\omega^r\left(f,\frac{1}{n}\right)_B \quad \textit{for all} \quad r \geqq 1 \tag{6.4}$$

where $\omega^r(f,t)_B$ is given by (1.1) *and* (1.2) *and $E_n(f)_B$ by* (1.7).

PROOF. As shown earlier (see [4]), (6.2) implies the existence of a multiplier operator $V_n f$ satisfying (5.3), (5.4) and (5.5), and therefore (5.6). We may assume that $V_n f$ takes the form given in (3.11), but that is not necessary as long as $V_n f$ is a multiplier operator. (All the operators V_n of de la Vallée-Poussin type i.e. satisfying (5.3), (5.4) and (5.5) which we ever encountered were multiplier operators.) We now prove by induction

$$\|V_n f - f\|_B \leqq C(r)\omega^r\left(f,\frac{1}{n}\right)_B \quad \text{for all} \quad f \in B \quad \text{and} \quad n \geqq 1. \tag{6.5}$$

Assuming (6.5) for $r = k$ and all n, we set $g = f - V_n f$ and write

$$g - V_{[\frac{n}{2}]}g = f - V_n f - V_{[\frac{n}{2}]}f + V_{[\frac{n}{2}]}V_n f = f - V_n f = g.$$

Using (6.5) on g and (6.1), we have

$$\|f-V_nf\|_B=\|g\|_B=\left\|g-V_{[\frac{n}{2}]}g\right\|_B\leqq E_{[\frac{n}{2}]}(g)_B$$

$$\leqq C(k)\omega^k\left(g,\frac{1}{n}\right)_B\leqq C(k)Cn^{-k}\int_{1/n}^{1}\frac{\omega^{k+1}(g,u)_B}{u^{k+1}}\,du$$

$$\leqq C(k)Cn^{-k}\int_{1/n}^{L/n}\frac{\omega^{k+1}(g,u)_B}{u^{k+1}}\,du+C(k)Cn^{-k}\int_{L/n}^{1}\frac{\omega^{k+1}(g,u)_B}{u^{k+1}}\,du$$

$$\leqq\frac{1}{k}C(k)C\omega^{k+1}\left(g,\frac{L}{n}\right)_B+C(k)C\frac{1}{k}\,\frac{1}{L^k}2^{k+1}\|g\|_B.$$

Observe that C and $C(k)$ are constants independent of L,n and g, and hence we may choose $L>1$ so big that $C(k)C\frac{1}{L^k}2^{k+1}\leqq\frac{1}{2}$. (If $\frac{L}{n}\geqq 1$ the second term does not appear.) Therefore, using

$$\omega^{k+1}\left(g,\frac{L}{n}\right)_B\leqq(L+1)^{k+1}\omega^{k+1}\left(g,\frac{1}{n}\right)_B$$

(see [8]) we have

$$\|f-V_nf\|_B=\|g\|_B\leqq\frac{2}{k}C(k)C(L+1)^{k+1}\omega^{k+1}\left(g,\frac{1}{n}\right)_B. \tag{6.6}$$

Note that $L=\left(2^{k+2}C(k)C\right)^{1/k}$ will do in (6.6) and is not dependent on n,g or f. Since $T_\rho V_nf=V_nT_\rho f$ for any n and $\rho\in SO(d)$, we have

$$\omega^{k+1}(g,u)_B\leqq\omega^{k+1}(f,u)_B+\omega^{k+1}(V_nf,u)_B$$

$$\leqq\omega^{k+1}(f,u)_B+A\omega^{k+1}(f,u)_B,$$

and hence (6.5) for $r=k+1$ follows. $\square$

Using Theorems 2.3, 3.1, 5.1, 5.3, [4, Theorem 2.2] and [7, Theorem 3.2], we have for any $B\subset L_1(S^{d-1})$ satisfying (1.3) and (1.4) and for odd dimension (4.1) the Bernstein-type inequality

$$\left\|(-\widetilde{\Delta})^\alpha\varphi_n\right\|_{B_T}\leqq Cn^{2\alpha}\|\varphi_n\|_{B_T}\quad\text{for}\quad\varphi_n\in\operatorname{span}\bigcup_{k=0}^{n}H_k. \tag{6.7}$$

Here we obtain also the following Bernstein-type inequality.

THEOREM 6.4. *For $B \subset L_1(S^{d-1})$ which satisfies* (1.3), (1.4) *and* (4.1) *we have*

$$\left\| \max_{\xi \perp x} \left| \left(\frac{\partial}{\partial \xi} \right)^r \varphi_n(x) \right| \right\|_{B_T} \leqq C n^r \|\varphi_n\|_{B_T}, \quad for \quad \varphi_n \in \operatorname{span} \bigcup_{k=0}^{n} H_k. \tag{6.8}$$

PROOF. Actually we may repeat the proof in [10, Theorem 6.3] verbatim. Note that we use there only the boundedness of $V_n f$, its kernel and (5.1). □

We can now prove

THEOREM 6.5. *For $f \in B \subset L_1(S^{d-1})$ which satisfies* (1.3), (1.4) *and* (4.1)

$$\omega^r \left(f, \frac{1}{n} \right)_B \approx \inf \left(\|f - g\|_B + n^{-r} \left\| \max_{\xi \perp x} \left| \frac{\partial^r}{\partial \xi^r} g(x) \right| \right\|_B \right) \equiv K_r(f, n^{-r})_B \tag{6.9}$$

and

$$\omega^r \left(f, \frac{1}{n} \right)_B \approx \|f - \varphi_n\|_B + n^{-r} \left\| \max_{\xi \perp x} \left| \frac{\partial^r}{\partial \xi^r} \varphi_n \right| \right\|_B \equiv K_r^*(f, n^{-r})_B \tag{6.10}$$

where φ_n is the best (or near best) approximant to f in $\operatorname{span}\big(\bigcup_{k=1}^{n} H_k \big)$.

PROOF. As $K_r(f, n^{-r})_B \leqq K_r^*(f, n^{-r})_B$, it is sufficient to show (I) $\omega^r\big(f, \frac{1}{n}\big)_B \leqq K_r(f, n^{-r})_B$ and (II) $K_r^*(f, n^{-r})_B \leqq \omega^r\big(f, \frac{1}{n}\big)_B$. To prove (I) it is sufficient to show $\omega^r\big(g, \frac{1}{n}\big)_B \leqq n^{-r} \big\| \max_{\xi \perp x} \big| \frac{\partial^r}{\partial \xi^r} g(x) \big| \big\|_B$ which follows [9, p. 28] as the integral in (8.13) there can be considered as a Banach valued Riemann integral. To prove (II) we note first that $\|f - \varphi_n\|_B \leqq \omega^r\big(f, \frac{1}{n}\big)_B$ by Theorem 6.3. We then follow word for word the proof in [9, p. 27]. □

7. Comparisons and concluding remarks

In a recent article [10] the condition on the space B was that $B \in SHBS$, that is, the space B, besides satisfying (1.3), (1.4), and $B \subset L_1(S^{d-1})$, also satisfies $C^m(S^{d-1}) \subset B$, and B can be represented as a space of functions on $SO(d)$ which satisfy

$$\big\| f(\cdot \boldsymbol{v}_1) \big\|_B = \big\| f(\cdot \boldsymbol{v}_2) \big\|_B, \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in S^{d-1} \tag{7.1}$$

and

$$\big\| f(\cdot \boldsymbol{v}) - f(\cdot \boldsymbol{u}) \big\|_B \to 0 \quad \text{as} \quad |\boldsymbol{u} - \boldsymbol{v}| \to 0. \tag{7.2}$$

Here it was already shown that for even d the extra conditions ((7.1) and (7.2)) are not needed. For odd d we replaced (7.1), (7.2) and $C^m(S^{d-1}) \subset B$ by having the space described as B_T (given by (3.7)) with B satisfying (4.1). We do not have a good natural example which differentiates between these spaces of functions. To compare the two sets of conditions we observe that:

(I) The condition here ((4.1) on B) is not enough for us to show that S_θ is a bounded operator on B for odd d.

(II) On the other hand, for the proof of the equivalence between $\omega^r(f,t)_B$ and the appropriate K-functionals or for the proof of the realization result we needed (4.1) anyway.

While the Jackson-type result was not proved for $B \in HSBS$ for odd d, it now follows from other ideas in this paper.

THEOREM 7.1. *For* $f \in B$, B *satisfying* (1.3), (1.4), (7.1), (7.2) *and*

$$C^m(S^{d-1}) \subset B \subset L_1(S^{d-1}), \quad d \geqq 3 \tag{7.3}$$

(for some m*) we have*

$$E_n(f)_B \leqq C\omega^r(f, 1/n)_B. \tag{7.4}$$

PROOF. For even d (7.4) follows from Theorems 2.3, 3.4 and 6.3. Using Theorem 6.3, it is enough to prove (7.4) for $r = 1$. For odd d the operator $A_\theta f \equiv A_\theta(f, x)$ given in (4.3) satisfies (4.5) and (4.6), and hence we need to show only

$$E_n(f)_B \leqq C_1 \|f - A_{1/n} f\|_B. \tag{7.5}$$

Using $V_n f$ given by (3.11) which is well defined for our space as $\|C_n^\ell f\|_B \leqq C_2 \|f\|_B$ for some ℓ as shown in [10], it is sufficient to show

$$\|f - V_n f\|_B \leqq C_3 \|f - A_{1/n} f\|_B. \tag{7.6}$$

To prove (7.6) we follow the proof of (5.3) in [6] observing that the multiplier result needed (used there for $L_p(S^{d-1})$) is valid for $B \in SHBS$ (see Theorem 5.1 of [10]), and in fact whenever one can show $\|C_n^\ell f\|_B \leqq C\|f\|_B$ for some ℓ. Using the description of $A_\theta f$ in Theorem 4.1, $A_\theta f$ is a multiplier operator with

$$m_k(\theta) = C \int_0^{\pi/2} \cos^{d-2} \varphi P_k^\lambda\big(\cos \psi(\theta, \varphi)\big)\, d\varphi,$$

where $P_k^\lambda(t)$ are the ultraspherical polynomials with $\lambda = \frac{d-2}{2}$. The rest of the proof now follows [6]. $\square$

REMARK 7.2. In all theorems of this paper $\omega^r(f,t)_B$ can be replaced by

$$(7.6)' \qquad \omega_*^r(f,t)_B = \sup_{0<\theta\leqq t} \left\{ \|\Delta_\delta^r f\|_B : \delta = QM_\theta Q^{-1},\ Q \in SO(d) \right\}$$

where M_θ is given by (2.1) for even d and by (4.2) for odd d. In fact we use only such matrices in this paper. In Section 6 we used the Marchaud inequality of [8, Theorem 3.1] and $\omega^r(f,2t)_B \leqq 2^r\omega^r(f,t)_B$, both valid for $\omega_*^r(f,t)_B$ as well.

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