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Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights

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Abstract

In one-dimensional case, various important, weighted polynomial inequalities, such as Bernstein, Marcinkiewicz–Zygmund, Nikolskii, Schur, Remez, etc., have been proved under the doubling condition or the slightly stronger A_∞ condition on the weights by Mastroianni and Totik in a recent paper [G. Mastroianni, V. Totik, Weighted polynomial inequalities with doubling and A_∞ weights, Constr. Approx. 16 (1) (2000) 37–71]. The main purpose of this paper is to prove multivariate analogues of these results. We establish analogous weighted polynomial inequalities on some multivariate domains, such as the unit sphere $\mathbb{S}^{d-1}$, the unit ball B^d, and the general compact symmetric spaces of rank one. Moreover, positive cubature formulae based on function values at scattered sites are established with respect to the doubling weights on these multivariate domains. Some of these multi-dimensional results are new even in the unweighted case. Our proofs are based on the investigation of a new maximal function for spherical polynomials.

Keywords: Spherical harmonics; Multivariate polynomial inequalities; Positive cubature formulae; Doubling weights; A_∞ weights

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1. Introduction

Various important, weighted, algebraic and trigonometric polynomial inequalities such as Bernstein, Marcinkiewicz, Nikolskii, Schur, Remez, etc., have been proved for $1 \leqslant p \leqslant \infty$ by G. Mastroianni and V. Totik in a recent remarkable paper [13] under minimal assumption on the weights. It turns out that in most cases this minimal assumption is the doubling condition. Sometimes, however, as for the Remez and Nikolskii inequalities, one needs the slightly stronger A_∞ condition. In [8] Erdélyi showed that most of the inequalities proved in [13] hold even if $0 < p < 1$, while in [9] he established the important Markov–Bernstein-type inequalities for trigonometric polynomials with respect to doubling weights on a finite interval $[-\omega, \omega]$. We refer to [8,9,12–14] for further information.

Our main purpose in this paper is to show multivariate analogues of the weighted polynomial inequalities proved in [8,13]. We will illustrate our method mainly for the spherical polynomials on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$. However, polynomial inequalities on other multivariate domains, such as compact two-point homogeneous manifolds, and the unit ball B^d of $\mathbb{R}^d$ will also be deduced. We shall discuss polynomial inequalities only for $0 < p < \infty$, as in most cases those for $p = \infty$ can be derived directly from known weighted inequalities for trigonometric polynomials and the fact that any spherical polynomial of degree at most n on $\mathbb{S}^{d-1}$ restricted to a great circle of $\mathbb{S}^{d-1}$ is a trigonometric polynomial of degree at most n.

We organize this paper as follows. Section 2 contains some basic notations and facts concerning harmonic analysis on the unit sphere $\mathbb{S}^{d-1}$. In Section 3, we introduce a new maximal function for spherical polynomials on $\mathbb{S}^{d-1}$ and prove a fundamental theorem (Theorem 3.1) related to this new maximal function, as well as some of its useful corollaries. Based on the results obtained in Section 3, we obtain Marcinkiewicz–Zygmund (MZ) type inequalities and positive cubature formulae with doubling weights, Bernstein-type and Schur-type inequalities with doubling weights, as well as Remez-type and Nikolskii-type inequalities with A_∞ weights for spherical polynomials on $\mathbb{S}^{d-1}$ in Sections 4, 5 and 6, respectively. After that, in Section 7, a few remarks concerning weighted polynomial inequalities on general compact two-point homogeneous manifolds are given without detailed proofs. Finally, in Section 8, we deduce analogous weighted polynomial inequalities on the unit ball B^d of $\mathbb{R}^d$ from those already proven weighted inequalities for spherical polynomials on the sphere $\mathbb{S}^{d-1}$.

Throughout the paper, the letter C denotes a general positive constant depending only on the parameters indicated as subscripts, and the notation $A \sim B$ means that there exist two inessential positive constants C_1, C_2 such that $C_1 A \leqslant B \leqslant C_2 A$.

2. Harmonic analysis on $\mathbb{S}^{d-1}$

This section is devoted to a brief description of some basic facts and notations concerning harmonic analysis on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d\colon\ |x| = 1\}$ of $\mathbb{R}^d$. Most of the material in this section can be found in [1,16,21].

Let $d\sigma(x)$ be the usual rotation-invariant measure on $\mathbb{S}^{d-1}$ normalized by

$$\int_{\mathbb{S}^{d-1}} d\sigma(x) = 1.$$

Given a weight function w, we denote by $L_{p,w} \equiv L_{p,w}(\mathbb{S}^{d-1})$ $(0 < p < \infty)$ the Lebesgue space on $\mathbb{S}^{d-1}$ endowed with the quasi-norm

$$\|f\|_{p,w} = \left(\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\, d\sigma(x) \right)^{\frac{1}{p}},$$

and we write, for a measurable subset E of $\mathbb{S}^{d-1}$,

$$w(E) := \int_E w(x)\, d\sigma(x).$$

We denote by $d(x, y)$ the geodesic distance $\arccos x \cdot y$ between two points x and y on $\mathbb{S}^{d-1}$, and by $B(x, r) := \{y \in \mathbb{S}^{d-1}\colon d(x, y) \leqslant r\}$ the spherical cap with center $x \in \mathbb{S}^{d-1}$ and radius $r \in (0, \pi)$. Also, for a measurable subset $E \subset \mathbb{S}^{d-1}$, we denote by χ_E the characteristic function of E and $|E|$ the Lebesgue measure $\sigma(E)$ of E. Given an integer $n \geqslant 0$, the restriction to $\mathbb{S}^{d-1}$ of a harmonic homogeneous polynomial in d variables of total degree n is called a spherical harmonic of degree n, while the restriction to $\mathbb{S}^{d-1}$ of a polynomial in d variables of degree at most n is called a spherical polynomial of degree at most n. We denote by $\mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^{d-1}$ and Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$. It is well known that Π_n^d can be written as a direct sum $\bigoplus_{k=0}^n \mathcal{H}_k^d$ of the spaces of spherical harmonics and that the orthogonal projection Y_k of $L_2(\mathbb{S}^{d-1})$ onto $\mathcal{H}_k^d$ can be expressed as follows:

$$\begin{aligned} Y_k(f)(x) = {} & \frac{(2k+d-2)\Gamma\left(\frac{d-1}{2}\right)\Gamma(k+d-2)}{\Gamma(d-1)\Gamma\left(k+\frac{d-1}{2}\right)} \\ & \times \int_{\mathbb{S}^{d-1}} f(y) P_k^{\left(\frac{d-3}{2}, \frac{d-3}{2}\right)}(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \end{aligned}$$

where and throughout the paper, $P_k^{(\alpha,\beta)}$ denotes the usual Jacobi polynomial as defined in [19, pp. 58–60]. Let η be a C^∞-function on $[0, \infty)$ with the properties that $\eta(x) = 1$ for $0 \leqslant x \leqslant 1$ and $\eta(x) = 0$ for $x \geqslant 2$. We define, for an integer $n \geqslant 1$,

$$K_n(t) = \sum_{k=0}^{2n} \eta\left(\frac{k}{n}\right) \frac{(2k+d-2)\Gamma\left(\frac{d-1}{2}\right)\Gamma(k+d-2)}{\Gamma(d-1)\Gamma\left(k+\frac{d-1}{2}\right)} P_k^{\left(\frac{d-3}{2}, \frac{d-3}{2}\right)}(t), \quad t \in [-1, 1]. \tag{2.1}$$

Then, evidently, for all $f \in \Pi_n^d$,

$$f(x) = \int_{\mathbb{S}^{d-1}} f(y) K_n(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{2.2}$$

We will keep the notations K_n and η for the rest of this paper.

Given $\varepsilon > 0$, we say a finite subset $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separable if

$$\min_{\substack{\omega,\omega' \in \Lambda \\ \omega \neq \omega'}} d(\omega, \omega') \geqslant \varepsilon,$$

while we say it is maximal ε-separable if it is ε-separable and satisfies

$$\max_{x \in \mathbb{S}^{d-1}} \min_{\omega \in \Lambda} d(x, \omega) < \varepsilon.$$

A weight function w on $\mathbb{S}^{d-1}$ is a doubling weight if there exists a constant $L > 0$ (called the doubling constant) such that for any $x \in \mathbb{S}^{d-1}$ and $t > 0$

$$w\big(B(x, 2t)\big) \leqslant L w\big(B(x, t)\big).$$

Following [13], we set $w_0(x) = w_1(x)$ and

$$w_n(x) = n^{d-1} \int_{B(x, \frac{1}{n})} w(y)\, d\sigma(y), \quad n = 1, 2, \ldots, \ x \in \mathbb{S}^{d-1}.$$

From the definition, it is easily seen that for a doubling weight w and an integer $n \geqslant 0$,

$$w_n(x) \leqslant L\big(1 + n d(x, y)\big)^s w_n(y) \quad \text{for all } x, y \in \mathbb{S}^{d-1}, \tag{2.3}$$

where L denotes the doubling constant of w and $s = \log L / \log 2$.

Many of the weights on $\mathbb{S}^{d-1}$ that appear in analysis satisfy the doubling condition; in particular, all weights of the form

$$h_{\alpha,\mathbf{v}}(x) = \prod_{j=1}^{m} |x \cdot v_j|^{\alpha_j}, \quad x \in \mathbb{S}^{d-1}, \tag{2.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j > 0$, $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ and $v_j \in \mathbb{S}^{d-1}$. (The proof of this fact will be given in Section 5.) It is worthwhile to point out that the weights $h_{\alpha,\mathbf{v}}$ play an important role in the theory of multivariate orthogonal polynomials. (For details, we refer to a series of interesting papers [5,11,22–26] by Y. Xu and his collaborators.)

3. A fundamental theorem and its useful corollaries

Let w be a doubling weight on $\mathbb{S}^{d-1}$ and L be its doubling constant. Suppose that $s = \log L/\log 2$ and w_n, K_n are as defined in Section 2. For $\beta > 0$ and $f \in C(\mathbb{S}^{d-1})$, we define

$$f_{\beta,n}^*(x) = \max_{y\in\mathbb{S}^{d-1}} \bigl|f(y)\bigr|\bigl(1 + nd(x,y)\bigr)^{-\beta s}, \quad x \in \mathbb{S}^{d-1},\ n = 0, 1, \dots. \tag{3.1}$$

We will keep these notations for the rest of the paper.

Our main result in this section is the following theorem, which will play a fundamental role in the proofs of the following sections.

Theorem 3.1. *For $0 < p \leqslant \infty$, $f \in \Pi_n^d$ and $\beta > \frac{1}{p}$, we have*

$$\|f\|_{p,w} \leqslant \|f_{\beta,n}^*\|_{p,w} \leqslant C\|f\|_{p,w},$$

where $C > 0$ depends only on d, L and β when β is close to $\frac{1}{p}$.

For the proof of Theorem 3.1, we need the following lemma, which was proved in [3, Lemma 3.3].

Lemma 3.2. *For $\theta \in [0, \pi]$ and any positive integer ℓ,*

$$\bigl|K_n^{(i)}(\cos\theta)\bigr| \leqslant C_{\ell,i} n^{d-1+2i} \min\bigl\{1, (n\theta)^{-\ell}\bigr\}, \quad i = 0, 1, \dots,\ n = 1, 2, \dots,$$

where $K_n^{(0)}(t) = K_n(t)$, $K_n^{(i)}(t) = (\frac{d}{dt})^i\{K_n(t)\}$ for $i \geqslant 1$.

The point of Lemma 3.2 is that the positive integer ℓ can be chosen as large as we like.

Proof of Theorem 3.1. The first inequality $\|f\|_{p,w} \leqslant \|f_{\beta,n}^*\|_{p,w}$ is evident. For the proof of the second inequality, we define, for $g \in L_{1,w}$,

$$M_w(g)(x) = \sup_{0<r\leqslant\pi} \frac{1}{w(B(x,r))} \int_{B(x,r)} \bigl|g(y)\bigr| w(y)\, d\sigma(y).$$

M_w is the weighted Hardy–Littlewood maximal function and it is known that for any doubling weight w and all $1 < p \leqslant \infty$,

$$\bigl\|M_w(g)\bigr\|_{p,w} \leqslant C\left(\frac{p}{p-1}\right)^{\frac{1}{p}} \|g\|_{p,w}, \tag{3.2}$$

where $C > 0$ depends only on the doubling constant of w, and it is understood that $(\frac{p}{p-1})^{1/p} = 1$ when $p = \infty$. In the case of $\mathbb{R}^d$, the proof of (3.2) can be found in [20, pp. 223–225], and the proof given there works equally well for the case of $\mathbb{S}^{d-1}$.

We claim that for all $f \in \Pi_n^d$ and $\beta > 0$

$$f_{\beta,n}^*(x) \leqslant C_1 \big(M_w\big(|f|^{\frac{1}{\beta}}\big)(x)\big)^{\beta} \tag{3.3}$$

where $C_1 > 0$ depends only on d, L and β when β is big. Combining (3.2) with (3.3) we will deduce the desired inequality

$$\|f_{\beta,n}^*\|_{p,w} \leqslant C\|f\|_{p,w}$$

for $\beta > 1/p$.

For the proof of the claim (3.3), we set, for $\delta > 0$ and $y, u \in \mathbb{S}^{d-1}$,

$$A_{n,\delta}(y,u) := \max_{z \in B(y, \frac{\delta}{n})} \big|K_n(y \cdot u) - K_n(z \cdot u)\big|. \tag{3.4}$$

Then, using Lemma 3.2 with $i = 0, 1$, it is easy to verify that for any integer $\ell > 0$,

$$A_{n,\delta}(y,u) \leqslant C_{d,\ell} \begin{cases} n^{d-1}, & \text{if } \theta \in [0, \frac{4\delta}{n}], \\ \delta n^{d-1} \min\{1, (n\theta)^{-\ell}\}, & \text{if } \theta \in [\frac{4\delta}{n}, \pi], \end{cases} \tag{3.5}$$

where $\theta = d(y,u)$. Now we use (2.2) to obtain that for $f \in \Pi_n^d$ and $x, y \in \mathbb{S}^{d-1}$,

$$\begin{aligned} \max_{z \in B(y, \frac{\delta}{n})} \frac{|f(y) - f(z)|}{(1 + nd(x,y))^{\beta s}} &\leqslant f_{\beta,n}^*(x) \int_{\mathbb{S}^{d-1}} \left(\frac{1 + nd(x,u)}{1 + nd(x,y)}\right)^{\beta s} A_{n,\delta}(y,u)\, d\sigma(u) \\ &\leqslant f_{\beta,n}^*(x) \int_{\mathbb{S}^{d-1}} \big(1 + nd(y,u)\big)^{\beta s} A_{n,\delta}(y,u)\, d\sigma(u) \leqslant C_\beta \delta f_{\beta,n}^*(x), \end{aligned}$$

where the last inequality follows by (3.5), and $C_\beta > 0$ is a constant increasing with β. It then follows that for $x, y \in \mathbb{S}^{d-1}$ and $\delta \in (0, \frac{1}{4})$,

$$\begin{aligned} &\big|f(y)\big|^{\frac{1}{\beta}} \big(1 + nd(x,y)\big)^{-s} \\ &\quad \leqslant 2^{\frac{1}{\beta}} \big(1 + nd(x,y)\big)^{-s} \min_{z \in B(y, \frac{\delta}{n})} \big|f(z)\big|^{\frac{1}{\beta}} + \big(2C_\beta \delta f_{\beta,n}^*(x)\big)^{\frac{1}{\beta}} \\ &\quad \leqslant 2^{\frac{1}{\beta}} \big(1 + nd(x,y)\big)^{-s} \left(\int_{B(y, \frac{\delta}{n})} w(z)\, d\sigma(z) \right)^{-1} \int_{B(y, \frac{\delta}{n})} \big|f(z)\big|^{\frac{1}{\beta}} w(z)\, d\sigma(z) \\ &\qquad + \big(2C_\beta \delta f_{\beta,n}^*(x)\big)^{\frac{1}{\beta}} \\ &\quad =: I + \big(2C_\beta \delta f_{\beta,n}^*(x)\big)^{\frac{1}{\beta}}. \end{aligned} \tag{3.6}$$

To estimate I, we consider the following two cases.

Case 1. $\theta := d(x,y) \leqslant \frac{4\delta}{n}$. In this case, $B(y, \frac{\delta}{n}) \subset B(x, \frac{5\delta}{n})$ and

$$\int\limits_{B(y,\frac{\delta}{n})} w(z)\,d\sigma(z) \geqslant L^{-4} \int\limits_{B(y,\frac{16\delta}{n})} w(z)\,d\sigma(z) \geqslant L^{-4} \int\limits_{B(x,\frac{5\delta}{n})} w(z)\,d\sigma(z).$$

It follows that

$$\begin{aligned} I &\leqslant 2^{\frac{1}{\beta}} L^4 \Bigg(\int\limits_{B(x,\frac{5\delta}{n})} w(z)\,d\sigma(z) \Bigg)^{-1} \int\limits_{B(x,\frac{5\delta}{n})} \big|f(z)\big|^{\frac{1}{\beta}} w(z)\,d\sigma(z) \\ &\leqslant 2^{\frac{1}{\beta}} L^4 M_w\big(|f|^{\frac{1}{\beta}}\big)(x); \end{aligned}$$

Case 2. $\frac{4\delta}{n} \leqslant \theta = d(x,y) \leqslant \pi$. In this case, $B(y, \frac{\delta}{n}) \subset B(x, 2\theta)$ and

$$\begin{aligned} \int\limits_{B(y,\frac{\delta}{n})} w(z)\,d\sigma(z) &\geqslant L^{-1} \left(\frac{3\theta n}{\delta}\right)^{-s} \int\limits_{B(y,3\theta)} w(z)\,d\sigma(z) \\ &\geqslant L^{-1} \left(\frac{3\theta n}{\delta}\right)^{-s} \int\limits_{B(x,2\theta)} w(z)\,d\sigma(z). \end{aligned}$$

It follows that

$$\begin{aligned} I &\leqslant L 2^{\frac{1}{\beta}} \left(\frac{3\theta n}{\delta}\right)^{s} (1+n\theta)^{-s} \Bigg(\int\limits_{B(x,2\theta)} w(z)\,d\sigma(z) \Bigg)^{-1} \int\limits_{B(x,2\theta)} \big|f(z)\big|^{\frac{1}{\beta}} w(z)\,d\sigma(z) \\ &\leqslant L 2^{\frac{1}{\beta}} \left(\frac{3}{\delta}\right)^{s} M_w\big(|f|^{\frac{1}{\beta}}\big)(x). \end{aligned}$$

Therefore, in either case, we have

$$I \leqslant 2^{\frac{1}{\beta}} C_2 \delta^{-s} M_w\big(|f|^{\frac{1}{\beta}}\big)(x), \tag{3.7}$$

where $C_2 > 0$ depends only on the doubling constant L. Now substituting (3.7) into (3.6), letting $\delta = (4C_\beta)^{-1}$, and taking the supremum over all $y \in \mathbb{S}^{d-1}$, we deduce

$$\big(f^*_{\beta,n}(x)\big)^{\frac{1}{\beta}} \leqslant C_2 2^{\frac{1}{\beta}} (4C_\beta)^s M_w\big(|f|^{\frac{1}{\beta}}\big)(x) + 2^{-\frac{1}{\beta}} \big(f^*_{\beta,n}(x)\big)^{\frac{1}{\beta}},$$

and the claim (3.3) with $C_1 = \frac{4^{s\beta+1} C_2^{\beta} C_\beta^{s\beta}}{(2^{1/\beta}-1)^{\beta}}$ follows. This completes the proof. □

As an immediate consequence of Theorem 3.1, we have the following result, which seems to be of independent interest.

Corollary 3.3. *For any $\frac{\delta}{n}$-separable subset $\Lambda \subset \mathbb{S}^{d-1}$, $f \in \Pi_n^d$ and $0 < p < \infty$,*

$$\left(\sum_{\omega\in\Lambda}\left|\operatorname{osc}(f)(\omega)\right|^p w\left(B\left(\omega, \frac{\delta}{n}\right)\right)\right)^{\frac{1}{p}} \leqslant C\delta\|f\|_{p,w}, \tag{3.8}$$

where

$$\operatorname{osc}(f)(\omega) = \max_{x,y\in B(\omega,\frac{\delta}{n})}\left|f(x) - f(y)\right|, \tag{3.9}$$

$C > 0$ depends only on d, L and p when p is small.

In the unweighted case, (3.8) was proved for $1 \leqslant p < \infty$ in [3], but the proof there does not work for $0 < p < 1$.

Proof. Using (2.2), we obtain, for any $\omega \in \Lambda$,

$$\max_{y,z\in B(\omega,\frac{\delta}{n})}\left|f(y) - f(z)\right| \leqslant 2\int_{\mathbb{S}^{d-1}}\left|f(u)\right|A_{n,\delta}(\omega, u)\,d\sigma(u),$$

where $A_{n,\delta}$ is defined by (3.4). Thus, by (3.5) and a straightforward computation, we have

$$\operatorname{osc}(f)(\omega) \leqslant 2f^*_{2/p,n}(\omega)\int_{\mathbb{S}^{d-1}}\left(1 + nd(u,\omega)\right)^{\frac{2s}{p}}A_{n,\delta}(\omega, u)\,d\sigma(u) \leqslant C\delta f^*_{2/p,n}(\omega),$$

where $C > 0$ depends only on d, L and p when p is small. Noticing that

$$f^*_{2/p,n}(y) \sim f^*_{2/p,n}(\omega) \quad \text{for } y \in B\left(\omega, \frac{\delta}{n}\right),$$

we obtain

$$\begin{aligned}\sum_{\omega\in\Lambda}\left|\operatorname{osc}(f)(\omega)\right|^p \int_{B(\omega,\frac{\delta}{n})} w(y)\,d\sigma(y) &\leqslant (C\delta)^p\sum_{\omega\in\Lambda}\int_{B(\omega,\frac{\delta}{n})}\left(f^*_{2/p,n}(y)\right)^p w(y)\,d\sigma(y)\\ &\leqslant (C\delta)^p\int_{\mathbb{S}^{d-1}}\left(f^*_{2/p,n}(y)\right)^p w(y)\,d\sigma(y)\\ &\leqslant (C\delta)^p\int_{\mathbb{S}^{d-1}}\left|f(y)\right|^p w(y)\,d\sigma(y),\end{aligned}$$

where in the second inequality we have used the $\frac{\delta}{n}$-separable property of the set Λ, and in the last inequality we have used Theorem 3.1. This completes the proof. $\square$

We also have the following useful corollary.

Corollary 3.4. *For $f \in \Pi_n^d$ and $0 < p < \infty$,*

$$C^{-1}\|f\|_{p,w_n} \leqslant \|f\|_{p,w} \leqslant C\|f\|_{p,w_n},$$

where $C > 0$ depends only on d, L and p when p is small.

Proof. We note that each w_n is again a doubling weight with a doubling constant depending only on d and that of w. Thus, by Theorem 3.1 it will suffice to prove that

$$\|f^*_{2/p,n}\|_{p,w} \sim \|f^*_{2/p,n}\|_{p,w_n}, \tag{3.10}$$

where $f^*_{\beta,n}$ is defined by (3.1) with $s > 0$ replaced by a possibly bigger number s' depending only on the doubling constant L. To show (3.10) we let $\Lambda \subset \mathbb{S}^{d-1}$ be a maximal $\frac{1}{n}$-separable subset. Then noticing that for $x \in B(\omega, \frac{1}{n})$,

$$f^*_{2/p,n}(x) \sim f^*_{2/p,n}(\omega) \quad \text{and} \quad w_n(x) \sim w_n(\omega),$$

we obtain

$$\begin{aligned}
\|f^*_{2/p,n}\|^p_{p,w} &\sim \sum_{\omega\in\Lambda} \int_{B(\omega,\frac{1}{n})} \bigl(f^*_{2/p,n}(x)\bigr)^p w(x)\,d\sigma(x) \\
&\sim n^{-(d-1)} \sum_{\omega\in\Lambda} \bigl(f^*_{2/p,n}(\omega)\bigr)^p w_n(\omega) \sim \sum_{\omega\in\Lambda} \int_{B(\omega,\frac{1}{n})} \bigl(f^*_{2/p,n}(x)\bigr)^p w_n(x)\,d\sigma(x) \\
&\sim \|f^*_{2/p,n}\|_{p,w_n}
\end{aligned}$$

proving (3.10). This completes the proof. $\square$

4. Marcinkiewicz–Zygmund (MZ) inequalities and positive cubature formulae with doubling weights

Our first result in this section is the following

Theorem 4.1. *Let w be a doubling weight on $\mathbb{S}^{d-1}$. Then there exists a positive constant ε depending only on d and the doubling constant of w such that for any $\delta \in (0, \varepsilon)$ and any maximal $\frac{\delta}{n}$-separable subset $\Lambda \subset \mathbb{S}^{d-1}$ there exists a sequence of positive numbers $\lambda_\omega \sim w(B(\omega, \frac{\delta}{n}))$, $(\omega \in \Lambda)$ for which the following holds for all $f \in \Pi_n^d$:*

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\,d\sigma(x) = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega). \tag{4.1}$$

Our next result connects positive cubature formulae and MZ inequalities.

Theorem 4.2. *Let w be a doubling weight on $\mathbb{S}^{d-1}$ and μ be a finite positive measure on $\mathbb{S}^{d-1}$. Suppose that the following equality holds for all $f \in \Pi_{3n}^d$:*

$$\int_{\mathbb{S}^{d-1}} f(x)\, w(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)\, d\mu(x). \tag{4.2}$$

Then for all $0 < p < \infty$ and $f \in \Pi_n^d$, we have

$$\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\, d\sigma(x) \sim \int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\mu(x), \tag{4.3}$$

with the constants of equivalence depend only on d, p and the doubling constant of w.

Remarks. (i) Of particular interest is the case when the measure μ in Theorem 4.2 is supported in a finite subset of $\mathbb{S}^{d-1}$. In this case, equality (4.2) is called a positive cubature formula, while equivalence (4.3) is called a Marcinkiewicz–Zygmund (MZ) type inequality.

(ii) In the unweighted case, positive cubature formulae and MZ inequalities (for $1 \leqslant p \leqslant \infty$) on $\mathbb{S}^{d-1}$ were first established by Mhaskar et al. in the fundamental paper [15], while the estimate $\lambda_\omega \sim |B(\omega, \frac{\delta}{n})|$ for the coefficients in the cubature formula (4.1) (with $w \equiv 1$) was obtained recently by Narcowich et al. in the paper [17]. Also in the unweighted case, a different proof of MZ inequalities and positive cubature formulae on $\mathbb{S}^{d-1}$ was given in [3]. Compared with that of [15], the proof in [3] is simpler and works for all compact two-point homogeneous manifolds.

(iii) In the unweighted case, Theorem 4.2 can be easily proved by the standard duality technique. This technique, however, does not work in the weighted case considered here since the spaces of spherical harmonics are not mutually orthogonal with respect to a measure other than $d\sigma(x)$.

For the proof of Theorem 4.1, we need two lemmas, the first of which is from [15, Proposition 4.1]. Let X be a finite-dimensional normed linear space, X^* be its dual, and $Z \subset X^*$ be a finite subset with cardinality m. We say Z is a norming set for X if the operator $x \mapsto (y^*(x))_{y^* \in Z}$ from X to $\mathbb{R}^m$ is injective. A functional $x^* \in X^*$ is said to be positive with respect to Z if for all $x \in X$, $x^*(x) \geqslant 0$ whenever $\min_{y^* \in Z} y^*(x) \geqslant 0$.

Lemma 4.3 *((Mhaskar et al. [15])). Let X be a finite-dimensional normed linear space, X^* be its dual, $Z \subset X^*$ be a finite, norming set for X, and $x^* \in X^*$ be positive with respect to Z. Suppose further that $\sup_{x \in X} \min_{y^* \in Z} y^*(x) > 0$. Then there exists a sequence of nonnegative numbers ℓ_{y^*}, $(y^* \in Z)$ such that for any $x \in X$,*

$$x^*(x) = \sum_{y^* \in Z} \ell_{y^*} y^*(x).$$

Our next lemma can be stated as follows.

Lemma 4.4. *Let w be a doubling weight and μ be a finite positive measure on $\mathbb{S}^{d-1}$. If* (4.2) *holds for all $f \in \Pi_n^d$, then the following condition must be satisfied*:

$$\mu\left(B\left(x, \frac{2}{n}\right)\right) \leqslant Cw\left(B\left(x, \frac{2}{n}\right)\right) \quad \textit{for all } x \in \mathbb{S}^{d-1},$$

where C depends only on d and the doubling constant of w.

Proof. For a fixed $x \in \mathbb{S}^{d-1}$, we set

$$g_x(y) = \frac{K_{[n/4]}(x \cdot y)}{K_{[n/4]}(1)},$$

where $K_{[n/4]}$ is as defined in (2.1). Note that by Bernstein's inequality for trigonometric polynomials,

$$\left|K_{[n/4]}(x \cdot y) - K_{[n/4]}(1)\right| \leqslant \frac{n}{4}\big(d(x, y)\big)\|K_{[n/4]}\|_\infty = \frac{n}{4}\big(d(x, y)\big)K_{[n/4]}(1).$$

This means that for $d(x, y) \leqslant \frac{2}{n}$,

$$g_x(y) \geqslant \frac{1}{2}.$$

It then follows that

$$\begin{aligned}
\frac{1}{4}\mu\left(B\left(x, \frac{2}{n}\right)\right) &\leqslant \int_{\mathbb{S}^{d-1}} \left|g_x(y)\right|^2 d\mu(y) = \int_{\mathbb{S}^{d-1}} \left|g_x(y)\right|^2 w(y)\, d\sigma(y) \quad \text{(by (4.2))} \\
&\leqslant C \int_{\mathbb{S}^{d-1}} \left|g_x(y)\right|^2 w_n(y)\, d\sigma(y) \quad \text{(by Corollary 3.4)} \\
&\leqslant Cw_n(x) \int_{\mathbb{S}^{d-1}} \left|g_x(y)\right|^2 \big(1 + nd(x, y)\big)^s\, d\sigma(y) \quad \text{(by (2.3))} \\
&\leqslant Cn^{-(d-1)} w_n(x) \quad \text{(by Lemma 3.2 and the fact that } K_{[n/4]}(1) \sim n^{d-1}) \\
&\leqslant C \int_{B(x, \frac{2}{n})} w(y)\, d\sigma(y),
\end{aligned}$$

proving the lemma. □

Now we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We get the idea from [17]. Let $C_2 \geqslant 1$ be a constant depending only on d and L such that the conclusion of Corollary 3.3 with $p=1$ and $C=C_2$ holds. Set $\varepsilon = \frac{1}{6C_2}$ and suppose $\Lambda \subset \mathbb{S}^{d-1}$ is a maximal $\frac{\delta}{n}$-separable subset with $0<\delta<\varepsilon$. Set $n_1 = \left[\frac{\varepsilon n}{\delta}\right]$, $\delta_1 = \frac{n_1}{n}\delta$ and

$$\Lambda(x) = \sum_{\omega\in\Lambda} \chi_{B(\omega,\delta_1/n_1)}(x), \quad x \in \mathbb{S}^{d-1}.$$

Then $n_1 \geqslant n$, $0<\delta_1 \leqslant \varepsilon$ and $1 \leqslant \Lambda(x) \leqslant C_d$.

Now consider the following linear functional on $\Pi_{n_1}^d$:

$$\ell(f) = 2\int_{\mathbb{S}^{d-1}} f(y)w(y)\,d\sigma(y) - \sum_{\omega\in\Lambda}\left(\int_{B(\omega,\frac{\delta_1}{n_1})} \frac{w(x)}{\Lambda(x)}\,d\sigma(x)\right) f(\omega).$$

It will be shown that there exists a sequence of nonnegative numbers μ_ω, $\omega\in\Lambda$ such that

$$\ell(f) = \sum_{\omega\in\Lambda} \mu_\omega f(\omega) \quad \text{for all } f \in \Pi_{n_1}^d. \tag{4.4}$$

We claim that (4.4) is enough for the proof of Theorem 4.1. In fact, once (4.4) is proved then setting

$$\lambda_\omega = \frac{1}{2}\mu_\omega + \frac{1}{2}\int_{B(\omega,\frac{\delta_1}{n_1})} \frac{w(x)}{\Lambda(x)}\,d\sigma(x),$$

we obtain the cubature formula (4.1) for all $f \in \Pi_{n_1}^d \supset \Pi_n^d$. Furthermore, by Lemma 4.4 and the definition of λ_ω, it is easily seen that the equivalence

$$\lambda_\omega \sim \int_{B(\omega,\frac{1}{n_1})} w(y)\,d\sigma(y) \sim \int_{B(\omega,\frac{\delta}{n})} w(y)\,d\sigma(y)$$

holds for all $\omega \in \Lambda$.

The proof of (4.4) is based on Corollary 3.3. In fact, by Corollary 3.3, it is easily seen that each $f \in \Pi_{n_1}^d$ is uniquely determined by its restriction to the set Λ. (This can also be seen from the proof below.) Thus, in view of Lemma 4.3, it will suffice to prove that for any $f \in \Pi_{n_1}^d$ with $\min_{\omega\in\Lambda} f(\omega) \geqslant 0$,

$$\ell(f) \geqslant 0.$$

To see this, we note that if $\omega \in \Lambda$ and $f(\omega) \geqslant 0$, then for all $x \in B(\omega, \frac{\delta_1}{n_1})$,

$$\begin{aligned} f(x) &\geqslant \max_{z\in B(\omega,\frac{\delta_1}{n_1})} \left|f(z)\right| - \max_{z\in B(\omega,\frac{\delta_1}{n_1})} \left(\left|f(z)\right| - f(\omega) + f(\omega) - f(x)\right) \\ &\geqslant \max_{z\in B(\omega,\frac{\delta_1}{n_1})} \left|f(z)\right| - 2\operatorname{osc}(f)(\omega), \end{aligned}$$

where

$$\operatorname{osc}(f)(\omega) = \max_{y,z\in B(\omega,\frac{\delta_1}{n_1})} \left|f(y) - f(z)\right|. \tag{4.5}$$

Thus, for $f \in \Pi_{n_1}^d$ with $\min_{\omega\in\Lambda} f(\omega) \geqslant 0$, we have

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} f(x)w(x)\,d\sigma(x) \\ &\quad= \sum_{\omega\in\Lambda} \int_{B(\omega,\frac{\delta_1}{n_1})} \frac{f(x)}{\Lambda(x)} w(x)\,d\sigma(x) \\ &\quad\geqslant \sum_{\omega\in\Lambda} \Big(\max_{z\in B(\omega,\frac{\delta_1}{n_1})} \left|f(z)\right|\Big) \int_{B(\omega,\frac{\delta_1}{n_1})} \frac{w(x)}{\Lambda(x)}\,d\sigma(x) - 2\sum_{\omega\in\Lambda} \operatorname{osc}(f)(\omega) \int_{B(\omega,\frac{\delta_1}{n_1})} \frac{w(x)}{\Lambda(x)}\,d\sigma(x) \\ &\quad\geqslant \sum_{\omega\in\Lambda} \int_{B(\omega,\frac{\delta_1}{n_1})} \left|f(x)\right| \frac{w(x)}{\Lambda(x)}\,d\sigma(x) - 2\sum_{\omega\in\Lambda} \operatorname{osc}(f)(\omega) \int_{B(\omega,\frac{\delta_1}{n_1})} w(x)\,d\sigma(x), \end{aligned}$$

which, by Corollary 3.3 and the fact that Λ is maximal $\frac{\delta_1}{n_1}$-separable, is greater than or equal

$$(1 - 2C_2\delta_1)\|f\|_{1,w}. \tag{4.6}$$

But, on the other hand, again by Corollary 3.3, we have

$$\begin{aligned} &\left| \int_{\mathbb{S}^{d-1}} f(x)w(x)\,d\sigma(x) - \sum_{\omega\in\Lambda} \Big(\int_{B(\omega,\frac{\delta_1}{n_1})} \frac{w(x)}{\Lambda(x)}\,d\sigma(x)\Big) f(\omega) \right| \\ &\quad\leqslant \sum_{\omega\in\Lambda} \int_{B(\omega,\frac{\delta_1}{n_1})} \left|f(x) - f(\omega)\right| \frac{w(x)}{\Lambda(x)}\,d\sigma(x) \\ &\quad\leqslant \sum_{\omega\in\Lambda} \operatorname{osc}(f)(\omega) \int_{B(\omega,\frac{\delta_1}{n_1})} w(x)\,d\sigma(x) \leqslant C_2\delta_1\|f\|_{1,w}. \end{aligned} \tag{4.7}$$

Therefore, combining (4.6) with (4.7), we have, for $f \in \Pi_{n_1}^d$ with $\min_{\omega\in\Lambda} f(\omega) \geqslant 0$,

$$\ell(f) \geqslant (1 - 3C_2\delta_1)\|f\|_{1,w} \geqslant 0$$

as desired. This completes the proof. $\square$

For the proof of Theorem 4.2, we need two more lemmas.

Lemma 4.5. *Suppose that w is a doubling weight with doubling constant L, $s = \log L/\log 2$, n is a positive integer, and μ is a finite positive measure on $\mathbb{S}^{d-1}$ satisfying*

$$\mu\left(B\left(x, \frac{1}{n}\right)\right) \leqslant K w\left(B\left(x, \frac{1}{n}\right)\right) \quad \text{for all } x \in \mathbb{S}^{d-1}. \tag{4.8}$$

Then for any $0 < p < \infty$ and $f \in \Pi_m^d$ with $m \geqslant n$, we have

$$\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\mu(x) \leqslant C_{p,L,d} K \left(\frac{m}{n}\right)^{s+1} \|f\|_{p,w}^p.$$

Proof. Let $\beta = \frac{1}{p}\frac{s+1}{s}$ and let Λ be a maximal $\frac{1}{n}$-separable subset of $\mathbb{S}^{d-1}$. Note that, for any $x \in \mathbb{S}^{d-1}$ and $m \geqslant n$,

$$f_{\beta,m}^*(x) = \max_{y\in\mathbb{S}^{d-1}} \big(1 + m d(x, y)\big)^{-\beta s} \left|f(y)\right| \geqslant \left(\frac{m}{n}\right)^{-\frac{s+1}{p}} f_{\beta,n}^*(x). \tag{4.9}$$

It follows that for $f \in \Pi_m^d$

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\mu(x) &\leqslant C \sum_{\omega\in\Lambda} \big(f_{\beta,n}^*(\omega)\big)^p \int_{B(\omega,\frac{1}{n})} d\mu(x) \\
&\leqslant CK \sum_{\omega\in\Lambda} \big(f_{\beta,n}^*(\omega)\big)^p \int_{B(\omega,\frac{1}{n})} w(x)\, d\sigma(x) \quad \text{(by (4.8))} \\
&\leqslant CK \int_{\mathbb{S}^{d-1}} \big(f_{\beta,n}^*(y)\big)^p w(y)\, d\sigma(y) \\
&\leqslant CK \left(\frac{m}{n}\right)^{s+1} \|f_{\beta,m}^*\|_{p,w}^p \quad \text{(by (4.9))} \\
&\leqslant CK \left(\frac{m}{n}\right)^{s+1} \|f\|_{p,w}^p \quad \text{(by Theorem 3.1)}.
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.6. *Suppose that α is a fixed nonnegative number, n is a positive integer and f is a nonnegative function on $\mathbb{S}^{d-1}$ satisfying*

$$f(x) \leqslant C_1\bigl(1+nd(x,y)\bigr)^{\alpha} f(y) \quad \text{for all } x, y \in \mathbb{S}^{d-1}. \tag{4.10}$$

Then for any $0 < p < \infty$, there exists a nonnegative spherical polynomial $g \in \Pi_n^d$ such that

$$C^{-1} f(x) \leqslant g(x)^p \leqslant C f(x) \quad \text{for any } x \in \mathbb{S}^{d-1}, \tag{4.11}$$

where $C > 0$ depends only on d, C_1, p and α. In addition, if $e \in \mathbb{S}^{d-1}$ is a fixed point and $f(x) = F(x \cdot e)$ is a nonnegative zonal function on $\mathbb{S}^{d-1}$ satisfying (4.10)*, then the function g in* (4.11) *can be chosen to be a zonal polynomial of the form $G(x \cdot e)$.*

Proof. We get the idea from [13, Lemma 3.2]. We set $m = [\frac{\alpha}{p}] + d + 1$, $n_1 = [\frac{n}{2m}]$, and define

$$T_n(\cos\theta) = \gamma_n \left(\frac{\sin\bigl(n_1 + \frac{1}{2}\bigr)\theta}{\sin\frac{\theta}{2}} \right)^{2m}, \tag{4.12}$$

where γ_n is chosen so that

$$\int_0^{\pi} T_n(\cos\theta) \sin^{d-2}\theta \, d\theta = 1.$$

Then it is easy to verify that $\gamma_n \sim n^{d-1-2m}$ and

$$T_n(\cos\theta) \leqslant C n^{d-1} \min\bigl\{1, (n\theta)^{-2m}\bigr\}. \tag{4.13}$$

Now we claim that the function

$$g(x) = \int_{\mathbb{S}^{d-1}} f(y)^{\frac{1}{p}} T_n(x \cdot y) \, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{4.14}$$

has the desired properties. In fact, since by definition (4.12) we can write

$$T_n(\cos\theta) = \sum_{i=0}^{n} C_{n,i} \cos^i \theta,$$

where $C_{n,i}$ are some constants, it follows that g is a nonnegative spherical polynomial of degree at most n. Also, using (4.10) and (4.13), we have, for any $x \in \mathbb{S}^{d-1}$,

$$g(x) \leqslant C_1^{\frac{1}{p}} f(x)^{\frac{1}{p}} \int_{\mathbb{S}^{d-1}} \bigl(1 + nd(x,y)\bigr)^{\frac{\alpha}{p}} T_n(x \cdot y) \, d\sigma(y) \leqslant C f(x)^{\frac{1}{p}},$$

proving the upper estimate.

Meanwhile, the lower estimate is straightforward:

$$g(x) \geqslant \int\limits_{d(x,y)\leqslant \frac{1}{2n}} f(y)^{\frac{1}{p}} T_n(x\cdot y)\,d\sigma(y) \geqslant Cf(x)^{\frac{1}{p}} \int\limits_0^{\frac{1}{2n}} n^{d-1}\theta^{d-2}\,d\theta \geqslant Cf(x)^{\frac{1}{p}}.$$

To complete the proof, we only need to note that, by definition (4.14), if $f(x) = F(x\cdot e)$ is a zonal function with pole at a fixed point $e \in \mathbb{S}^{d-1}$ so is $g(x)$. $\square$

Now we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. The inequality

$$\int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\mu(x) \leqslant C \int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x)$$

follows directly from Lemmas 4.4 and 4.5. Thus, it remains to prove the inverse inequality

$$\int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x) \leqslant C \int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\mu(x). \tag{4.15}$$

In the case $w \equiv 1$, (4.15) can be easily deduced by the standard duality argument (see, for example, [3]). This argument, however, does not work for the weighted case. Here, we have to use a different approach.

Using (2.2) and Hölder's inequality, we obtain that for $x \in \mathbb{S}^{d-1}$ and any $f \in \Pi_n^d$,

$$\left|f(x)\right| \leqslant C\bigg(\int\limits_{\mathbb{S}^{d-1}} \left|f(y)\right|^2 \left|K_n(x\cdot y)\right| d\sigma(y)\bigg)^{\frac{1}{2}}.$$

It then follows by (2.3) that, for $0 < p < \infty$,

$$\begin{aligned}&\left|f(x)\right|^p w_n(x)\\&\quad\leqslant C\bigg(\int\limits_{\mathbb{S}^{d-1}} \left|f(y)\right|^2 \left|K_n(x\cdot y)\right| \big(1+nd(x,y)\big)^{\frac{2}{p}s} \big(w_n(y)\big)^{\frac{2}{p}}\,d\sigma(y)\bigg)^{\frac{p}{2}}.\end{aligned} \tag{4.16}$$

Note, by Lemma 4.6, however, that there exist a nonnegative spherical polynomial $Q_1 \in \Pi_{[n/2]}^d$ and a nonnegative zonal spherical polynomial $Q_2(x\cdot y) \in \Pi_{[n/2]}^d$ such that

$$\begin{aligned}&Q_1(y) \sim \big(w_n(y)\big)^{\frac{2}{p}-1},\\&Q_2(y\cdot x) \sim n^{d-1}\big(1+nd(y,x)\big)^{-\kappa} \quad \text{for any } y \in \mathbb{S}^{d-1},\end{aligned} \tag{4.17}$$

where $\kappa > (d-1)\max\{\frac{2}{p}, 1\}$ is a fixed integer. Thus, using (4.16) and Lemma 3.2 with $\ell = \kappa + [\frac{2s}{p}] + 1$, we deduce

$$\begin{aligned}
\left|f(x)\right|^p w_n(x) &\leqslant C\Bigl(\int_{\mathbb{S}^{d-1}} \left|f(y)\right|^2 Q_2(x\cdot y)Q_1(y)w_n(y)\,d\sigma(y)\Bigr)^{\frac{p}{2}} \\
&\leqslant C\Bigl(\int_{\mathbb{S}^{d-1}} \left|f(y)\right|^2 Q_2(x\cdot y)Q_1(y)w(y)\,d\sigma(y)\Bigr)^{\frac{p}{2}} \quad \text{(by Corollary 3.4)} \\
&= C\Bigl(\int_{\mathbb{S}^{d-1}} \left|f(y)\right|^2 Q_2(x\cdot y)Q_1(y)\,d\mu(y)\Bigr)^{\frac{p}{2}} \quad \text{(by (4.2)).} \qquad (4.18)
\end{aligned}$$

To show (4.15) for $0 < p \leqslant 2$, we let Λ be a maximal $\frac{1}{n}$-separable subset of $\mathbb{S}^{d-1}$. We then obtain from (4.18) that

$$\begin{aligned}
\left|f(x)\right|^p w_n(x) &\leqslant C\sum_{\omega\in\Lambda}\Bigl|\int_{B(\omega,\frac{1}{n})} \left|f(y)\right|^2 Q_2(x\cdot y)Q_1(y)\,d\mu(y)\Bigr|^{\frac{p}{2}} \\
&\leqslant C\sum_{\omega\in\Lambda}\bigl(f^*_{2/p,n}(\omega)\bigr)^{(2-p)\frac{p}{2}}\bigl(Q_2(x\cdot\omega)\bigr)^{\frac{p}{2}}\bigl(w_n(\omega)\bigr)^{1-\frac{p}{2}} \\
&\quad\times\Bigl(\int_{B(\omega,\frac{1}{n})} \left|f(y)\right|^p\,d\mu(y)\Bigr)^{\frac{p}{2}}.
\end{aligned}$$

Integrating with respect to $x \in \mathbb{S}^{d-1}$, we obtain

$$\begin{aligned}
&\|f\|_{p,w}^p \leqslant C\|f\|_{p,w_n}^p \\
&\quad\leqslant Cn^{(d-1)(\frac{p}{2}-1)}\sum_{\omega\in\Lambda}\bigl(f^*_{2/p,n}(\omega)\bigr)^{(2-p)\frac{p}{2}}\bigl(w_n(\omega)\bigr)^{1-\frac{p}{2}}\Bigl(\int_{B(\omega,\frac{1}{n})} \left|f(y)\right|^p\,d\mu(y)\Bigr)^{\frac{p}{2}} \\
&\quad\leqslant C\Bigl(\int_{\mathbb{S}^{d-1}} \left|f(y)\right|^p\,d\mu(y)\Bigr)^{\frac{p}{2}} \\
&\qquad\times\Bigl(\sum_{\omega\in\Lambda}\int_{B(\omega,\frac{1}{n})} \left|f^*_{2/p,n}(y)\right|^p w_n(y)\,d\sigma(y)\Bigr)^{1-\frac{p}{2}} \quad \text{(by Hölder's inequality)}
\end{aligned}$$

$$\leqslant C\bigg(\int_{\mathbb{S}^{d-1}} \big|f(y)\big|^p \,d\mu(y)\bigg)^{\frac{p}{2}} \|f^*_{2/p,n}\|_{p,w_n}^{p(1-\frac{p}{2})}$$

$$\leqslant C\bigg(\int_{\mathbb{S}^{d-1}} \big|f(y)\big|^p \,d\mu(y)\bigg)^{\frac{p}{2}} \|f\|_{p,w}^{p(1-\frac{p}{2})} \quad \text{(by Theorem 3.1 and Corollary 3.4).}$$

The desired inequality

$$\|f\|_{p,w} \leqslant C\bigg(\int_{\mathbb{S}^{d-1}} \big|f(x)\big|^p \,d\mu(x)\bigg)^{\frac{1}{p}}$$

in the case $0 < p \leqslant 2$ then follows.

It remains to show (4.15) for $2 < p < \infty$. In this case, using (4.18) and Hölder's inequality, we obtain

$$\big|f(x)\big|^p w_n(x) \leqslant C\bigg(\int_{\mathbb{S}^{d-1}} \big|f(y)\big|^p Q_2(x\cdot y)\,d\mu(y)\bigg) \times \bigg(\int_{\mathbb{S}^{d-1}} Q_2(x\cdot y)\big|Q_1(y)\big|^{\frac{p}{p-2}}\,d\mu(y)\bigg)^{\frac{p}{2}-1}. \tag{4.19}$$

Note that by Lemma 4.6 and (4.17), there exists a nonnegative spherical polynomial $Q_3 \in \Pi_n^d$ such that

$$Q_3(y) \sim Q_1(y)^{\frac{p}{p-2}} \sim w_n(y)^{-1} \quad \text{for all } y \in \mathbb{S}^{d-1}.$$

It then follows by Lemmas 4.4 and 4.5 that

$$\bigg(\int_{\mathbb{S}^{d-1}} Q_2(x\cdot y)\big|Q_1(y)\big|^{\frac{p}{p-2}}\,d\mu(y)\bigg)^{\frac{p}{2}-1} \leqslant C\bigg(\int_{\mathbb{S}^{d-1}} Q_2(x\cdot y)Q_3(y)w_n(y)\,d\sigma(y)\bigg)^{\frac{p}{2}-1}$$
$$\leqslant C\bigg(\int_{\mathbb{S}^{d-1}} Q_2(x\cdot y)\,d\sigma(y)\bigg)^{\frac{p}{2}-1} \leqslant C, \tag{4.20}$$

where the last inequality follows by (4.17). Now combining (4.19) with (4.20) and integrating with respect to $x \in \mathbb{S}^{d-1}$, we conclude, for $2 < p < \infty$,

$$\|f\|_{p,w}^p \leqslant C\|f\|_{p,w_n}^p \leqslant C\int_{\mathbb{S}^{d-1}} \big|f(y)\big|^p \,d\mu(y)$$

as desired.

This completes the proof. □

5. Bernstein-type and Schur-type inequalities with doubling weights

Given a positive integer ℓ and two vectors $x, \xi \in \mathbb{S}^{d-1}$ with $x \cdot \xi = 0$, we define the ℓth tangent directional derivative $(\frac{\partial}{\partial \xi})^\ell f(x)$ of $f \in C^\ell(\mathbb{S}^{d-1})$ in the direction ξ at the point x by

$$\left(\frac{\partial}{\partial \xi}\right)^\ell f(x) = \left\{\left(\frac{\partial}{\partial \theta}\right)^\ell \big(f(x\cos\theta + \xi\sin\theta)\big)\right\}\bigg|_{\theta=0}.$$

One of the main results in this section is the following Bernstein-type inequality.

Theorem 5.1. *Let $\ell > 0$ be an integer, w be a doubling weight and $0 < p < \infty$. Then for all $f \in \Pi_n^d$,*

$$\left(\int_{\mathbb{S}^{d-1}} \sup_{\substack{x\cdot\xi=0\\ \xi\in\mathbb{S}^{d-1}}} \left|\left(\frac{\partial}{\partial \xi}\right)^\ell f(x)\right|^p w(x)\,d\sigma(x)\right)^{\frac{1}{p}} \leqslant Cn^\ell \|f\|_{p,w}$$

where $C > 0$ depends only on d, ℓ, the doubling constant of w and p when p is small.

In the unweighted case, Theorem 5.1 for $1 \leqslant p \leqslant \infty$ is due to Ditzian [4].

We denote by Δ the usual Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. It is well known that if $f \in C^2(\mathbb{S}^{d-1})$ and $\{\xi_i\}_{i=1}^{d-1} \cup \{x\} \subset \mathbb{S}^{d-1}$ is an orthonormal basis for $\mathbb{R}^d$ then

$$\Delta f(x) = \sum_{i=1}^{d-1}\left(\frac{\partial}{\partial \xi_i}\right)^2 f(x).$$

This implies that for $f \in C^2(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$,

$$\left|\Delta f(x)\right| \leqslant (d-1) \sup_{\substack{\xi\cdot x=0\\ \xi\in\mathbb{S}^{d-1}}} \left|\left(\frac{\partial}{\partial \xi}\right)^2 f(x)\right|.$$

Therefore, as an immediate consequence of Theorem 5.1, we have

Corollary 5.2. *Let w be a doubling weight, ℓ be a positive integer and $0 < p < \infty$. Then for all $f \in \Pi_n^d$, we have*

$$\left\|\Delta^\ell f\right\|_{p,w} \leqslant Cn^{2\ell} \|f\|_{p,w},$$

where $\Delta^{i+1} f = \Delta(\Delta^i f)$ *for* $i \geqslant 1$, $C > 0$ *depends only on* d, ℓ, *the doubling constant of* w *and* p *when* p *is small.*

Theorem 5.1 and Corollary 5.2 for $0 < p < 1$ are new even in the unweighted case. For the proof of Theorem 5.1, we need the following

Lemma 5.3. *Let* x, y *be two fixed points on* $\mathbb{S}^{d-1}$, $\xi \in \mathbb{S}^{d-1}$ *be such that* $\xi \cdot x = 0$ *and let*

$$\varphi(\theta) \equiv \varphi_{x,y,\xi}(\theta) = K_n(x \cdot y \cos\theta + \xi \cdot y \sin\theta).$$

Then for any positive integers v *and* m, *we have*

$$\left|\varphi^{(v)}(0)\right| \leqslant Cn^{d-1+v} \min\left\{1, \left(nd(x,y)\right)^{-m}\right\}, \tag{5.1}$$

where $C > 0$ *depends only on* v *and* m.

Proof. By induction on v it can be easily seen that $\varphi^{(v)}(\theta)$ can be written in the form

$$\varphi^{(v)}(\theta) = \sum_{i=1}^{v} \sum_{\substack{j_0+j_1+j_2+j_3=i \\ j_1+j_3 \geqslant 2i-v \\ j_0,j_1,j_2,j_3 \in \mathbb{Z}_+}} C_{j_0,j_1,j_2,j_3} K_n^{(i)}\big(t(\theta)\big)\big(t(\theta)\big)^{j_0}\big(t'(\theta)\big)^{j_1} \\ \times \big(t''(\theta)\big)^{j_2}\big(t'''(\theta)\big)^{j_3}, \tag{5.2}$$

where $t(\theta) = x \cdot y \cos\theta + y \cdot \xi \sin\theta$, and C_{j_0,j_1,j_2,j_3} are some absolute constants. We note that for $y \in \mathbb{S}^{d-1}$ and $\xi \in \mathbb{S}^{d-1}$ with $\xi \cdot x = 0$, $|y \cdot \xi| \leqslant \sqrt{1-(x \cdot y)^2}$. Thus, using Lemma 3.2 with $\ell = m + v$, we conclude that for $1 \leqslant i \leqslant v$ and any $j_0, j_1, j_2, j_3 \in \mathbb{Z}_+$ satisfying $j_0 + j_1 + j_2 + j_3 = i$ and $j_1 + j_3 \geqslant 2i - v$,

$$\left|K_n^{(i)}\big(t(0)\big)\big(t(0)\big)^{j_0}\big(t'(0)\big)^{j_1}\big(t''(0)\big)^{j_2}\big(t'''(0)\big)^{j_3}\right| \leqslant Cn^{d-1+v} \min\left\{1, \left(nd(x,y)\right)^{-m}\right\}.$$

The desired inequality (5.1) then follows by (5.2). This completes the proof. □

Proof of Theorem 5.1. By the definition and (2.2), we have, for $f \in \Pi_n^d$ and $x, \xi \in \mathbb{S}^{d-1}$ with $x \cdot \xi = 0$,

$$\left(\frac{\partial}{\partial \xi}\right)^{\ell} f(x) = \int_{\mathbb{S}^{d-1}} f(y)\varphi_{x,y,\xi}^{(\ell)}(0)\, d\sigma(y)$$

with $\varphi_{x,y,\xi}$ as defined in Lemma 5.3. It then follows by Lemma 5.3 with $m > \frac{2}{p}s + d - 1$ that

$$\begin{aligned}\left|\left(\frac{\partial}{\partial\xi}\right)^{\ell} f(x)\right| &\leqslant Cn^{d-1+\ell}\int_{\mathbb{S}^{d-1}} \left|f(y)\right|\left(1+nd(x,y)\right)^{-m}\,d\sigma(y)\\ &\leqslant Cn^{d-1+\ell} f^*_{2/p,n}(x)\int_{\mathbb{S}^{d-1}} \left(1+nd(x,y)\right)^{-m+\frac{2}{p}s}\,d\sigma(y)\\ &\leqslant Cn^{\ell} f^*_{2/p,n}(x).\end{aligned}$$

This combined with Theorem 3.1 gives the desired Bernstein's inequality and therefore completes the proof. □

Our next result is the following Schur-type inequality.

Theorem 5.4. *Let* $0 < p < \infty$, w *be a doubling weight and let* $h_{\alpha,\mathbf{v}}$ *be defined by* (2.4) *with* $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_j > 0$, $\mathbf{v} = (v_1, \dots, v_m)$ *and* $v_j \in \mathbb{S}^{d-1}$. *Then for all* $f \in \Pi_n^d$,

$$\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x) \leqslant Cn^{|\alpha|}\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p h_{\alpha,\mathbf{v}}(x)w(x)\,d\sigma(x),$$

where $|\alpha| = \sum_{j=1}^m \alpha_j$, *C depends only on d, m, α, the doubling constant of w, and p when p is small.*

To the best of our knowledge, Theorem 5.4 is new even in the case $w(x) \equiv 1$.

Theorem 5.4 is a direct consequence of Corollary 3.4 and the following lemma.

Lemma 5.5. *Let w be a doubling weight. Then* $h_{\alpha,\mathbf{v}}(x)w(x)$ *is again a doubling weight, and moreover*

$$w_n(x) \leqslant Cn^{|\alpha|}(h_{\alpha,\mathbf{v}}w)_n(x), \quad x \in \mathbb{S}^{d-1},$$

where $C > 0$ *depends only on d, m, α and the doubling constant of w.*

Proof. Without loss of generality, we may assume that $v_i \neq v_j$ if $i \neq j$. For simplicity, we set, for fixed $\theta \in (0,\pi)$ and $x \in \mathbb{S}^{d-1}$,

$$\mathcal{A} = \left\{i\colon 1 \leqslant i \leqslant m,\ |x\cdot v_i| < 4\theta\right\}, \qquad \mathcal{B} = \left\{i\colon 1 \leqslant i \leqslant m,\ |x\cdot v_i| \geqslant 4\theta\right\}.$$

We then claim that

$$\begin{aligned}\int_{B(x,\theta)} h_{\alpha,\mathbf{v}}(y)w(y)\,d\sigma(y) &\sim \int_{B(x,2\theta)} h_{\alpha,\mathbf{v}}(y)w(y)\,d\sigma(y)\\ &\sim \left(\prod_{i\in\mathcal{A}}\theta^{\alpha_i}\right)\left(\prod_{j\in\mathcal{B}}|x\cdot v_j|^{\alpha_j}\right)\int_{B(x,\theta)} w(y)\,d\sigma(y), \quad (5.3)\end{aligned}$$

where the constants of equivalence depend only on d, $h_{\alpha,\mathbf{v}}$ and the doubling constant of w. The desired conclusion of Lemma 5.5 will follow directly from this claim.

For the proof of the claim (5.3), we first note that, for $y \in B(x, 2\theta)$,

$$h_{\alpha,\mathbf{v}}(y) \sim \left(\prod_{i\in\mathcal{A}} |y\cdot v_i|^{\alpha_i}\right)\left(\prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j}\right) \leqslant \left(\prod_{i\in\mathcal{A}} (6\theta)^{\alpha_i}\right)\left(\prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j}\right). \quad (5.4)$$

Next, we let $\varepsilon_{d,m} > 0$ be a sufficiently small constant depending only on d and m and set, for $1 \leqslant j \leqslant m$,

$$E_j = \left\{y \in B\left(x, \frac{\theta}{4}\right): \left|d(y, v_j) - \frac{\pi}{2}\right| \leqslant \varepsilon_{d,m}\theta\right\}.$$

Then a straightforward calculation shows that for each $1 \leqslant j \leqslant m$, $|E_j| \leqslant C_d \varepsilon_{d,m}\theta^{d-1}$ and therefore

$$\sum_{j=1}^{m} |E_j| \leqslant C_d \varepsilon_{d,m} m\theta^{d-1} \leqslant \frac{1}{2}\left|B\left(x, \frac{\theta}{4}\right)\right|$$

provided that $\varepsilon_{d,m}$ is small enough. Thus, there must exist a point $y_0 \in B(x, \frac{\theta}{4})$ such that

$$|y_0 \cdot v_j| \geqslant \sin(\varepsilon_{d,m}\theta), \quad 1 \leqslant j \leqslant m.$$

It follows that $B(y_0, \frac{\varepsilon_{d,m}\theta}{2}) \subset B(x, \theta)$ and for any $y \in B(y_0, \frac{\varepsilon_{d,m}\theta}{2})$ and $i \in \mathcal{A}$,

$$5\theta > |y \cdot v_i| \geqslant \sin\left(\frac{\varepsilon_{d,m}\theta}{2}\right) \geqslant \frac{\varepsilon_{d,m}\theta}{\pi}.$$

This together with (5.4) means that

$$h_{\alpha,\mathbf{v}}(y) \sim \left(\prod_{i\in\mathcal{A}} \theta^{\alpha_i}\right) \prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j} \quad \text{for } y \in B\left(y_0, \frac{\varepsilon_{d,m}\theta}{2}\right). \quad (5.5)$$

Therefore, we have

$$\begin{aligned}
&\left(\prod_{i\in\mathcal{A}} \theta^{\alpha_i}\right)\left(\prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j}\right) \int_{B(x,\theta)} w(y)\, d\sigma(y) \\
&\quad \leqslant C\left(\prod_{i\in\mathcal{A}} \theta^{\alpha_i}\right)\left(\prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j}\right) \int_{B(y_0, \frac{\varepsilon_{d,m}\theta}{2})} w(y)\, d\sigma(y) \\
&\quad \leqslant C \int_{B(y_0, \frac{\varepsilon_{d,m}\theta}{2})} h_{\alpha,\mathbf{v}}(y) w(y)\, d\sigma(y)
\end{aligned}$$

$$\leqslant C \int_{B(x,\theta)} h_{\alpha,\mathbf{v}}(y) w(y)\, d\sigma(y) \leqslant C \int_{B(x,2\theta)} h_{\alpha,\mathbf{v}}(y) w(y)\, d\sigma(y)$$

$$\leqslant C\biggl(\prod_{i\in\mathcal{A}} \theta^{\alpha_i}\biggr)\biggl(\prod_{j\in\mathcal{B}} |x\cdot v_j|^{\alpha_j}\biggr) \int_{B(x,\theta)} w(y)\, d\sigma(y),$$

where in the second inequality we have used (5.5), while in the last inequality we have used (5.4). This proves the claim (5.3) and therefore completes the proof. □

6. Remez-type and Nikolskii-type inequalities with A_∞ weights

We say a weight w on $\mathbb{S}^{d-1}$ is an A_∞ weight if there exists a constant $\beta \geqslant 1$ (called A_∞ constant) such that

$$\int_{B(x,r)} w(y)\, d\sigma(y) \leqslant \beta \biggl(\frac{|B(x,r)|}{|E|}\biggr)^{\beta} \int_E w(y)\, d\sigma(y)$$

for all spherical caps $B(x,r) \subset \mathbb{S}^{d-1}$ and all measurable subsets $E \subset B(x,r)$.

We start with the following Remez-type inequality.

Theorem 6.1. *Let w be an A_∞ weight on $\mathbb{S}^{d-1}$ and let $0 < t^{d-1} \leqslant \frac{1}{2}$. Then for any $0 < p < \infty$, $f \in \Pi_n^d$ and $E \subset \mathbb{S}^{d-1}$ with $|E| = t^{d-1}$, we have*

$$\int_{\mathbb{S}^{d-1}} \bigl|f(x)\bigr|^p w(x)\, d\sigma(x) \leqslant C^{nt+1} \int_{\mathbb{S}^{d-1}\setminus E} \bigl|f(x)\bigr|^p w(x)\, d\sigma(x),$$

where $C > 0$ depends only on d, p and the A_∞ constant of w.

To the best of our knowledge, Theorem 6.1 is new even in the unweighted case.

It was shown in [13, pp. 54–56] that Remez-type inequality does not, in general, hold for an arbitrary doubling weight.

Proof. First, we show that for any $f \in \Pi_n^d$,

$$\|f\|_{C(\mathbb{S}^{d-1})} \leqslant C^{nt} \sup_{x\in\mathbb{S}^{d-1}\setminus E} \bigl|f(x)\bigr|, \tag{6.1}$$

where $E \subset \mathbb{S}^{d-1}$ and $|E| = t^{d-1} \leqslant \frac{4}{5}$. Let $x_0 \in \mathbb{S}^{d-1}$ be such that $|f(x_0)| = \|f\|_{C(\mathbb{S}^{d-1})}$. We denote by $C(x_0, y)$ the great circle on $\mathbb{S}^{d-1}$ passing through x_0 and $y \in \mathbb{S}^{d-1}\setminus\{x_0\}$, and by $d\gamma_{x_0,y}$ the one-dimensional Lebesgue measure on $C(x_0, y)$ normalized by $\gamma_{x_0,y}(C(x_0, y)) = 2\pi$. Since the restriction of $f \in \Pi_n^d$ to any great circle is a trigonometric polynomial of degree at most n, by the Remez-type inequality for the usual trigonometric polynomials (see [2,7]) it will suffice to show that there exists a point $y_0 \in \mathbb{S}^{d-1}\backslash\{x_0\}$ such that

$$\gamma_{x_0,y_0}\big(E \cap C(x_0, y_0)\big) \leqslant \min\{C_d t,\ 2\pi - \varepsilon_d\}, \tag{6.2}$$

where $C_d > 0$ and $\varepsilon_d \in (0, \pi)$ denote two constants depending only on d. For the proof of (6.2), we set $E^c = \mathbb{S}^{d-1}\backslash E$ and

$$S(x_0) = \big\{y \in \mathbb{S}^{d-1}\colon\ y \cdot x_0 = 0\big\}.$$

We denote by $d\sigma_{x_0}(y)$ the Lebesgue measure on $S(x_0)$ normalized by $\int_{S(x_0)} d\sigma_{x_0}(y) = 1$. It will be shown that

$$\int\limits_{S(x_0)} \big(\gamma_{x_0,y}\big(E \cap C(x_0, y)\big)\big)^{d-1}\, d\sigma_{x_0}(y) \leqslant C'_d t^{d-1}, \tag{6.3}$$

and

$$\int\limits_{S(x_0)} \gamma_{x_0,y}\big(E^c \cap C(x_0, y)\big)\, d\sigma_{x_0}(y) \geqslant \varepsilon'_d > 0, \tag{6.4}$$

from which we will conclude that there must exist a $y_0 \in S(x_0)$ such that

$$\gamma_{x_0,y_0}\big(E \cap C(x_0, y_0)\big) \leqslant C_d t \quad \text{and} \quad \gamma_{x_0,y_0}\big(E^c \cap C(x_0, y_0)\big) \geqslant \varepsilon_d > 0,$$

and the desired inequality (6.1) will then follow. In fact, noticing that

$$\gamma_{x_0,y}\big(E \cap C(x_0, y)\big) = \int\limits_{-\pi}^{\pi} \chi_E\big(x_0 \cos\theta + y \sin\theta\big)\, d\theta,$$

and setting

$$E(x_0, y) = \big\{\theta \in [-\pi, \pi]\colon\ |\sin\theta| \geqslant \sin\big[8^{-1}\gamma_{x_0,y}\big(C(x_0, y) \cap E\big)\big]\big\},$$

we have

$$\begin{aligned} t^{d-1} = |E| &= C''_d \int\limits_{S(x_0)} \int\limits_{-\pi}^{\pi} \chi_E(x_0 \cos\theta + y \sin\theta)\big|\sin^{d-2}\theta\big|\, d\theta\, d\sigma_{x_0}(y) \\ &\geqslant C''_d \int\limits_{S(x_0)} \int\limits_{E(x_0,y)} \chi_E(x_0 \cos\theta + y \sin\theta)\big|\sin^{d-2}\theta\big|\, d\theta\, d\sigma_{x_0}(y) \\ &\geqslant \frac{1}{2} C''_d \int\limits_{S(x_0)} \left(\sin\left(\frac{\gamma_{x_0,y}(E \cap C(x_0, y))}{8}\right)\right)^{d-2} \gamma_{x_0,y}\big(E \cap C(x_0, y)\big)\, d\sigma_{x_0}(y), \end{aligned}$$

which implies (6.3), and meanwhile, recalling that $\sigma(\mathbb{S}^{d-1}) = 1$, we have

$$\frac{1}{5} \leqslant \left|E^c\right| = C_d'' \int\limits_{S(x_0)} \int\limits_{-\pi}^{\pi} \chi_{E^c}(x_0\cos\theta + y\sin\theta)\left|\sin^{d-2}\theta\right| d\theta\, d\sigma_{x_0}(y)$$

$$\leqslant C_d'' \int\limits_{S(x_0)} \gamma_{x_0,y}\big(E^c \cap C(x_0, y)\big)\, d\sigma_{x_0}(y),$$

which gives (6.4). This completes the proof of (6.2) and hence (6.1).

Next, we show that for $0 < p < \infty$ and $f \in \Pi_n^d$,

$$\int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p d\sigma(x) \leqslant C^{nt+1} \int\limits_{\mathbb{S}^{d-1}\setminus E} \left|f(x)\right|^p d\sigma(x), \tag{6.5}$$

where $E \subset \mathbb{S}^{d-1}$ and $|E| = t^{d-1} \leqslant \frac{3}{4}$. We get the idea from [6,7]. Set $\alpha = (\frac{16}{15})^{\frac{1}{d-1}}$, and

$$F = \left\{x \in \mathbb{S}^{d-1}\colon \left|f(x)\right| \geqslant \|f\|_\infty C^{-\alpha nt}\right\},$$

with $C > 0$ being the same as in (6.1). Then by the already proven inequality (6.1) it follows that $|F| \geqslant (\alpha t)^{d-1}$ and hence $|F \cap E^c| \geqslant \frac{1}{15}|E|$. Therefore, we have

$$\int\limits_E \left|f(x)\right|^p d\sigma(x) \leqslant |E|\|f\|_\infty^p \leqslant 15C^{\alpha ntp} \int\limits_{E^c\cap F} \left|f(x)\right|^p d\sigma(x)$$

$$\leqslant 15C^{\alpha pnt} \int\limits_{\mathbb{S}^{d-1}\setminus E} \left|f(x)\right|^p d\sigma(x)$$

and (6.5) then follows.

Finally, we show that for any A_∞ weight w, $0 < p < \infty$ and all $f \in \Pi_n^d$,

$$\int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\, d\sigma(x) \leqslant C^{nt+1} \int\limits_{\mathbb{S}^{d-1}\setminus E} \left|f(x)\right|^p w(x)\, d\sigma(x), \tag{6.6}$$

where $|E| = t^{d-1} \leqslant \frac{1}{2}$. Let $\{\omega_i\}_{i=1}^{M(n,\delta)}$ be a maximal $\frac{\delta}{n}$-separable subset of $\mathbb{S}^{d-1}$ with $\delta > 0$ to be specified later, and let

$$B_1^* = B\left(\omega_1, \frac{\delta}{n}\right) - \bigcup_{j=2}^{M(n,\delta)} B\left(\omega_j, \frac{\delta}{4n}\right)$$

and

$$B_i^* = B\left(\omega_i, \frac{\delta}{n}\right) - \left[\left(\bigcup_{k=1}^{i-1} B_k^*\right) \cup \left(\bigcup_{j=i+1}^{M(n,\delta)} B\left(\omega_j, \frac{\delta}{4n}\right)\right)\right], \quad 2 \leqslant i \leqslant M(n,\delta).$$

Then the following properties can be easily verified:

$$B_i^* \cap B_j^* = \emptyset \quad \text{if } i \neq j;$$
$$B\left(\omega_i, \frac{\delta}{4n}\right) \subset B_i^* \subset B\left(\omega_i, \frac{\delta}{n}\right) \quad \text{for } 1 \leqslant i \leqslant M(n,\delta);$$
$$\bigcup_{i=1}^{M(n,\delta)} B_i^* = \mathbb{S}^{d-1}.$$

Now setting

$$\Lambda^* = \left\{i\colon 1 \leqslant i \leqslant M(n,\delta),\ \left|B_i^* \cap E\right| > \frac{2}{3}|B_i^*|\right\},$$

we have

$$\sum_{i\in\Lambda^*} |B_i^*| \leqslant \frac{3}{2}|E| = \frac{3}{2}t^{d-1},$$

and hence, using Corollary 3.4, Lemma 4.6 and the already proven inequality (6.5), we obtain

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x) &\leqslant C^{nt+1} \int_{\mathbb{S}^{d-1}\setminus\bigcup_{i\in\Lambda^*} B_i^*} \left|f(x)\right|^p w_n(x)\,d\sigma(x)\\
&\leqslant C^{nt+1} \sum_{i\notin\Lambda^*} \int_{B(\omega_i,\frac{\delta}{n})} \left|f(x)\right|^p w_n(x)\,d\sigma(x)\\
&\leqslant C^{nt+1} \sum_{i\notin\Lambda^*} \left|f(\xi_i)\right|^p \int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x)\\
&\quad + C^{nt+1} \sum_{i\notin\Lambda^*} \left|\operatorname{osc}(f)(\omega_i)\right|^p \int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x),
\end{aligned}$$

where

$$\left|f(\xi_i)\right| = \min_{x\in B(\omega_i,\frac{\delta}{n})} \left|f(x)\right|$$

and

$$\operatorname{osc}(f)(\omega_i) = \max_{x,y\in B(\omega_i,\frac{\delta}{n})} \left|f(x) - f(y)\right|.$$

Since w_n is a doubling weight with the doubling constant depending only on that of w, it follows by Corollaries 3.3 and 3.4 that

$$\begin{aligned}\sum_{i\notin\Lambda^*} \left|\operatorname{osc}(f)(\omega_i)\right|^p \int\limits_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x) &\leqslant C\delta^p \int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w_n(x)\,d\sigma(x)\\ &\leqslant C\delta^p \int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x).\end{aligned}$$

On the other hand, however, by the A_∞-property of w it follows that for $i\notin\Lambda^*$,

$$\begin{aligned}\left|f(\xi_i)\right|^p \int\limits_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x) &\leqslant C\delta^{d-1}\left|f(\xi_i)\right|^p \int\limits_{B(\omega_i,\frac{1}{n})} w(x)\,d\sigma(x)\\ &\leqslant C\delta^{d-1-s}\left|f(\xi_i)\right|^p \int\limits_{B(\omega_i,\frac{\delta}{n})} w(x)\,d\sigma(x)\\ &\leqslant C\delta^{d-1-s}\left|f(\xi_i)\right|^p \int\limits_{B(\omega_i,\frac{\delta}{n})\setminus E} w(x)\,d\sigma(x)\\ &\leqslant C\delta^{d-1-s} \int\limits_{B(\omega_i,\frac{\delta}{n})\setminus E} \left|f(x)\right|^p w(x)\,d\sigma(x),\end{aligned}$$

where in the first inequality we have used the fact that $w_n(x)\sim w_n(\omega_i)$ for $x\in B(\omega_i,\frac{\delta}{n})$, in the second inequality we have used the doubling property of w, and in the third inequality we have used the definition of Λ^* and the A_∞ property of w.

Therefore, noticing that $\sum_{i=1}^{M(n,\delta)} \chi_{B(\omega_i,\frac{\delta}{n})}(x)\leqslant C_d$, we deduce

$$\begin{aligned}\int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x) &\leqslant C^{nt+1}\delta^{d-1-s} \int\limits_{\mathbb{S}^{d-1}\setminus E} \left|f(x)\right|^p w(x)\,d\sigma(x)\\ &\quad + \delta^p C^{nt+1} \int\limits_{\mathbb{S}^{d-1}} \left|f(x)\right|^p w(x)\,d\sigma(x).\end{aligned}$$

Now letting $\delta = (\frac{1}{2})^{\frac{1}{p}} C^{-(nt+1)\frac{1}{p}}$, we obtain the desired inequality (6.6) and therefore complete the proof. $\square$

As a consequence of Theorem 6.1, we have the following Nikolskii-type inequality:

Corollary 6.2. *Let w be an A_∞ weight and let $0 < p < q < \infty$. Then for all $f \in \Pi_n^d$,*

$$\Bigl(\int_{\mathbb{S}^{d-1}} \bigl|f(x)\bigr|^q w(x)\,d\sigma(x)\Bigr)^{\frac{1}{q}} \leqslant C n^{(d-1)(\frac{1}{p}-\frac{1}{q})}\Bigl(\int_{\mathbb{S}^{d-1}} \bigl|f(x)\bigr|^p w(x)^{\frac{p}{q}}\,d\sigma(x)\Bigr)^{\frac{1}{p}}.$$

The proof of Corollary 6.2 is almost identical to that of [13, Theorem 5.5], therefore we omit the detail.

7. Weighted polynomial inequalities on compact two-point homogeneous manifolds

Let X be a compact two-point homogeneous space of dimension $d-1$. Besides the sphere $\mathbb{S}^{d-1}$, these spaces are: the real projective space $P^{d-1}(\mathbb{R})$; the complex projective space $P^{d-1}(\mathbb{C})$; the quaternionic projective space $P^{d-1}(\mathbb{H})$; and the Cayley projective plane $P^{16}(\mathit{Cayley})$ (with $d=17$). These spaces are the compact symmetric spaces of rank one and their geometry is quite similar to that of the sphere $\mathbb{S}^{d-1}$.

Let $d\sigma(x)$ denote the Riemannian measure on X normalized by $\int_X d\sigma(x)=1$, and let $d(\cdot,\cdot)$ be the Riemannian metric on X normalized so that all geodesics on X have the same length 2π. We denote by $B(x,r)$ the ball centered at $x\in X$ and having radius $r>0$, i.e., $B(x,r)=\{y\in X\colon\ d(x,y)\leqslant r\}$, and $|E|$ the measure $\sigma(E)$ of a measurable subset $E\subset X$. The definitions of doubling weights and A_∞ weights can be easily extended to the space X.

It is known that the spectrum of the Laplace–Beltrami operator Δ on X is discrete, real, non-positive and can be arranged in decreasing order

$$0=\lambda_0>\lambda_1>\lambda_2>\cdots.$$

We denote by $\mathcal{H}_k$ the eigenspace of Δ corresponding to the eigenvalue λ_k. For an integer $N\geqslant 0$, we put $\Pi_N\equiv\Pi_N(X)=\bigoplus_{k=0}^N\mathcal{H}_k$, and we call the functions in Π_N the spherical polynomials of degree at most N. In the case $X=\mathbb{S}^{d-1}$, these functions coincide with the ordinary spherical polynomials. We refer to [1, Section 7], [10, Chapter I, Section 4] and [3] for more background information.

Most of the inequalities that have been proved in the preceding sections for the spherical polynomials on $\mathbb{S}^{d-1}$ can be extended to the spherical polynomials on X. The proofs go through with hardly any change.

8. Weighted polynomial inequalities on the unit ball B^d

Our main purpose in this section is to establish analogous weighted polynomial inequalities on the unit ball $B^d=\{x\in\mathbb{R}^d\colon\ |x|\leqslant 1\}$. We refer to [24,25] for Fourier analysis on B^d. Here we only introduce some basic concepts that will be needed. We denote by dx the usual Lebesgue measure on B^d and $|E|$ the measure of a subset $E\subset B^d$. Given an integer $n \geqslant 0$, let $\mathcal{P}_n^d$ denote the space of all polynomials in d variables of total degree $\leqslant n$ on $\mathbb{R}^d$. We introduce the following metric on B^d:

$$\rho(x, y) = \sqrt{|x-y|^2 + \left(\sqrt{1-|x|^2} - \sqrt{1-|y|^2}\right)^2} \quad \text{for } x, y \in B^d.$$

For $r > 0$, $x \in B^d$ we set

$$B_\rho(x, r) = \left\{y \in B^d \colon\ \rho(x, y) \leqslant r\right\}.$$

We say a weight w on B^d is a doubling weight if for any $x \in B^d$ and $r > 0$,

$$\int\limits_{B_\rho(x,2r)} w(y)\,dy \leqslant L \int\limits_{B_\rho(x,r)} w(y)\,dy,$$

while we say a finite subset $\Lambda \subset B^d$ is maximal $(\frac{\delta}{n}, \rho)$-separable if $B^d \subset \bigcup_{y\in\Lambda} B_\rho(y, \frac{\delta}{n})$ and $\min_{y\neq y'\in\Lambda} \rho(y, y') \geqslant \frac{\delta}{n}$.

There is an important connection between the integration on B^d and the integration on the unit sphere $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \colon\ |x| = 1\}$, proved by Xu (see [22, Lemma 2.1]): for any integrable function f on B^d, we have

$$\int\limits_{B^d} f(x)\,dx = C_d \int\limits_{\mathbb{S}^d} T(f)(z)|z_{d+1}|\,d\sigma(z), \tag{8.1}$$

where and throughout this section $T(f)$ is defined for a function f on B^d by

$$T(f)(z) = f(x) \quad \text{for } z = (x, z_{d+1}) \in \mathbb{S}^d,$$

and $d\sigma(z)$ denotes the rotation invariant measure on $\mathbb{S}^d$ normalized by $\int_{\mathbb{S}^d} d\sigma(z) = 1$. By (8.1) it is easily seen that w has the doubling property on B^d if and only if $|z_{d+1}|T(w)(z)$ does so on $\mathbb{S}^d$. Therefore, by Lemma 5.5 it follows that all the classical weights

$$w_\alpha(x) = |x_1|^{\alpha_1} \ldots |x_d|^{\alpha_d}\left(1-|x|^2\right)^{\alpha_{d+1}-\frac{1}{2}}, \quad x \in B^d, \tag{8.2}$$

with $\alpha = (\alpha_1, \ldots, \alpha_d, \alpha_{d+1})$, $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant d+1$, have the doubling property on B^d.

Invoking (8.1) and Theorems 4.1 and 4.2, we have the following cubature formulae and Marcinkiewicz–Zygmund inequalities on B^d.

Theorem 8.1. *Let w be a doubling weight on B^d. Then there exists a constant $\gamma > 0$ depending only on d and the doubling constant of w such that for any $0 < \delta < \gamma$ and* *any maximal* $(\frac{\delta}{n}, \rho)$*-separable subset* $\Lambda \subset B^d$ *there exists a sequence of positive numbers* $\lambda_\omega \sim \int_{B_\rho(\omega,\delta/n)} w(y)\,dy$, $\omega \in \Lambda$, *such that for any* $f \in \mathcal{P}_{3n}^d$,

$$\int_{B^d} f(x)w(x)\,dx = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega)$$

and moreover, for any $f \in \mathcal{P}_n^d$ *and* $0 < p < \infty$,

$$\int_{B^d} \left|f(x)\right|^p w(x)\,dx \sim \sum_{\omega\in\Lambda} \left|f(\omega)\right|^p \biggl(\int_{B_\rho(\omega,\frac{\delta}{n})} w(y)\,dy\biggr),$$

where the constants of equivalence depend only on d, *the doubling constant of* w *and* p.

Cubature formulae with different properties on B^d were previously constructed by many authors (see, for instance, [18,22,24]). Moreover, it was shown by Xu [22] that cubature formulae on B^d are, in fact, closely related to those on $\mathbb{S}^d$. For the MZ inequalities, however, the result obtained in Theorem 8.1 seems new even in the unweighted case.

Next, we show the following analogue of Bernstein-type inequality:

Theorem 8.2. *Let* w *be a doubling weight on* B^d *and let* $0 < p < \infty$. *Then for all* $f \in \mathcal{P}_n^d$ *we have*

$$\biggl(\int_{B^d} \bigl(\varphi\bigl(|x|\bigr)\bigr)^{|\alpha|p} \left|D^\alpha f(x)\right|^p w(x)\,dx\biggr)^{\frac{1}{p}} \leqslant Cn^{|\alpha|}\biggl(\int_{B^d} \left|f(x)\right|^p w(x)\,dx\biggr)^{\frac{1}{p}},$$

where $\varphi(t) = \sqrt{1-t^2}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_d})^{\alpha_d}$, *and* $|\alpha| = \sum_{j=1}^d \alpha_j$.

In the case of the interval $[-1, 1]$, an inequality similar to that in Theorem 8.2 for $p \geqslant 1$ was proved by Mastroianni and Totik [13, Theorem 7.3]. For $d > 1$, however, to the best of our knowledge, Theorem 8.2 seems new even in the unweighted case.

Proof. First, note by Lemma 5.5, that for any $\beta \geqslant 0$, $(\varphi(|x|))^\beta w(x)$ is again a doubling weight. Therefore, it suffices to show that for $1 \leqslant i \leqslant d$ and $f \in \mathcal{P}_n^d$

$$\biggl(\int_{B^d} \bigl(\varphi\bigl(|x|\bigr)\bigr)^p \left|\frac{\partial f(x)}{\partial x_i}\right|^p w(x)\,dx\biggr)^{\frac{1}{p}} \leqslant Cn\biggl(\int_{B^d} \left|f(x)\right|^p w(x)\,dx\biggr)^{\frac{1}{p}}. \tag{8.3}$$

We claim that for each $1 \leqslant i \leqslant d$ and any $z = (x, z_{d+1}) \in \mathbb{S}^d$,

$$\varphi\big(|x|\big)\left|\frac{\partial f(x)}{\partial x_i}\right| \leqslant \sqrt{d} \sup_{\substack{\xi\in\mathbb{S}^d\\ \xi\cdot z=0}} \left|\frac{\partial T(f)(z)}{\partial \xi}\right|, \tag{8.4}$$

where $\partial T(f)(z)/\partial\xi$ denotes the tangent derivative of $T(f)$ at z in the direction ξ, as defined in Section 5. Once the claim (8.4) is proved, then using (8.1), Lemma 5.5 and Theorem 5.1, we have

$$\begin{aligned}
\int_{B^d} \left|\varphi\big(|x|\big)\frac{\partial f(x)}{\partial x_i}\right|^p w(x)\,dx &\leqslant C_d^p \int_{\mathbb{S}^d} \Bigg(\sup_{\substack{\xi\in\mathbb{S}^d\\ \xi\cdot z=0}} \left|\frac{\partial T(f)(z)}{\partial \xi}\right|\Bigg)^p |z_{d+1}| T(w)(z)\,d\sigma(z) \\
&\leqslant C_d^p n^p \int_{\mathbb{S}^d} \big|T(f)(z)\big|^p |z_{d+1}| T(w)(z)\,d\sigma(z) \\
&= C_d^p n^p \int_{B^d} \big|f(x)\big|^p w(x)\,dx
\end{aligned}$$

proving the desired inequality (8.3).

Therefore, it remains to prove the claim (8.4). Without loss of generality, we may assume $0 < |x| < 1$. We then take an orthonormal basis $\{\xi_j\}_{j=1}^{d+1} = \{\xi_j(z)\}_{j=1}^{d+1}$ of $\mathbb{R}^{d+1}$, where $z = (x, z_{d+1}) \in \mathbb{S}^d$,

$$\begin{aligned}
\xi_j &= \xi_j(z) = \big(a_{j,1}(z), \ldots, a_{j,d}(z), 0\big) \in \mathbb{S}^d, \quad 1 \leqslant j \leqslant d-1, \\
\xi_d &= \xi_d(z) = \left(\frac{x_1}{|x|}, \ldots, \frac{x_d}{|x|}, -\frac{|x|}{z_{d+1}}\right)|z_{d+1}| \in \mathbb{S}^d, \\
\xi_{d+1} &= \xi_{d+1}(z) = z \in \mathbb{S}^d.
\end{aligned}$$

Then by the definition, it is easily seen that

$$\begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_d} \end{pmatrix} = Q(z) \begin{pmatrix} \frac{\partial T(f)(z)}{\partial \xi_1} \\ \vdots \\ \frac{\partial T(f)(z)}{\partial \xi_{d-1}} \\ \frac{1}{\varphi(|x|)} \frac{\partial T(f)(z)}{\partial \xi_d} \end{pmatrix},$$

where

$$Q(z) = \begin{pmatrix} a_{1,1}(z) & a_{2,1}(z) & \ldots & a_{d-1,1}(z) & \frac{x_1}{|x|} \\ a_{1,2}(z) & a_{2,2}(z) & \ldots & a_{d-1,2}(z) & \frac{x_2}{|x|} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1,d}(z) & a_{2,d}(z) & \ldots & a_{d-1,d}(z) & \frac{x_d}{|x|} \end{pmatrix}.$$

Since $Q(z)$ is an orthogonal matrix it follows that

$$\max_{1\leqslant i\leqslant d}\left|\frac{\partial f(x)}{\partial x_i}\right| \leqslant \sqrt{d}\big(1-|x|^2\big)^{-\frac{1}{2}}\Bigg(\sup_{\substack{\xi\in\mathbb{S}^d\\ \xi\cdot z=0}}\left|\frac{\partial T(f)(z)}{\partial\xi}\right|\Bigg),$$

which proves the claim (8.4) and hence completes the proof. □

We say a weight w on B^d is an A_∞ weight if $|z_{d+1}|T(w)(z)$ is an A_∞ weight on $\mathbb{S}^d$. All the classical weights $w_\alpha(x)$ defined by (8.2) are A_∞ weights, as can be easily shown.

The Remez-type inequality on the unit ball B^d reads as follows.

Theorem 8.3. *Suppose w is an A_∞ weight on B^d, $0 < p < \infty$, $E \subset B^d$, and $|E| = (\frac{A}{n})^d \leqslant \frac{1}{2}|B^d|$. Then for any $f \in \mathcal{P}_n^d$,*

$$\int_{B^d}\big|f(x)\big|^p w(x)\,dx \leqslant C^{\sqrt{nA}+1}\int_{B^d\setminus E}\big|f(x)\big|^p w(x)\,dx.$$

Theorem 8.3 seems new even in the unweighted cases.

Proof. Let

$$\tilde{E} = \big\{(x, z_{d+1}) \in \mathbb{S}^d\colon\ x \in E\big\}.$$

We claim that

$$\sigma(\tilde{E}) \leqslant C_d\sqrt{|E|}. \tag{8.5}$$

(In fact, it can be shown that $\sigma(\tilde{E}) \sim \sqrt{|E|}$.)

To show (8.5), we let $\varepsilon_d > 0$ be a sufficiently small absolute constant so that

$$\sigma\big(\big\{(x, z_{d+1}) \in \mathbb{S}^d\colon\ |z_{d+1}| \leqslant \varepsilon_d\sigma(\tilde{E})\big\}\big) \leqslant \frac{1}{2}\sigma(\tilde{E}).$$

Then

$$|E| = C_d\int_{\tilde{E}}|z_{d+1}|\,d\sigma(z) \geqslant \int_{\tilde{E}\cap\{z\in\mathbb{S}^d\colon\ |z_{d+1}|>\varepsilon_d\sigma(\tilde{E})\}}|z_{d+1}|\,d\sigma(z) \geqslant \frac{1}{2}C_d\varepsilon_d\big(\sigma(\tilde{E})\big)^2$$

and inequality (8.5) follows.

Next, note, by the definition, that for any A_∞ weight w on B^d, $|z_{d+1}|T(w)(z)$ is an A_∞ weight on $\mathbb{S}^d$. Thus, using Theorem 6.1, it follows that

$$\begin{aligned}\int\limits_{B^d} |f(x)|^p w(x)\,dx &= C_d \int\limits_{\mathbb{S}^d} |T(f)(z)|^p T(w)(z)|z_{d+1}|\,d\sigma(z)\\ &\leqslant C^{n(\sigma(\tilde{E}))^{\frac{1}{d}}+1} \int\limits_{\mathbb{S}^d\setminus\tilde{E}} |T(f)(z)|^p T(w)(z)|z_{d+1}|\,d\sigma(z)\\ &\leqslant C^{\sqrt{nA}+1} \int\limits_{B^d\setminus E} |f(x)|^p w(x)\,dx\end{aligned}$$

proving the desired Remez-type inequality. □

Finally, a simple use of (8.1) and Corollary 6.2 gives the following Nikolskii-type inequality:

Theorem 8.4. *Let* $0 < p < q < \infty$ *and let* w *be an* A_∞ *weight on* B^d. *Then for any* $f \in \mathcal{P}_n^d$,

$$\left(\int\limits_{B^d} |f(x)|^q w(x)\,dx\right)^{\frac{1}{q}} \leqslant C n^{d(\frac{1}{p}-\frac{1}{q})} \left(\int\limits_{B^d} |f(x)|^p w(x)^{\frac{p}{q}} \varphi(|x|)^{\frac{p}{q}-1}\,dx\right)^{\frac{1}{p}}.$$

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