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Approximation by the Bernstein–Durrmeyer Operator on a Simplex

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Abstract For the Jacobi-type Bernstein–Durrmeyer operator $M_{n,\kappa}$ on the simplex T^d of $\mathbb{R}^d$, we proved that for $f \in L^p(W_\kappa; T^d)$ with $1 < p < \infty$,

$$K_{2,\Phi}\big(f, n^{-1}\big)_{\kappa,p} \le c\|f - M_{n,\kappa} f\|_{\kappa,p} \le c' K_{2,\Phi}\big(f, n^{-1}\big)_{\kappa,p} + c' n^{-1}\|f\|_{\kappa,p},$$

where W_κ denotes the usual Jacobi weight on T^d, $K_{2,\Phi}(f, t^2)_{\kappa,p}$ and $\|\cdot\|_{\kappa,p}$ denote the second-order Ditzian–Totik K-functional and the L^p-norm with respect to the weight W_κ on T^d, respectively, and the constants c and c' are independent of f and n. This confirms a conjecture of Berens, Schmid, and Yuan Xu (J. Approx. Theory 68(3), 247–261, 1992). Also, a related conjecture of Ditzian (Acta Sci. Math. (Szeged) 60(1–2), 225–243, 1995; J. Math. Anal. Appl. 194(2), 548–559, 1995) was settled in our proof of this result.

Keywords Bernstein–Durrmeyer operator · K-functionals · Ditzian–Totik moduli of smoothness · Orthogonal polynomials · Simplex

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1 Introduction

Given $\kappa = (\kappa_1, \ldots, \kappa_{d+1}) \in \mathbb{R}^{d+1}$ with $\min_{1\le j\le d+1} \kappa_j \ge 0$, consider the Jacobi weight function

$$W_\kappa(x) := \prod_{i=1}^{d} x_i^{\kappa_i - 1/2}\bigl(1-|x|\bigr)^{\kappa_{d+1}-1/2}, \tag{1.1}$$

on the simplex

$$T^d := \Bigl\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \min_{1\le j\le d} x_j \ge 0,\ 1-|x| \ge 0\Bigr\},$$

where, and throughout we write $|x| = \sum_{j=1}^d |x_j|$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $L^p(W_\kappa; T^d)$, $(1 \le p < \infty)$, denote the usual weighted Lebesgue space endowed with the norm

$$\|f\|_{\kappa,p} := \left(\int_{T^d} \bigl|f(y)\bigr|^p W_\kappa(y)\,dy\right)^{1/p}.$$

Given $f \in L^p(W_\kappa; T^d)$, its weighted Ditzian–Totik K-functional of order $r \in \mathbb{N}$ is defined by

$$K_{r,\Phi}\bigl(f, t^r\bigr)_{\kappa,p} := \inf_{g\in C^r(T^d)}\Bigl(\|f-g\|_{\kappa,p} + t^r \max_{1\le i\le j\le d}\bigl\|\varphi_{ij}^r D_{ij}^r g\bigr\|_{\kappa,p}\Bigr), \tag{1.2}$$

where $\varphi_{ii}(x) = \sqrt{x_i(1-|x|)}$, $\varphi_{ij}(x) = \sqrt{x_i x_j}$, $i \ne j$, and

$$D_{ii} = \frac{\partial}{\partial x_i}, \qquad D_{ij} = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}, \quad i \ne j. \tag{1.3}$$

These K-functionals are equivalent to the weighted Ditzian–Totik moduli of smoothness, which are computable. See [2, 12] and [11].

For each $n \in \mathbb{N}$, the weighted Bernstein–Durrmeyer operator $M_{n,\kappa}$ on T^d is given by

$$M_{n,\kappa}(f)(x) = \sum_{|\mathbf{k}|\le n} p_{\mathbf{k}n}(x)\left(\int_{T^d} p_{\mathbf{k}n}(y)W_\kappa(y)\,dy\right)^{-1}\int_{T^d} f(y)p_{\mathbf{k}n}(y)W_\kappa(y)\,dy,$$

where $x = (x_1, \ldots, x_d) \in T^d$, $\mathbf{k} = (k_1, \ldots, k_d)$, $|\mathbf{k}| = |k_1| + |k_2| + \cdots + |k_d|$, and

$$p_{\mathbf{k}n}(x) = \frac{n!}{(n-|\mathbf{k}|)!\prod_{j=1}^d k_j!} x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}\bigl(1-|x|\bigr)^{n-|\mathbf{k}|}.$$

Our main result can be stated as follows:

Theorem 1.1 *Suppose* $1 < p < \infty$ *and* $f \in L^p(W_k; T^d)$. *Then*

$$c^{-1}K_{2,\Phi}\bigl(f, n^{-1}\bigr)_{\kappa,p} \le \|f - M_{n,\kappa}f\|_{\kappa,p} \le cK_{2,\Phi}\bigl(f, n^{-1}\bigr)_{\kappa,p} + cn^{-1}\|f\|_{\kappa,p}, \tag{1.4}$$

where the constant c *is independent of* n *and* f.

Several remarks are in order.

Remark 1.1 In the unweighted case (corresponding to the case of $\kappa = (\frac{1}{2}, \dots, \frac{1}{2})$), the first inequality in (1.4),

$$c^{-1} K_{2,\Phi}\big(f, n^{-1}\big)_{\kappa,p} \le \|f - M_{n,\kappa} f\|_{\kappa,p}, \tag{1.5}$$

was proved for $p = 2$ and conjectured for all $1 < p < \infty$ by Berens, Schmid and Xu [2, p. 260] in 1992, while the second inequality in (1.4) was proved by them [2] for $1 \le p < \infty$. An alternative proof of (1.5) for $p = 2$ in the unweighted case was given by Chen and Ditzian [4] in 1993.

Remark 1.2 It was shown in [15] (in the unweighted case) and [9] (in the weighted case) that for $1 \le p \le \infty$,

$$\|f - M_{n,\kappa} f\|_{\kappa,p} \sim \inf_{g \in C^2(T^d)} \big\{\|f - g\|_{\kappa,p} + n^{-1} \big\|P(D)_\kappa g\big\|_{\kappa,p}\big\} \equiv: \widetilde{K}_2\big(f, n^{-1}\big)_{\kappa,p}, \tag{1.6}$$

where $P(D)_\kappa$ is a differential operator closely related to the orthogonal polynomial expansions with respect to the weight W_κ on T^d (see Sect. 2 for details), whose precise definition is as follows (see [13, p. 46, (2.3.11)]):[1]

$$\begin{aligned} P(D)_\kappa := {} & \sum_{i=1}^{d} x_i(1 - x_i) \frac{\partial^2}{\partial x_i^2} - 2 \sum_{1 \le i < j \le d} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \\ & + \sum_{i=1}^{d} \left(\left(\kappa_i + \frac{1}{2} \right) - \left(|\kappa| + \frac{d+1}{2} \right) x_i \right) \frac{\partial}{\partial x_i}, \end{aligned} \tag{1.7}$$

with $|\kappa| = \sum_{j=1}^{d+1} \kappa_j$. Thus, Theorem 1.1 combined with (1.6) yields that for $1 < p < \infty$,

$$K_{2,\Phi}\big(f, n^{-1}\big)_{\kappa,p} \le c \widetilde{K}_2\big(f, n^{-1}\big)_{\kappa,p} \le c^2 K_{2,\Phi}\big(f, n^{-1}\big)_{\kappa,p} + c^2 n^{-1} \|f\|_{\kappa,p}.$$

Remark 1.3 Another interesting fact on the differential operator $P(D)_\kappa$ defined by (1.7) is the following elegant decomposition of Ditzian (see [10])[2]

$$P(D)_\kappa = \sum_{\xi \in E} P(D)_{\kappa,\xi}, \tag{1.8}$$

where

$$P(D)_{\kappa,\xi} := W_\kappa(x)^{-1} \frac{\partial}{\partial \xi} \big(\widetilde{d}(\xi, x) W_\kappa(x)\big) \frac{\partial}{\partial \xi}, \tag{1.9}$$

[1] There is a small misprint in (2.3.11) of [13]: the expression $(|\kappa| + \frac{d-1}{2})x_i$ there should be replaced by $(|\kappa| + \frac{d+1}{2})x_i$.

[2] In the unweighted case, an equivalent version of (1.8) was first obtained by Berens, Schmid and Xu in [2, p. 257]. (See (2.3) of Sect. 2 for details.)

$\widetilde{d}(\xi, x) := \sup\{ts:\ t, s \ge 0,\ x + t\xi, x - s\xi \in T^d\}$, and E is the set of all unit directions parallel to the edges of T^d in which we do not distinguish ξ and $-\xi$. Such a decomposition plays an important role in our proof, and some related known facts on this decomposition will be sketched in Sect. 2. It will be shown in Sect. 3 (see (3.3)) that for $f \in C^2(T^d)$,

$$\left\| P(D)_\kappa f \right\|_{\kappa,p} \sim \max_{\xi \in E} \left\| P(D)_{\kappa,\xi} f \right\|_{\kappa,p}, \quad 1 < p < \infty. \tag{1.10}$$

We point out that (1.10) was proved for $p = 2$ and conjectured for all $1 < p < \infty$ by Ditzian [10] in 1995. For more background information on this problem, we refer to the recent survey paper [11] of Ditzian.

We organize the paper as follows. Section 2 contains some necessary material on harmonic analysis on T^d. In Sect. 3, we prove Theorem 1.1, assuming a key pointwise estimate, Theorem 3.3, whose proof is given in Sect. 4.

2 Harmonic Analysis on T^d

In this section, we shall briefly describe some known facts on harmonic analysis on the simplex T^d. Most of the material in this section can be found in the book [13] by Dunkl and Xu.

Let $\mathcal{V}_0^d(W_\kappa)$ denote the space of all constant functions on T^d and let $\mathcal{V}_n^d(W_\kappa)$, $(n \ge 1)$, denote the space of polynomials of degree n on T^d that are orthogonal to all polynomials of lower degree with respect to the weight function $W_\kappa(x)$ on T^d defined by (1.1). The Hilbert space theory shows that

$$L^2\big(W_k; T^d\big) = \sum_{n=0}^{\infty} \mathcal{V}_n^d(W_\kappa), \quad \text{and} \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n^\kappa(f),$$

where $\operatorname{proj}_n^\kappa : L^2(W_\kappa; T^d) \mapsto \mathcal{V}_n^d(W_\kappa)$ is the orthogonal projection operator, which can be written as an integral operator

$$\operatorname{proj}_n^\kappa(f, x) = a_\kappa \int_{T^d} f(y) P_n(W_\kappa; x, y) W_\kappa(y)\, dy, \quad x \in T^d,$$

with $a_\kappa^{-1} = \int_{T^d} W_\kappa(y)\, dy$. A key point is that the kernel $P_n(W_\kappa; \cdot, \cdot)$ satisfies the following formula of Yuan Xu (see [17, 18]):

$$\begin{aligned} P_n(W_\kappa; x, y) = {} & \frac{(2n + \lambda_\kappa)\Gamma(\frac{1}{2})\Gamma(n + \lambda_k)}{\Gamma(\lambda_\kappa + 1)\Gamma(n + \frac{1}{2})} \\ & \times c_\kappa \int_{[-1,1]^{d+1}} P_n^{(\lambda_k - \frac{1}{2}, -\frac{1}{2})}\big(2u(x, y, t)^2 - 1\big) \prod_{i=1}^{d+1} \big(1 - t_i^2\big)^{\kappa_i - 1}\, dt, \end{aligned} \tag{2.1}$$

where $u(x,y,t)=\sqrt{x_1y_1}t_1+\cdots+\sqrt{x_dy_d}t_d+\sqrt{1-|x|}\sqrt{1-|y|}t_{d+1}$, $\lambda_\kappa=\frac{d-1}{2}+|\kappa|$, $|\kappa|=\sum_{j=1}^{d+1}\kappa_j$, and $P_k^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree k and indices (α,β) as defined in [16]. If some $\kappa_i=0$, then the formula holds under the limit relation

$$\lim_{\delta\to 0} c_\delta \int_{-1}^{1} f(t)\big(1-t^2\big)^{\delta-1}\,dt=\frac{f(1)+f(-1)}{2},$$

where $c_\delta=\frac{\Gamma(\delta+\frac12)}{\sqrt{\pi}\Gamma(\delta)}$.

The space $\mathcal{V}_n^d(W_\kappa)$ can be characterized as an eigenspace of the differential operator $P(D)_\kappa$ defined by (1.7). In fact, it is known (see [13, p. 46, (2.3.11)]) that a $C^2(T^d)$-function P belongs to the space $\mathcal{V}_n^d(W_\kappa)$ if and only if

$$P(D)_\kappa P=-n\left(n+|\kappa|+\frac{d-1}{2}\right)P. \tag{2.2}$$

Thus, given $r\in\mathbb{R}-\{0\}$, we can define the operator $(-P(D)_\kappa)^r$ in a distributional sense by

$$\big(-P(D)_\kappa\big)^r(f)=\sum_{n=1}^{\infty}\left(n\left(n+|\kappa|+\frac{d-1}{2}\right)\right)^r \operatorname{proj}_n^\kappa(W_\kappa;f).$$

The differential operator $P(D)_\kappa$ can be decomposed as in (1.9). Note that for $\xi\in E$ and $\widetilde{d}(\xi,x)$ given in (1.9), either $\xi=e_i$ and $\widetilde{d}(\xi,x)=x_i(1-|x|)$ or $\xi=\frac{1}{\sqrt2}(e_i-e_j)$ and $\widetilde{d}(\xi,x)=2x_ix_j$. (Here e_i is the unit vector in the x_i direction.) See [9, p. 228]. Thus, the decomposition (1.9) can be written more explicitly (but less compactly) as

$$P(D)_\kappa=\sum_{1\le i\le j\le d}U_{ij,\kappa}, \tag{2.3}$$

where

$$U_{ii,\kappa}=W_\kappa(x)^{-1}\frac{\partial}{\partial x_i}\big(x_i\big(1-|x|\big)W_\kappa(x)\big)\frac{\partial}{\partial x_i},\quad 1\le i\le d, \tag{2.4}$$

$$U_{ij,\kappa}=W_\kappa(x)^{-1}D_{ij}\big(x_ix_jW_\kappa(x)\big)D_{ij},\quad 1\le i<j\le d. \tag{2.5}$$

The operators $U_{ij,\kappa}$ are self-adjoint on $L^2(W_\kappa;T^d)$ in the sense that for any $f,g\in C^2(T^d)$,

$$\langle U_{ij,\kappa}f,g\rangle_\kappa=\langle f,U_{ij,\kappa}g\rangle_\kappa, \tag{2.6}$$

where and throughout

$$\langle f,g\rangle_\kappa=\int_{T^d}f(x)g(x)W_\kappa(x)\,dx.$$

Moreover, each space $\mathcal{V}_n^d(W_\kappa)$ of orthogonal polynomials is invariant under $U_{ij,\kappa}$; that is, $U_{ij,\kappa}(\mathcal{V}_n^d(W_\kappa)) \subset \mathcal{V}_n^d(W_\kappa)$. All these properties can be found in [9, pp. 228–229]. Finally, we point out that in the unweighted case (corresponding to $\kappa = (\frac{1}{2}, \dots, \frac{1}{2})$), the decomposition (2.3) was proved earlier by Berens, Schmid and Xu [2, p. 257].

The following Marcinkiewitcz-type multiplier theorem was proved recently in [7, Theorem 5.2]:

Theorem 2.1 *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers satisfying*

$$\text{(i)} \quad \sup_j |\mu_j| \le c < \infty,$$

$$\text{(ii)} \quad \sup_j 2^{j(\ell-1)} \sum_{k=2^j}^{2^{j+1}} \left|\Delta^\ell u_k\right| \le c < \infty,$$

where ℓ is the smallest integer $\ge \frac{d+1}{2} + \sum_{j=1}^{d+1} \kappa_j$, $\Delta^0 \mu_k = \mu_k$, $\Delta^1 \mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{i+1} = \Delta(\Delta^i)$. Then for $f \in L^p(W_\kappa; T^d)$ with $1 < p < \infty$, we have

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^\kappa(f) \right\|_{\kappa,p} \le C \|f\|_{\kappa,p},$$

where C is independent of f and $\{\mu_j\}$ but depends on the constant c in (i) *and* (ii).

Let $\eta \in C^\infty[0,\infty)$ be such that $\eta(x) = 1$ for $0 \le x \le \frac{1}{2}$ and $\eta(x) = 0$ for $x \ge 1$. Let $\theta(x) = \eta(x) - \eta(2x)$. Define $\theta_0^\kappa(f) = \operatorname{proj}_0^\kappa(f)$, and

$$\theta_j^\kappa(f) := \sum_{n=0}^{\infty} \theta\left(\frac{n}{2^j}\right) \operatorname{proj}_n^\kappa(f), \qquad j \ge 1.$$

As a consequence of Theorem 2.1, we have the following Littlewood–Paley-type inequality:

Corollary 2.2 *Let $f \in L^p(W_\kappa; T^d)$ with $1 < p < \infty$. Then*

$$\|f\|_{\kappa,p} \sim \left\| \left(\sum_{j=0}^{\infty} \left(\theta_j^\kappa(f)\right)^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p}. \tag{2.7}$$

The proof of Corollary 2.2 is quite standard, and it can be found in [6, Sect. 2]. Consider the following metric on T^d:

$$\rho(x,y) := \arccos\left(\sum_{j=1}^{d+1} \sqrt{x_j y_j} \right), \qquad x, y \in T^d, \tag{2.8}$$

where here and elsewhere $x_{d+1} = 1 - |x|$ and $y_{d+1} = 1 - |y|$. It's easily seen that for $x, y \in T^d$,

$$\|\xi - \eta\| \le \rho(x, y) \le \frac{\pi}{2}\|\xi - \eta\|, \tag{2.9}$$

where $\xi = (\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_{d+1}})$, $\eta = (\sqrt{y_1}, \sqrt{y_2} \dots, \sqrt{y_{d+1}})$, and $\|\xi - \eta\|$ denotes the Euclidean norm of $\xi - \eta$. Let $C(x, t) = \{y \in T^d : \rho(x, y) \le t\}$, where $x \in T^d$ and $t \in (0, \frac{\pi}{2}]$. As was shown in [5],

$$\int_{C(x,\theta)} W_\kappa(y)\, dy \sim \theta^d \prod_{j=1}^{d+1} \big(|x_j| + \theta^2\big)^{\kappa_j}, \quad \text{for } \theta \in \left(0, \frac{\pi}{2}\right], \tag{2.10}$$

where $x = (x_1, \dots, x_d) \in T^d$ and $x_{d+1} = 1 - |x|$. In particular, this means that the weight W_κ satisfies the doubling condition with respect to the metric ρ. Thus, $(T^d; \rho; W_\kappa\, dx)$ is a space of homogeneous type in the sense of Coiffman and Weiss (see [14]). Thus, following [14], we define

$$V(x, y) = \int_{C(x,\rho(x,y))} W_\kappa(z)\, dz, \quad x, y \in T^d, \tag{2.11}$$

and

$$V_t(x) = \int_{C(x,t)} W_\kappa(z)\, dz, \quad x \in T^d,\ 0 < t \le \frac{\pi}{2}. \tag{2.12}$$

The Hardy–Littlewood maximal function with respect to the weight W_κ is defined as usual by

$$M_\kappa(f)(x) := \sup_{t>0} \frac{1}{\text{meas}_\kappa(C(x,t))} \int_{C(x,t)} \big|f(y)\big| W_\kappa(y)\, dy, \quad x \in T^d,$$

where for a measurable set $E \subset T^d$,

$$\text{meas}_\kappa(E) := \int_E W_\kappa(y)\, dy.$$

Since the weight W_κ satisfies the doubling condition, the following Fefferman–Stein-type inequality is well known:

$$\left\| \left(\sum_{j=1}^{\infty} \big|M_\kappa(f_j)\big|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p} \le c \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p}, \quad 1 < p < \infty. \tag{2.13}$$

We also need the following lemma:

Lemma 2.3 *For $f \in L(W_\kappa; T^d)$ and $\ell > 0$,*

$$\int_{T^d} \frac{|f(y)|}{V(x, y) + V_{n^{-1}}(x)} \big(1 + n\rho(x, y)\big)^{-\ell} W_\kappa(y)\, dy \le c M_\kappa(f)(x),$$
$$x \in T^d,\ n = 1, 2, \dots,$$

where c is independent of f, x and n.

Proof By definition, we have

$$
\begin{aligned}
&\int_{T^d} \frac{|f(y)|}{V(x,y)+V_{n^{-1}}(x)} \big(1+n\rho(x,y)\big)^{-\ell} W_\kappa(y)\,dy \\
&= \left\{ \int_{C(x,n^{-1})} + \sum_{j=0}^{\infty} \int_{\{y:\ \frac{2^j}{n} < \rho(x,y) \le \frac{2^{j+1}}{n}\}} \right\} \\
&\quad \times \frac{|f(y)|}{V(x,y)+V_{n^{-1}}(x)} \big(1+n\rho(x,y)\big)^{-\ell} W_\kappa(y)\,dy \\
&\le \frac{1}{V_{n^{-1}}(x)} \int_{C(x,n^{-1})} \big|f(y)\big| W_\kappa(y)\,dy \\
&\quad + \sum_{j=0}^{\infty} 2^{-j\ell} \int_{\{y: \frac{2^j}{n} < \rho(x,y) \le \frac{2^{j+1}}{n}\}} \frac{|f(y)|}{V(x,y)} W_\kappa(y)\,dy \\
&\le M_\kappa(f)(x) + c \sum_{j=0}^{\infty} \frac{2^{-j\ell}}{\mathrm{meas}_\kappa(C(x,\frac{2^{j+1}}{n}))} \int_{C(x,\frac{2^{j+1}}{n})} \big|f(y)\big| W_\kappa(y)\,dy \\
&\le M_\kappa(f)(x) + c \sum_{j=0}^{\infty} 2^{-j\ell} M_\kappa(f)(x) \le c M_\kappa(f)(x),
\end{aligned}
$$

where we have used the doubling property of the weight function W_κ in the third step. □

3 Proof of Theorem 1.1

For the proof of Theorem 1.1, by (1.6), it is sufficient to show that for $g \in C^2(T^d)$ and $1 < p < \infty$,

$$
\max_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_{\kappa,p} \le c \big\| P(D)_\kappa g\big\|_{\kappa,p} \le c^2 \|g\|_{\kappa,p} + c^2 \max_{1\le i\le j\le d} \big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_{\kappa,p}. \tag{3.1}
$$

Equation (3.1) is a consequence of the following two propositions:

Proposition 3.1 *For $1 \le i \le j \le d$, $1 < p < \infty$, and $g \in C^2(T^d)$,*

$$
\big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_{\kappa,p} \le c \|U_{ij,\kappa} g\|_{\kappa,p} \le c^2 \|g\|_{\kappa,p} + c^2 \big\|\varphi_{ij}^2 D_{ij}^2 g\big\|_{\kappa,p}. \tag{3.2}
$$

Proposition 3.2 *For $g \in C^2(T^d)$,*

$$
\max_{1\le i\le j\le d} \|U_{ij,\kappa} g\|_{\kappa,p} \sim \|P(D)_\kappa g\|_{\kappa,p}, \quad 1 < p < \infty, \tag{3.3}
$$

where the operators $U_{ij,\kappa}$, $1 \le i \le j \le d$ are defined by (2.4) and (2.5).

We point out that (3.2) in the unweighted case was proved in [2], while (3.3) was proved for $p=2$ and conjectured for all $1<p<\infty$ by Ditzian [10] in 1995.

The proofs of Propositions 3.1 and 3.2 will be given in Sects. 3.1 and 3.2, respectively.

3.1 Proof of Proposition 3.1

Following the idea of [2], we shall use a recent result of [6, Theorem 7.1]: if $\alpha,\beta>-1$, $w^{(\alpha,\beta)}(t)=(1-t)^\alpha(1+t)^\beta$, and $h\in C^2[-1,1]$, then for $1<p<\infty$,

$$\begin{aligned}\big\|(1-t^2)h''(t)\big\|_{p,\alpha,\beta} &\le c\big\|D^{(\alpha,\beta)}h\big\|_{p,\alpha,\beta}\\ &\le c^2\|h\|_{p,\alpha,\beta}+c^2\big\|(1-t^2)h''(t)\big\|_{p,\alpha,\beta},\end{aligned} \tag{3.4}$$

where $\|h\|_{p,\alpha,\beta}=(\int_{-1}^1|h(t)|^p w^{(\alpha,\beta)}(t)\,dt)^{\frac{1}{p}}$ and

$$D^{(\alpha,\beta)}=w^{(\alpha,\beta)}(t)^{-1}\left(\frac{d}{dt}(1-t^2)w^{(\alpha,\beta)}(t)\frac{d}{dt}\right).$$

Let us first consider the case of $i=j$. For simplicity, we may assume without loss of generality that $i=j=1$. Let $h_y(t)=g(\frac{(1-|y|)(1+t)}{2},y)$ for $t\in[-1,1]$ and a fixed $y\in T^{d-1}$. We also let $x=(x_1,y)$ with $0\le x_1\le 1-|y|$. Then a straightforward computation shows that under the transformation $x_1=\frac{(1-|y|)(1+t)}{2}$

$$x_1x_{d+1}\left(\frac{\partial}{\partial x_1}\right)^2 g(x)=(1-t^2)h_y''(t), \tag{3.5}$$

$$W_\kappa(x)^{-1}\left(\frac{\partial}{\partial x_1}(x_1x_{d+1})W_\kappa(x)\frac{\partial}{\partial x_1}\right)g(x)=D^{(\kappa_{d+1}-\frac12,\kappa_1-\frac12)}h_y(t), \tag{3.6}$$

where $x_{d+1}=1-|x|=1-x_1-|y|$. It follows that for $1<p<\infty$,

$$\begin{aligned}&\int_{T^d}\left|x_1x_{d+1}\left(\frac{\partial}{\partial x_1}\right)^2 g(x)\right|^p W_\kappa(x)\,dx\\ &=\int_{T^{d-1}}\int_0^{1-|y|}\left|x_1x_{d+1}\left(\frac{\partial}{\partial x_1}\right)^2 g(x)\right|^p W_\kappa(x)\,dx_1\,dy \quad (\text{here } x=(x_1,y))\\ &=2^{-\kappa_1-\kappa_{d+1}}\int_{T^{d-1}}\left[\int_{-1}^1\left|(1-t^2)h_y''(t)\right|^p w^{(\kappa_{d+1}-\frac12,\kappa_1-\frac12)}(t)\,dt\right]\\ &\quad\times(1-|y|)^{\kappa_1+\kappa_{d+1}}\prod_{j=2}^d|y_j|^{\kappa_j-\frac12}\,dy\end{aligned}$$

$$\le c\int_{T^{d-1}}\left[\int_{-1}^{1}\left|\left(D^{(\kappa_{d+1}-\frac12,\kappa_1-\frac12)}h_y\right)(t)\right|^p w^{(\kappa_{d+1}-\frac12,\kappa_1-\frac12)}(t)\,dt\right]$$
$$\times\left(1-|y|\right)^{\kappa_1+\kappa_{d+1}}\prod_{j=2}^{d}|y_j|^{\kappa_j-\frac12}\,dy$$
$$=c\int_{T^{d-1}}\int_0^{1-|y|}\left|U_{11,\kappa}g(x)\right|^p W_\kappa(x)\,dx_1\,dy=c\int_{T^d}\left|U_{11,\kappa}g(x)\right|^p W_\kappa(x)\,dx,$$

where we have used (3.5) and a change of variable $x_1=\frac{1+t}{2}(1-|y|)$ in the second step, (3.4) in the third step, (3.6) and a change of variable $x_1=\frac{1+t}{2}(1-|y|)$ in the fourth step. Similarly, one can show

$$\|U_{11,\kappa}g\|_{\kappa,p}\le c^2\|g\|_{\kappa,p}+c^2\left\|\varphi_{11}^2D_{11}^2g\right\|_{\kappa,p}.$$

This proves (3.2) for $i=j$.

Next, we show (3.2) for $1\le i<j\le d$. We define an operator T_j by

$$T_j(f)(x)=f\left(x_1,\dots,x_{j-1},1-|x|,x_{j+1},\dots,x_d\right), \tag{3.7}$$

where $x=(x_1,\dots,x_d)\in T^d$. It's easily seen that $T_jD_{ij}=D_{ii}T_j$, which implies that $T_j(\varphi_{ij}^2(x)D_{ij}^2)=\varphi_{ii}^2(x)(D_{ii}^2T_j)$ and $T_jU_{ij,\kappa}=U_{ii,\tau}T_j$, where

$$\tau=(\kappa_1,\dots,\kappa_{j-1},\kappa_{d+1},\kappa_{j+1},\dots,\kappa_{d-1},\kappa_d,\kappa_j). \tag{3.8}$$

Since the Lebesgue measure on T^d is invariant under the transformation $x\mapsto(x_1,\dots,x_{j-1},1-|x|,x_{j+1},\dots,x_d)$, it follows that $\|\varphi_{ij}^2D_{ij}^2g\|_{\kappa,p}=\|\varphi_{ii}^2D_{ii}^2(T_jg)\|_{\tau,p}$ and

$$\|U_{ij,\kappa}g\|_{\kappa,p}=\left\|U_{ii,\tau}(T_jg)\right\|_{\tau,p}. \tag{3.9}$$

However, invoking (3.2) for the already proven case $i=j$, we obtain

$$c^{-1}\left\|\varphi_{ii}^2D_{ii}^2(T_jg)\right\|_{\tau,p}\le\left\|U_{ii,\tau}(T_jg)\right\|_{\tau,p}\le c\left\|\varphi_{ii}^2D_{ii}^2(T_jg)\right\|_{\tau,p}+c\|T_jg\|_{\tau,p}.$$

The desired inequality (3.2) for $i<j$ then follows, since $\|T_jg\|_{\tau,p}=\|g\|_{\kappa,p}$. This completes the proof. □

3.2 Proof of Proposition 3.2

The inequality

$$\left\|P(D)_\kappa g\right\|_{\kappa,p}\le c\max_{1\le i\le j\le d}\|U_{ij,\kappa}g\|_{\kappa,p}$$

follows directly from (2.3) and the triangle inequality. To show the converse inequality, we first observe that (3.9) is true for $i<j$, and that $P(D)_\tau T_j=T_jP(D)_\kappa$, on

account of (2.2). Thus, for the proof of the desired converse inequality, it suffices to prove

$$\|U_{ii,\kappa} g\|_{\kappa,p} \le c \big\| P(D)_\kappa g \big\|_{\kappa,p}, \quad 1 \le i \le d. \tag{3.10}$$

The following theorem plays a crucial role in the proof of (3.10):

Theorem 3.3 *Let $\phi \in C^\infty[0,\infty)$ be such that supp $\phi \subset [\frac{1}{8}, 8]$. Given $n \in \mathbb{N}$ and $x, y \in T^d$, define*

$$\Phi_{n,\kappa}(x,y) := \sum_{j=0}^{\infty} \phi\left(\frac{j}{2^n}\right)\left(j\left(j + |\kappa| + \frac{d-1}{2}\right)\right)^{-\frac{1}{2}} P_j(W_\kappa; x, y), \tag{3.11}$$

where $P_j(W_\kappa; x, y)$ is defined by (2.1). Then for any positive integer ℓ, and $i = 1, \dots, d$,

$$\left| \sqrt{x_i x_{d+1}} \frac{\partial \Phi_{n,\kappa}(x,y)}{\partial x_i} \right| \le c\big(1 + 2^n \rho(x,y)\big)^{-\ell} \frac{1}{V(x,y) + V_{2^{-n}}(x) + V_{2^{-n}}(y)},$$

where $x_{d+1} = 1 - |x|$.

The proof of Theorem 3.3 will be given in the next section. For the moment, we take it for granted and proceed with the proof of (3.10).

Let us first define

$$A_{x_i} = \sqrt{x_i\big(1 - |x|\big)} \frac{\partial}{\partial x_i}, \quad 1 \le i \le d.$$

An integration by parts shows that for any $f_1, f_2 \in C^2(T^d)$,

$$\langle U_{ii,\kappa} f_1, f_2 \rangle_\kappa = -\langle A_{x_i} f_1, A_{x_i} f_2 \rangle_\kappa. \tag{3.12}$$

Let the operators θ_j^κ, $j = 0, 1, \dots$ be as defined in Sect. 2. As a consequence of Theorem 3.3, we have the following lemma:

Lemma 3.4 *If $f \in L(W_\kappa; T^d)$, $x \in T^d$ and $1 \le i \le d$, then*

$$\left| A_{x_i}\big(\theta_j^\kappa \big(-P(D)_\kappa\big)^{-\frac{1}{2}} f\big)(x) \right| \le c M_\kappa\big(\theta_j^\kappa f\big)(x).$$

Proof Without loss of generality, we may assume $j \ge 1$, since otherwise the lemma is trivial. Let $\phi \in C^\infty[0,\infty)$ satisfy $\phi(t) = 1$ for $\frac{1}{4} \le t \le 1$ and $\phi(t) = 0$ for $t \notin [\frac{1}{8}, 8]$. Let $\Phi_{n,\kappa}$ be as defined in Theorem 3.3. Then we have

$$\begin{aligned}\theta_j^\kappa\big(-P(D)_\kappa\big)^{-\frac{1}{2}} f(x) &= \sum_{n=1}^{\infty} \phi\left(\frac{n}{2^j}\right)\left(n\left(n + |\kappa| + \frac{d-1}{2}\right)\right)^{-\frac{1}{2}} \operatorname{proj}_n^\kappa\big(\theta_j^\kappa f\big)(x) \\ &= \int_{T^d} \theta_j^\kappa f(y) \Phi_{j,\kappa}(x,y) W_\kappa(y)\, dy.\end{aligned}$$

So, using Theorem 3.3 with ℓ sufficiently large, we obtain

$$
\begin{aligned}
&\left|A_{x_i}\left(\theta_j^\kappa\left(-P(D)_\kappa\right)^{-1/2} f\right)(x)\right| \\
&\quad \le \int_{T^d} \left|\theta_j^\kappa f(y)\right|\left|A_{x_i}\Phi_{j,\kappa}(x,y)\right| W_\kappa(y)\,dy \\
&\quad \le c\int_{T^d} \left|\theta_j^\kappa f(y)\right|\left(1+2^j\rho(x,y)\right)^{-\ell} \frac{1}{V(x,y)+V_{2^{-j}}(x)} W_\kappa(y)\,dy \\
&\quad \le c M_\kappa\left(\theta_j^\kappa f\right)(x),
\end{aligned}
$$

where we have used Lemma 2.3 in the last step. This completes the proof of Lemma 3.4. □

We are now in a position to prove (3.10). Clearly, it is sufficient to prove

$$
\left\|U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1} F\right\|_p \le c\|F\|_p \tag{3.13}
$$

whenever $F \in L^p(W_\kappa; T^d)$. Let $\frac{1}{p}+\frac{1}{p'}=1$ and $G \in L^{p'}(W_\kappa; T^d)$. Then

$$
\begin{aligned}
&\left\langle U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1} F, G\right\rangle_\kappa \\
&\quad =\sum_{j=0}^\infty \sum_{|\ell-j|\le 2} \left\langle U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1}\theta_j^\kappa F, \theta_\ell^\kappa(G)\right\rangle_\kappa \\
&\quad =\sum_{j=0}^\infty \sum_{|\ell-j|\le 2} \left\langle U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1/2}\theta_j^\kappa F, \left(-P(D)_\kappa\right)^{-1/2}\theta_\ell^\kappa(G)\right\rangle_\kappa \\
&\quad =-\sum_{j=0}^\infty \sum_{|\ell-j|\le 2} \left\langle A_{x_i}\left(-P(D)_\kappa\right)^{-1/2}\theta_j^\kappa F, A_{x_i}\left(-P(D)_\kappa\right)^{-1/2}\theta_\ell^\kappa(G)\right\rangle_\kappa,
\end{aligned}
$$

where we have used orthogonality, the identity $\sum_{j=0}^\infty \theta_j^\kappa(f)=f$ and the fact that $U_{ii,\kappa}(\mathcal{V}_n^d(W_\kappa)) \subset \mathcal{V}_n^d(W_\kappa)$ in the first step, the fact that $U_{ii,\kappa}P(D)_\kappa^r = P(D)_\kappa^r U_{ii,\kappa}$ in the second step,[3] and (3.12) in the third step. Thus, using Lemma 3.4 and Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|\left\langle U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1} F, G\right\rangle_\kappa\right| \\
&\quad \le c\int_{T^d} \sum_{j=0}^\infty \sum_{|\ell-j|\le 2} M_\kappa\left(\theta_j^\kappa F\right)(x) M_\kappa\left(\theta_\ell^\kappa G\right)(x) W_\kappa(x)\,dx \\
&\quad \le c\int_{T^d} \left(\sum_{j=0}^\infty \left|M_\kappa\left(\theta_j^\kappa F\right)\right|^2\right)^{\frac12} \left(\sum_{\ell=0}^\infty \left|M_\kappa\left(\theta_\ell^\kappa G\right)\right|^2\right)^{\frac12} W_\kappa(x)\,dx \\
&\quad \le c\left\|\left(\sum_{j=0}^\infty \left|M_\kappa\left(\theta_j^\kappa F\right)\right|^2\right)^{\frac12}\right\|_{\kappa,p} \left\|\left(\sum_{j=0}^\infty \left|M_\kappa\left(\theta_j^\kappa G\right)\right|^2\right)^{\frac12}\right\|_{\kappa,p'}.
\end{aligned}
$$

[3] This fact is a consequence of the fact that $U_{ii,\kappa}(\mathcal{V}_n^d(W_\kappa)) \subset \mathcal{V}_n^d(W_\kappa)$ for all $n \in \mathbb{Z}_+$.

Thus, using the Fefferman–Stein inequality (2.13) and the Littlewood–Paley inequality (2.7), we deduce

$$\begin{aligned}\left|\left\langle U_{ii,\kappa}\left(-P(D)_\kappa\right)^{-1}F, G\right\rangle_\kappa\right| &\le c\left\|\left(\sum_{j=0}^\infty\left|\theta_j^\kappa F\right|^2\right)^{\frac12}\right\|_{\kappa,p}\left\|\left(\sum_{j=0}^\infty\left|\theta_j^\kappa G\right|^2\right)^{\frac12}\right\|_{\kappa,p'}\\ &\le c\|F\|_{\kappa,p}\|G\|_{\kappa,p'}.\end{aligned}$$

The desired inequality (3.13) then follows by taking supremum over all $G \in L^{p'}(W_\kappa; T^d)$ with $\|G\|_{\kappa,p'} = 1$. This completes the proof. □

4 Proof of Theorem 3.3

To show Theorem 3.3, we need a series of lemmas. The first two lemmas were proved recently in [8].

Lemma 4.1 ([8]) *Assume $\delta_j > 0$, $a_j \ne 0$ and $\xi_j \in C^\infty[-1,1]$ for all $j = 1, 2, \dots, m$. Let $|a| := \sum_{j=1}^m |a_j| \le 1$. If $\alpha \ge \beta$, $\alpha \ge \delta - \frac12 := \sum_{j=1}^m \delta_j - \frac12$ and $|x| + |a| \le 1$, then*

$$\begin{aligned}&\left|\int_{[-1,1]^m} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^m a_j t_j + x\right)\prod_{j=1}^m \xi_j(t_j)\left(1-t_j^2\right)^{\delta_j-1}dt\right|\\ &\quad\le cn^{\alpha-2\delta}\frac{\prod_{j=1}^m(|a_j| + n^{-1}\sqrt{1-|a|-|x|} + n^{-2})^{-\delta_j}}{(1+n\sqrt{1-|a|-|x|})^{\alpha+\frac12-\delta}}.\end{aligned}\tag{4.1}$$

Lemma 4.2 ([8]) *Let $\delta_j > 0$, $a_j \ne 0$, $\xi_j \in C^\infty[-1,1]$ with* supp $\xi_j \subset [-\frac12, 1]$ *for $j = 1, 2, \dots, m$, and let $\sum_{j=1}^m |a_j| \le 1$. Define*

$$f_m(x) := \int_{[-1,1]^m} P_n^{(\alpha,\beta)}\left(\sum_{j=1}^m a_j t_j + x\right)\prod_{j=1}^m \xi_j(t_j)(1-t_j)^{\delta_j-1}dt$$

for $|x| \le 1 - \sum_{j=1}^m |a_j|$. If $\alpha \ge \beta$, then

$$\left|f_m(x)\right| \le cn^{\alpha-2\tau_m}\left(\prod_{j=1}^m |a_j|^{-\delta_j}\right)\left(1+n\sqrt{1-|A_m+x|}\right)^{-\alpha-\frac12+\tau_m},$$

where $A_m := \sum_{j=1}^m a_j$ and $\tau_m := \sum_{j=1}^m \delta_j$.

Lemma 4.3 *For $x, y \in T^d$ and $n \in \mathbb{N}$, let*

$$H(x,y) = \prod_{j=1}^{d+1}\left(\sqrt{x_j y_j} + n^{-1}\rho(x,y) + n^{-2}\right)^{-\kappa_j},\tag{4.2}$$

where $x_{d+1} = 1 - |x|$ *and* $y_{d+1} = 1 - |y|$. *Then*

$$H(x, y) \le cn^{-d}\bigl(1 + n\rho(x, y)\bigr)^{|\kappa|+d} \frac{1}{V(x, y) + V_{n^{-1}}(x) + V_{n^{-1}}(y)}.$$

Proof Let $I_j(x, y) = (\sqrt{x_j y_j} + n^{-1}\rho(x, y) + n^{-2})^{-\kappa_j}$. In the following proof, we shall use (2.9) repeatedly. If $\sqrt{x_j} \ge 4\rho(x, y)$, then $\sqrt{x_j} \sim \sqrt{y_j}$, and hence

$$I_j(x, y) \le c\bigl(y_j + x_j + n^{-2}\bigr)^{-\kappa_j} \sim \bigl(\sqrt{y_j} + \sqrt{x_j} + \rho(x, y) + n^{-1}\bigr)^{-2\kappa_j}.$$

If $\sqrt{x_j} \le 4\rho(x, y)$, then $\sqrt{y_j} \le 5\rho(x, y)$, and hence

$$\begin{aligned} I_j(x, y) &\le cn^{2\kappa_j}\bigl(n\rho(x, y) + 1\bigr)^{-\kappa_j} \\ &\le c\bigl(n^{-1} + \sqrt{x_j} + \sqrt{y_j} + \rho(x, y)\bigr)^{-2\kappa_j}\bigl(n\rho(x, y) + 1\bigr)^{\kappa_j}. \end{aligned} \tag{4.3}$$

Therefore, in either case, we have the above inequality (4.3), which implies

$$\begin{aligned} H(x, y) &= \prod_{j=1}^{d+1} I_j(x, y) \\ &\le cn^{-d}\bigl(\rho(x, y) + n^{-1}\bigr)^{-d}\left(\prod_{j=1}^{d+1}\bigl(n^{-1} + \sqrt{x_j} + \sqrt{y_j} + \rho(x, y)\bigr)^{-2\kappa_j}\right) \\ &\quad \times \bigl(n\rho(x, y) + 1\bigr)^{|\kappa|+d} \\ &\sim \frac{n^{-d}}{V(x, y) + V_{n^{-1}}(x) + V_{n^{-1}}(y)}\bigl(n\rho(x, y) + 1\bigr)^{|\kappa|+d}, \end{aligned}$$

where we have used (2.10), (2.11) and (2.12) in this last step. □

The following lemma plays a key role in the proof of Theorem 3.3.

Lemma 4.4 *For* $\alpha \ge |\kappa| - \frac{1}{2}$, *define*

$$D_{n,\kappa}^{(\alpha,\alpha)}(x, y) = \int_{[-1,1]^{d+1}} P_{2n}^{(\alpha,\alpha)}\left(\sum_{j=1}^{d+1}\sqrt{x_j y_j}\, t_j\right)\prod_{j=1}^{d+1}\bigl(1 - t_j^2\bigr)^{\kappa_j - 1}\, dt,$$

where $x, y \in T^d$, $x_{d+1} = 1 - |x|$ *and* $y_{d+1} = 1 - |y|$. *Then for* $1 \le i \le d$,

$$\left|\sqrt{x_i x_{d+1}}\frac{\partial}{\partial x_i} D_{n,\kappa}^{(\alpha,\alpha)}(x, y)\right| \le cn^{\alpha+1-2|\kappa|-d}\frac{(1 + n\rho(x, y))^{-\alpha-\frac{1}{2}+2|\kappa|+d}}{V(x, y) + V_{n^{-1}}(x) + V_{n^{-1}}(y)}.$$

Proof Without loss of generality, we may assume $i = 1$. Since all the estimates below are uniform in κ, we may also assume $\min_{1\le j\le d+1}\kappa_j > 0$. Let $H(x, y)$ be as defined in (4.2). Since

$$\frac{d}{ds}P_n^{(\alpha,\beta)}(s)=\frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(s),$$

a straightforward computation shows that

$$\begin{aligned}\sqrt{x_1x_{d+1}}\frac{\partial}{\partial x_1}D_{n,\kappa}^{(\alpha,\alpha)}(x,y)=O(n)\int_{[-1,1]^{d+1}}&P_{2n-1}^{(\alpha+1,\alpha+1)}(u)\\ &\times\big[\sqrt{x_{d+1}y_1}t_1-\sqrt{x_1y_{d+1}}t_{d+1}\big]\prod_{j=1}^{d+1}\big(1-t_j^2\big)^{\kappa_j-1}\,dt,\end{aligned}\tag{4.4}$$

where and throughout the proof, $u\equiv u(x,y,t)=\sum_{j=1}^{d+1}\sqrt{x_jy_j}t_j$.

By Lemma 4.3, it is sufficient to prove

$$\left|\sqrt{x_1x_{d+1}}\frac{\partial}{\partial x_1}D_{n,\kappa}^{(\alpha,\alpha)}(x,y)\right|\le cn^{\alpha+1-2|\kappa|}H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|}.\tag{4.5}$$

To this end, we consider the following three cases:

Case 1. $\min\{\sqrt{x_1},\ \sqrt{x_{d+1}}\}<4\rho(x,y)$.

By (2.9), either $\max\{\sqrt{x_1},\sqrt{y_1}\}<5\rho(x,y)$ or $\max\{\sqrt{x_{d+1}},\sqrt{y_{d+1}}\}<5\rho(x,y)$. In both cases, using Lemma 4.1, we have

$$\begin{aligned}&\left|\sqrt{x_1x_{d+1}}\frac{\partial}{\partial x_1}D_{n,\kappa}^{(\alpha,\alpha)}(x,y)\right|\\ &\quad\le cn\sqrt{x_{d+1}y_1}\left|\int_{[-1,1]^{d+1}}P_{2n-1}^{(\alpha+1,\alpha+1)}(u)t_1\prod_{j=1}^{d+1}\big(1-t_j^2\big)^{\kappa_j-1}\,dt\right|\\ &\qquad+cn\sqrt{x_1y_{d+1}}\left|\int_{[-1,1]^{d+1}}P_{2n-1}^{(\alpha+1,\alpha+1)}(u)t_{d+1}\prod_{j=1}^{d+1}\big(1-t_j^2\big)^{\kappa_j-1}\,dt\right|\\ &\quad\le c\big(n\rho(x,y)\big)n^{\alpha+1-2|\kappa|}H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{3}{2}+|\kappa|}\\ &\quad\le cn^{\alpha+1-2|\kappa|}H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|},\end{aligned}$$

which gives the desired estimate (4.5) in this case.

Case 2. $4\rho(x,y)\le\min\{\sqrt{x_1},\sqrt{x_{d+1}}\}<\frac{1}{n}$.

By (2.9), either $\max\{\sqrt{x_1},\sqrt{y_1}\}\le\frac{2}{n}$ or $\max\{\sqrt{x_{d+1}},\sqrt{y_{d+1}}\}\le\frac{2}{n}$. In both cases, using Lemma 4.1, we have

$$\begin{aligned}\left|\sqrt{x_1x_{d+1}}\frac{\partial}{\partial x_1}D_{n,\kappa}^{(\alpha,\alpha)}(x,y)\right|&\le c\left|\int_{[-1,1]^{d+1}}P_{2n-1}^{(\alpha+1,\alpha+1)}(u)t_1\prod_{j=1}^{d+1}\big(1-t_j^2\big)^{\kappa_j-1}\,dt\right|\\ &\quad+c\left|\int_{[-1,1]^{d+1}}P_{2n-1}^{(\alpha+1,\alpha+1)}(u)t_{d+1}\prod_{j=1}^{d+1}\big(1-t_j^2\big)^{\kappa_j-1}\,dt\right|\\ &\le cn^{\alpha+1-2|\kappa|}H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|},\end{aligned}$$

as desired.

Case 3. $\min\{\sqrt{x_1}, \sqrt{x_{d+1}}\} \ge \max\{4\rho(x,y), \frac{1}{n}\}$.

Let $\eta_1 \in C^\infty[-1,1]$ satisfy $\eta_1(t) = 1$ for $-\frac{1}{4} \le t \le 1$ and $\eta_1(t) = 0$ for $t \in [-1, -\frac{1}{2}]$, and let $\eta_{-1}(t) = 1 - \eta(t)$. Set

$$\mathcal{J}_1 := \{j:\ 1 \le j \le d+1,\ \sqrt{x_j y_j} \ge (n^{-1}\rho(x,y) + n^{-2})/8\},$$
$$\mathcal{J}_2 := \{j:\ 1 \le j \le d+1,\ \sqrt{x_j y_j} < (n^{-1}\rho(x,y) + n^{-2})/8\}.$$

Since $\frac{3}{4}\sqrt{x_j} \le \sqrt{y_j} \le \frac{5}{4}\sqrt{x_j}$ for $j = 1, d+1$, we have $\{1, d+1\} \subset \mathcal{J}_1$. Now we denote by d_1 and d_2 the cardinalities of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. Given $t = (t_1, \dots, t_{d+1}) \in \mathbb{R}^{d+1}$, we shall write $\mathbf{t}_{\mathcal{J}_1} = (t_j)_{j \in \mathcal{J}_1} \in \mathbb{R}^{d_1}$ and $\mathbf{t}_{\mathcal{J}_2} = (t_j)_{j \in \mathcal{J}_2} \in \mathbb{R}^{d_2}$. Using (4.4) and Fubini's theorem, we obtain

$$\begin{aligned}
&\sqrt{x_1 x_{d+1}} \frac{\partial}{\partial x_1} D_{n,\kappa}^{(\alpha,\alpha)}(x,y) \\
&\le cn \int_{[-1,1]^{d_2}} \left| \int_{[-1,1]^{d_1}} P_{2n-1}^{(\alpha+1,\alpha+1)}(u) \left[\sqrt{x_{d+1} y_1} t_1 - \sqrt{x_1 y_{d+1}} t_{d+1}\right] \right. \\
&\quad \left. \times \prod_{j \in \mathcal{J}_1} (1 - t_j^2)^{\kappa_j - 1} \, d\mathbf{t}_{\mathcal{J}_1} \right| \prod_{j \in \mathcal{J}_2} (1 - t_j^2)^{\kappa_j - 1} \, d\mathbf{t}_{\mathcal{J}_2} \\
&\le cn \sum_{\varepsilon \in \{\pm 1\}^{d_1}} \int_{[-1,1]^{d_2}} I_\varepsilon(\mathbf{t}_{\mathcal{J}_2}) \prod_{j \in \mathcal{J}_2} (1 - t_j^2)^{\kappa_j - 1} \, d\mathbf{t}_{\mathcal{J}_2},
\end{aligned}$$

where for $\varepsilon = (\varepsilon_j)_{j \in \mathcal{J}_1} \in \{\pm 1\}^{d_1}$,

$$\begin{aligned}
I_\varepsilon(\mathbf{t}_{\mathcal{J}_2}) = &\left| \int_{[-1,1]^{d_1}} P_{2n-1}^{(\alpha+1,\alpha+1)}(u) \left[\sqrt{x_{d+1} y_1} t_1 - \sqrt{x_1 y_{d+1}} t_{d+1}\right] \right. \\
&\quad \left. \times \prod_{j \in \mathcal{J}_1} \eta_{\varepsilon_j}(t_j) (1 - t_j^2)^{\kappa_j - 1} \, d\mathbf{t}_{\mathcal{J}_1} \right|.
\end{aligned}$$

Thus, for the proof of (4.5), it is sufficient to show that for any $\varepsilon = (\varepsilon_j)_{j \in \mathcal{J}_1} \in \{\pm 1\}^{d_1}$,

$$I_\varepsilon(\mathbf{t}_{\mathcal{J}_2}) \le c n^{\alpha - 2|\kappa|} H(x,y) \left(1 + n\rho(x,y)\right)^{-\alpha - \frac{1}{2} + |\kappa|}. \tag{4.6}$$

We first prove (4.6) for the case of $\varepsilon_1 = \varepsilon_{d+1}$. In this case, we have

$$I_\varepsilon(\mathbf{t}_{\mathcal{J}_2}) \le I_{\varepsilon,1}(\mathbf{t}_{\mathcal{J}_2}) + I_{\varepsilon,2}(\mathbf{t}_{\mathcal{J}_2}) + I_{\varepsilon,3}(\mathbf{t}_{\mathcal{J}_2}),$$

where

$$\begin{aligned}
I_{\varepsilon,1}(\mathbf{t}_{\mathcal{J}_2}) = \sqrt{x_{d+1} y_1} &\left| \int_{[-1,1]^{d_1}} P_{2n-1}^{(\alpha+1,\alpha+1)}(u) (t_1 - \varepsilon_1) \right. \\
&\quad \left. \times \prod_{j \in \mathcal{J}_1} \eta_{\varepsilon_j}(t_j) (1 - t_j^2)^{\kappa_j - 1} \, d\mathbf{t}_{\mathcal{J}_1} \right|,
\end{aligned}$$

$$I_{\varepsilon,2}(\mathbf{t}_{\mathcal{J}_2}) = \sqrt{x_1 y_{d+1}} \left| \int_{[-1,1]^{d_1}} P_{2n-1}^{(\alpha+1,\alpha+1)}(u)(t_{d+1} - \varepsilon_{d+1}) \right.$$
$$\left. \times \prod_{j\in\mathcal{J}_1} \eta_{\varepsilon_j}(t_j)\big(1-t_j^2\big)^{\kappa_j-1}\, d\mathbf{t}_{\mathcal{J}_1} \right|,$$
$$I_{\varepsilon,3}(\mathbf{t}_{\mathcal{J}_2}) = \left|\sqrt{x_{d+1}y_1} - \sqrt{x_1 y_{d+1}}\right|$$
$$\times \left| \int_{[-1,1]^{d_1}} P_{2n-1}^{(\alpha+1,\alpha+1)}(u) \prod_{j\in\mathcal{J}_1} \eta_{\varepsilon_j}(t_j)\big(1-t_j^2\big)^{\kappa_j-1}\, d\mathbf{t}_{\mathcal{J}_1} \right|.$$

Now using Lemma 4.2 and the fact that $\sqrt{x_1} \ge \frac{1}{n}$, we obtain

$$I_{\varepsilon,1}(\mathbf{t}_{\mathcal{J}_2}) \le cn^{\alpha-2|\kappa|_{\mathcal{J}_1}-1}\sqrt{x_{d+1}y_1}\bigg(\prod_{j\in\mathcal{J}_1}\big(\sqrt{x_j y_j}\big)^{-\kappa_j}\bigg)$$
$$\times \big(\sqrt{x_1 y_1}\big)^{-1}\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|_{\mathcal{J}_1}}$$
$$\le cn^{\alpha-2|\kappa|} H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|},$$

where $|\kappa|_{\mathcal{J}_1} = \sum_{j\in\mathcal{J}_1}\kappa_j$. Similarly, using Lemma 4.2 and the inequalities $\sqrt{x_{d+1}} \ge \frac{1}{n}$ and $|\sqrt{x_{d+1}y_1} - \sqrt{x_1 y_{d+1}}| \le c\rho(x,y)$, we can prove

$$I_{\varepsilon,i}(\mathbf{t}_{\mathcal{J}_2}) \le cn^{\alpha-2|\kappa|} H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac{1}{2}+|\kappa|}, \quad i=2,3.$$

This proves (4.6) for the case of $\varepsilon_1 = \varepsilon_{d+1}$.

Next, we show (4.6) for the case $\varepsilon_1 = -\varepsilon_{d+1}$. Note that for $t_j \in [-1,1]$ with $j \in \mathcal{J}_2$,

$$1 - \left|\sum_{j\in\mathcal{J}_1}\varepsilon_j\sqrt{x_j y_j} - \sum_{j\in\mathcal{J}_2} t_j\sqrt{x_j y_j}\right|$$
$$\ge 1 - \sum_{j=1}^{d+1}\sqrt{x_j y_j} + 2\min\big\{\sqrt{x_1 y_1}, \sqrt{x_{d+1}y_{d+1}}\big\}$$
$$\ge c\min\{x_1, x_{d+1}\}, \tag{4.7}$$

where we have used the assumption $\varepsilon_1 = -\varepsilon_{d+1}$ in the first step, and the facts that $\sqrt{x_1} \sim \sqrt{y_1}$, $\sqrt{x_{d+1}} \sim \sqrt{y_{d+1}}$ and $\sum_{j=1}^{d+1}\sqrt{x_j y_j} \le 1$ in the second step. Without loss of generality, we may assume $x_1 \le x_{d+1}$. Then using Lemma 4.2, we obtain

$$I_\varepsilon(\mathbf{t}_{\mathcal{J}_2}) \le c\big[\sqrt{x_{d+1}y_1} + \sqrt{x_1 y_{d+1}}\big] n^{\alpha+1-2|\kappa|_{\mathcal{J}_1}}\bigg(\prod_{j\in\mathcal{J}_1}\big(\sqrt{x_j y_j}\big)^{-\kappa_j}\bigg)$$
$$\times \big(1+n\sqrt{x_1}\big)^{-\alpha-\frac{3}{2}+|\kappa|_{\mathcal{J}_1}}$$

$$\leq c\sqrt{x_1}\big(n\sqrt{x_1}\big)^{-1}n^{\alpha+1-2|\kappa|_{\mathcal{J}_1}}\Bigg(\prod_{j\in\mathcal{J}_1}\big(\sqrt{x_jy_j}\big)^{-\kappa_j}\Bigg)\big(1+n\rho(x,y)\big)^{-\alpha-\frac12+|\kappa|_{\mathcal{J}_1}}$$

$$\leq cn^{\alpha-2|\kappa|}H(x,y)\big(1+n\rho(x,y)\big)^{-\alpha-\frac12+|\kappa|},$$

where we have used Lemma 4.2 and (4.7) in the first step, and the fact that $\sqrt{x_1}\sim\sqrt{y_1}\geq c\rho(x,y)$ in the second step. This proves the desired inequality (4.6), and hence completes the proof of Lemma 4.4. □

Now we are in a position to prove Theorem 3.3.

Proof of Theorem 3.3 Let

$$K_n(t):=\sum_{k=0}^{\infty}\phi\left(\frac{k}{2^n}\right)\big(k(k+\lambda_\kappa)\big)^{-\frac12}\frac{(2k+\lambda_\kappa)\Gamma(\frac12)\Gamma(k+\lambda_k)}{\Gamma(\lambda_\kappa+1)\Gamma(k+\frac12)}P_k^{(\lambda_k-\frac12,-\frac12)}(t), \quad (4.8)$$

where $\lambda_\kappa=|\kappa|+\frac{d-1}{2}$. Following the idea in the proof of Lemma 3.3 of [3, pp. 413–414], we define a sequence $\{a_{n,j}(\cdot)\}_{j=0}^{\infty}$ of C^∞-functions on $[0,\infty)$ by

$$a_{n,0}(u)=(2u+\lambda_\kappa)\phi\left(\frac{u}{2^n}\right)\big(u(u+\lambda_\kappa)\big)^{-\frac12},$$

$$a_{n,j+1}(u)=\frac{a_{n,j}(u)}{2u+\lambda_\kappa+j}-\frac{a_{n,j}(u+1)}{2u+\lambda_\kappa+j+2}, \qquad j\geq 0.$$

On one hand, since supp $\phi\subset[\frac18,8]$, by definition it's easily seen that for $j\geq 1$

$$\text{supp } a_{n,j}(\cdot)\subset\big[2^{n-3}-j,\ 2^{n+3}\big], \quad (4.9)$$

$$\left|\left(\frac{d}{du}\right)^m a_{n,j}(u)\right|\leq C_{m,j}2^{-n(m+2j)}, \qquad m=0,1,\ldots. \quad (4.10)$$

On the other hand, however, since for any integer $j\geq 0$ (see [16, (4.5.3), p. 71]),

$$\sum_{m=0}^{k}\frac{(2m+\alpha+\beta+j+1)\Gamma(m+\alpha+\beta+j+1)}{\Gamma(m+\beta+1)}P_m^{(\alpha+j,\beta)}(t)$$

$$=\frac{\Gamma(k+\alpha+\beta+j+2)}{\Gamma(k+\beta+1)}P_k^{(\alpha+j+1,\beta)}(t),$$

it follows by summation by parts finite times that

$$K_n(t)=C_\kappa\sum_{k=0}^{\infty}a_{n,j}(k)\frac{\Gamma(k+\lambda_\kappa+j)}{\Gamma(k+\frac12)}P_k^{(\lambda_\kappa-\frac12+j,-\frac12)}(t). \quad (4.11)$$

Note that by (2.1), (3.11) and (4.8),

$$\Phi_{n,\kappa}(x,y) = \int_{[-1,1]^{d+1}} K_n\big(2u^2-1\big) \prod_{j=1}^{d+1} \big(1-t_j^2\big)^{\kappa_j-1}\,dt.$$

Thus, using (4.11) and the quadratic formula

$$P_k^{(\alpha,-\frac{1}{2})}\big(2s^2-1\big) = O(1) P_{2k}^{(\alpha,\alpha)}(s),$$

we obtain

$$\begin{aligned}\Phi_{n,\kappa}(x,y) &= C_\kappa \sum_{k=0}^{\infty} a_{n,j}(k) \frac{\Gamma(k+\lambda_\kappa+j)}{\Gamma(k+\frac{1}{2})} \int_{[-1,1]^{d+1}} P_k^{(\lambda_\kappa-\frac{1}{2}+j,-\frac{1}{2})}\big(2u^2-1\big) \\ &\quad \times \prod_{j=1}^{d+1} \big(1-t_j^2\big)^{\kappa_j-1}\,dt \\ &= \sum_{k=0}^{\infty} a_{n,j}(k) O\big((k+1)^{\lambda_\kappa+j-\frac{1}{2}}\big) D_{k,\kappa}^{(\lambda_\kappa-\frac{1}{2}+j,\lambda_\kappa-\frac{1}{2}+j)}(x,y),\end{aligned}$$

where $D_k^{(\alpha,\alpha)}$ is as defined in Lemma 4.4. This implies

$$\begin{aligned}&\left|\sqrt{x_i x_{d+1}} \frac{\partial}{\partial x_i} \Phi_{n,\kappa}(x,y)\right| \\ &\quad \le c \sum_{k\sim 2^n} (k+1)^{\lambda_\kappa-j-\frac{1}{2}} \big|A_{x_i} D_{k,\kappa}^{(\lambda_\kappa-\frac{1}{2}+j,\lambda_\kappa-\frac{1}{2}+j)}(x,y)\big| \\ &\quad \le c\big(1+2^n\rho(x,y)\big)^{|\kappa|+\frac{d+1}{2}-j} \frac{1}{V(x,y)+V_{2^{-n}}(x)+V_{2^{-n}}(y)},\end{aligned}$$

where we have used (4.10) with $m=0$ in the first step, and Lemma 4.4 in the last step. The desired estimate then follows since the integer j can be chosen arbitrarily large. □

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