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CONSTRUCTIVE APPROXIMATION

Strong Convergence of Spherical Harmonic Expansions on $H^1(\mathbf{S}^{d-1})$

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Abstract. Let σ_k^δ denote the Cesàro means of order $\delta > -1$ of the spherical harmonic expansions on the unit sphere $\mathbf{S}^{d-1}$, and let $E_j(f, H^1)$ denote the best approximation of f in the Hardy space $H^1(\mathbf{S}^{d-1})$ by spherical polynomials of degree at most j. It is known that $\lambda := (d-2)/2$ is the critical index for the summability of the Cesàro means on $H^1(\mathbf{S}^{d-1})$. The main result of this paper states that, for $f \in H^1(\mathbf{S}^{d-1})$,

$$\sum_{j=0}^{N} \frac{1}{j+1} \|\sigma_j^\lambda(f) - f\|_{H^1} \approx \sum_{j=0}^{N} \frac{1}{j+1} E_j(f, H^1),$$

where "$\approx$" means that the ratio of both sides lies between two positive constants independent of f and N.

1. Introduction

In this Introduction we shall describe the main results and their background with a "minimum" of definitions. We shall give necessary details and appropriate definitions, as needed, in the following sections.

Let $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d : |x| = 1\}$ be the unit sphere in d-dimensional Euclidean space $\mathbf{R}^d$ equipped with the usual Lebesgue measure $d\sigma(x)$ normalized by $\int_{\mathbf{S}^{d-1}} d\sigma(x) = 1$, and let $H^p(\mathbf{S}^{d-1})$, $0 < p \le 1$, denote the Hardy spaces on $\mathbf{S}^{d-1}$. For an integer $n \ge 0$, let $\mathbf{P}_n$ denote the space of all spherical polynomials of degree at most n (i.e., the space of all polynomials in d-variables restricted to $\mathbf{S}^{d-1}$). By $E_n(f, H^p)$ we denote the best approximation of f in $H^p(\mathbf{S}^{d-1})$ by spherical polynomials of degree $\le n$:

$$E_n(f, H^p) := \inf\{\|f - g\|_{H^p} : g \in \mathbf{P}_n\}.$$

For any distribution f on $\mathbf{S}^{d-1}$ we associate its expansion in spherical harmonics:

$$f \sim \sum_{k=0}^{\infty} Y_k(f),$$

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where $Y_k(f)$ is the orthogonal projection of f on the space of spherical harmonics of degree k. Given $\delta > -1$, the Cesàro means of order δ of f are defined by

$$\sigma_N^\delta(f)(x) = \frac{1}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta Y_k(f)(x), \qquad N = 0, 1, 2, \ldots,$$

where

$$A_k^\delta = \binom{k+\delta}{k}, \qquad k = 0, 1, 2, \ldots.$$

The special value $\lambda := (d-2)/2$ is known as the critical index for the summability of σ_k^δ on $H^1(\mathbf{S}^{d-1})$. Indeed, it was proved in [7] that, for $\delta > \lambda$,

$$\sup_k \|\sigma_k^\delta\|_{(H^1, H^1)} < \infty,$$

whereas

$$\|\sigma_k^\lambda\|_{(H^1, L^1)} \geq c_d \log(k+1).$$

Throughout this paper, we will keep $\lambda := (d-2)/2$.

The purpose of this paper is to show the strong summability of the Cesàro mean σ_k^λ on $H^1(\mathbf{S}^{d-1})$. The background for this problem is as follows. In 1983, Smith [11] proved that, for every $f \in H^1(\mathbf{T})$,

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}\|S_k(f)\|_{H^1(\mathbf{T})} \leq C\|f\|_{H^1(\mathbf{T})},$$

where $\mathbf{T} = \mathbf{S}^1$ denotes the unit circle and $S_k(f)$ denotes the usual kth partial sum of Fourier series. A new proof of this inequality was given by Belinskii [1] in 1996. In the multidimensional case, this inequality was generalized by the authors in [10] in 1990 for $f \in H^p(\mathbf{T}^d)$, $0 < p < 1$, and by K. Y. Wang and the current author in a recent paper [9] for $f \in H^p(\mathbf{S}^d)$, $0 < p < 1$, for the summability at the critical index. However, its multidimensional generalization for the space H^1 seems to be much more complicated. Indeed, the two-dimensional result for rectangle partial sums with bounded ratio of sides was obtained in [15], while its multidimensional result for the cubic partial sums and modified H^1 was obtained by Belinskii in [2].

It was Bochner [3] who first pointed out that when the dimension $d > 1$, summability at the "critical index" $(d-1)/2$ was the correct analogue of the convergence, for phenomena near L^1. (For the unit sphere $\mathbf{S}^{d-1}$, the dimension is $d-1$ and the critical index is $(d-2)/2$.) In this sense, versions of many of the results for S_k are known for σ_k^λ on the multidimensional sphere $\mathbf{S}^{d-1}$. (See [4], [7] and [14].)

In this paper, we shall prove

Theorem 1. *For $f \in H^1(\mathbf{S}^{d-1})$, we have*

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\lambda(f)\|_{H^1}}{k} \leq C_d\|f\|_{H^1}.$$

As a consequence, we have

Corollary 2. *For $f \in H^1(\mathbf{S}^{d-1})$,*

$$\sum_{j=0}^{N} \frac{1}{j+1} \|\sigma_j^\lambda(f) - f\|_{H^1} \approx \sum_{j=0}^{N} \frac{1}{j+1} E_j(f, H^1),$$

with the constants of equivalence being independent of f and N.

We remark that in the case $d = 2$, Corollary 2 for the partial sums of Fourier series on $\mathbf{S}^1$ ($= \mathbf{T}$) is due to Belinskii [1].

It should be pointed out that although we only consider the situation for $\mathbf{S}^{d-1}$ here our method works equally well for the Bockner–Riesz means with critical index $(d-1)/2$ on $\mathbf{T}^d$. (The proof in this case will appear elsewhere.)

The paper is organized as follows. In Section 2, we give some definitions and describe the atomic characterization of the Hardy spaces. In Section 3, a new characterization of the Hardy spaces in terms of the maximal Cesàro operators and some of its useful corollaries are given. After that, in Section 4, we prove the main results, Theorem 1 and Corollary 2. In Section 5, we state some analogous results for the spaces $H^p(\mathbf{S}^{d-1})$, $0 < p < 1$. In Section 6, the final section, the main results are extended for the generalized Riesz operators.

2. Hardy Spaces $H^p(\mathbf{S}^{d-1})$, $0 < p \le 1$, on $\mathbf{S}^{d-1}$

The main purpose in this section is to give some definitions and describe the atomic characterization of the Hardy spaces $H^p(\mathbf{S}^{d-1})$, $0 < p \le 1$. Most material below can be found in [7] and [6].

Let $\mathcal{S} \equiv \mathcal{S}(\mathbf{S}^{d-1})$ denote the set of indefinitely differentiable functions on $\mathbf{S}^{d-1}$ endowed with the usual test function topology and let $\mathcal{S}' \equiv \mathcal{S}'(\mathbf{S}^{d-1})$ be the dual of $\mathcal{S}$. $\mathcal{S}$ is called the space of test functions and $\mathcal{S}'$ the space of distributions. (One may think of a function on $\mathbf{S}^{d-1}$ as a function defined on the annulus about $\mathbf{S}^{d-1}$ by extending the function to be constant along rays through the origin. This allows us to associate with

$$\gamma = (\gamma_1, \ldots, \gamma_d), \qquad D^\gamma = \left(\frac{\partial}{\partial x_1}\right)^{\gamma_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\gamma_d}, \qquad |\gamma| = \gamma_1 + \cdots + \gamma_d,$$

a differential operator of order $|\gamma|$ by differentiating in $\mathbf{R}^d$ and restricting to $\mathbf{S}^{d-1}$. The topology on $\mathcal{S}$ is that induced by the seminorms

$$N_m(\varphi) = \sum_{|\gamma|=m} \|D^\gamma \varphi\|_\infty, \qquad m = 0, 1, 2, \ldots).$$

The pairing of $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ is given by $\langle f, \varphi \rangle$. If f is an integrable function on $\mathbf{S}^{d-1}$, we set $\langle f, \varphi \rangle = \int_{\mathbf{S}^{d-1}} f(u)\varphi(u)\, d\sigma(u)$.

For $x \in \mathbf{S}^{d-1}$ and $z \in B_d := \{(z_1, \ldots, z_d) \in \mathbf{R}^d : z_1^2 + \cdots + z_d^2 \le 1\}$, let

$$P_z(x) = c_d \frac{1 - |z|^2}{|z - x|^d}.$$

P_z belongs to $\mathcal{S}$ and is called the Poisson kernel, c_d is chosen so that $\int_{\mathbf{S}^{d-1}} P_z(x)\,d\sigma(x) = 1$ for all $z \in B_d$. For $f \in \mathcal{S}'$, we call the function

$$F(z) = \langle f, P_z\rangle, \qquad z \in B_d,$$

the Poisson integral of f.

For a distribution f we define the *radial maximal function*, $P^+ f(x)$,

$$P^+ f(x) = \sup_{0\le r<1} |\langle f, P_{rx}\rangle|, \qquad x \in \mathbf{S}^{d-1}.$$

Definition 2.1 ([6]). The Hardy space $H^p(\mathbf{S}^{d-1})$ is the linear space of distributions f with $\|P^+ f\|_p < \infty$. We set $\|f\|_{H^p} = \|P^+ f\|_p$.

It is well known that if $p > 1$, $\|P^+ f\|_p$ is equivalent to $\|f\|_p$. Thus, $H^p(\mathbf{S}^{d-1})$ coincides with $L^p(\mathbf{S}^{d-1})$ if $p > 1$. For the remainder of this paper we assume $0 < p \le 1$.

We now turn to the "atomic" characterization of Hardy spaces. For $x \in \mathbf{S}^{d-1}$ and $r \in (0, \pi)$, by $B(x, r)$ we denote the spherical cap

$$B(x, r) := \{y \in \mathbf{S}^{d-1} : 0 \le \arccos xy \le r\}.$$

Definition 2.2 ([6]). A regular p-atom, $0 < p \le 1$, centered at $x \in \mathbf{S}^{d-1}$, is a function $a \in L^\infty(\mathbf{S}^{d-1})$ satisfying:

(i) supp $a \subset B(x, s)$ for some $s > 0$;
(ii) $\|a\|_\infty \le s^{-(d-1)/p}$; and
(iii) $\int_{\mathbf{S}^{d-1}} a(u)Y(u)\,d\sigma(u) = 0$, for every spherical harmonic of degree less than or equal to $[(d-1)(1/p-1)]$.

An *exceptional atom* is a function $a \in L^\infty(\mathbf{S}^{d-1})$ with $\|a\|_\infty \le 1$.

Theorem A ([6]). *Let $0 < p \le 1$. If $\{a_j\}_{j=0}^\infty$ is a sequence of exceptional or regular p-atoms, and $\{c_j\}_{j=0}^\infty$ is a sequence of complex numbers with*

$$\left(\sum_{j=0}^\infty |c_j|^p\right)^{1/p} < \infty,$$

then $\sum_{j=0}^\infty c_j a_j$ converges in H^p and

$$\left\|\sum_j c_j a_j\right\|_{H^p} \le A\left(\sum_j |c_j|^p\right)^{1/p},$$

where $A > 0$, depends on p and d.

Conversely, if $f \in H^p(\mathbf{S}^{d-1})$ there exists a sequence $\{a_j\}$ of exceptional or p-atoms, and a sequence $\{c_j\}$ of complex numbers such that

$$f = \sum_j c_j a_j \qquad \text{and} \qquad \left(\sum_j |c_j|^p\right)^{1/p} \le B\|f\|_{H^p},$$

where B depends on p and d.

The conclusion of Theorem A is often described as the "atomic" characterization of Hardy spaces.

3. A New Characterization of $H^p(\mathbf{S}^{d-1})$ and its Corollaries

In this section we will give a new characterization of $H^p(\mathbf{S}^{d-1})$, which is in terms of maximal Cesàro operators. We will also deduce some corollaries from this characterization, which will be used in the proof of Theorem 1.

First, we describe some necessary notations. Let $\{\mu_k\}$ be a sequence of complex numbers. Given a nonnegative integer ℓ, we define $\triangle^\ell \mu_k$ by

$$\triangle^0 \mu_k = \mu_k, \quad \triangle \mu_k = \mu_k - \mu_{k+1}, \quad \triangle^{i+1}\mu_k = \triangle(\triangle^i \mu_k), \quad i = 1, \ldots, \ell - 1,$$

and define $\overleftarrow{\triangle}^\ell \mu_k$ by

$$\overleftarrow{\triangle}^\ell \mu_k = (-1)^\ell \triangle^\ell \mu_k.$$

Given $f \in \mathcal{S}'$, its maximal Cesàro mean $\sigma_*^\delta(f)$ of order $\delta > -1$ is

$$\sigma_*^\delta(f) = \sup_k |\sigma_k^\delta(f)|.$$

In this section, we shall prove

Theorem 3. *Suppose* $0 < p \le 1$, $\delta > \delta(p) := (d-1)/p - d/2$, *and* f *is a distribution on* $\mathbf{S}^{d-1}$. *Then* $f \in H^p(\mathbf{S}^{d-1})$ *if and only if* $\sigma_*^\delta(f) \in L^p(\mathbf{S}^{d-1})$. *Furthermore,*

$$\|f\|_{H^p(\mathbf{S}^{d-1})} \approx \|\sigma_*^\delta(f)\|_{L^p(\mathbf{S}^{d-1})},$$

with the constants of equivalence being independent of f.

It is known that for $0 < p \le 1$ the special value $\delta(p) := (d-1)/p - d/2$ of δ in the above theorem is critical for the uniform summability of σ_k^δ on H^p in the sense that

$$\sup_k \|\sigma_k^\delta\|_{(H^p, H^p)} < \infty$$

whenever $\delta > \delta(p)$, whereas

$$\|\sigma_k^{\delta(p)}\|_{(H^p, L^p)} \ge C_p (\log(k+1))^{1/p}.$$

See [7].

We shall prove a second result as well.

Theorem 4. *For* $\delta > 0$, $\ell = [\delta] + 1$, *and* $x \in \mathbf{S}^{d-1}$, *we have*

$$\sigma_*^{\ell+4}(\sigma_L^\delta(f))(x) \le C_\delta(\sigma_*^\ell(f)(x) + |\sigma_L^\delta(f)(x)|). \tag{3.1}$$

Combining these last two theorems, we obtain the following corollary, which will play an important role in the proof of Theorem 1.

Corollary 5. *For* $0 < p \leq 1$, $f \in H^p$, *and* $\delta = \delta(p) := (d-1)/p - d/2$, *we have*

$$\|\sigma_k^\delta(f)\|_{H^p} \leq C_p(\|f\|_{H^p} + \|\sigma_k^\delta(f)\|_{L^p}).$$

Proof of Theorem 3. First, we assume $\sigma_*^\delta(f) \in L^p(\mathbf{S}^{d-1})$ and will prove

$$\|f\|_{H^p} = \|P^+(f)\|_{L^p} \leq \|\sigma_*^\delta(f)\|_{L^p}. \tag{3.2}$$

We note that by our assumption, $\sigma_*^\delta(f)(x) < \infty$ for a.e. $x \in \mathbf{S}^{d-1}$, and hence, for each $k \in \mathbf{Z}_+$ and a.e. $x \in \mathbf{S}^{d-1}$,

$$(k+1)^{-1-\delta}|Y_k(f)(x)| \leq C_\delta \sigma_*^\delta(f)(x) < \infty.$$

The last "$\leq$" holds since

$$Y_k(f)(x) = \overleftarrow{\triangle}^{[\delta]+2}[A_k^{[\delta]+1}\sigma_k^{[\delta]+1}(f)(x)]$$

and since

$$\sigma_*^{[\delta]+1}(f)(x) \leq \sigma_*^\delta(f)(x).$$

Thus, for every $r \in (0,1)$ and a.e. $x \in \mathbf{S}^{d-1}$,

$$\sum_{k=0}^\infty r^k |Y_k(f)(x)| < \infty.$$

Since

$$(1-r)^{-1-\delta} = \sum_{k=0}^\infty A_k^\delta r^k, \tag{3.3}$$

it follows that, for a.e. $x \in \mathbf{S}^{d-1}$ and every $r \in (0,1)$,

$$\begin{aligned}(1-r)^{-1-\delta}\sum_{k=0}^\infty r^k Y_k(f)(x) &= \left(\sum_{k=0}^\infty A_k^\delta r^k\right)\left(\sum_{k=0}^\infty r^k Y_k(f)(x)\right)\\ &= \sum_{k=0}^\infty A_k^\delta r^k \sigma_k^\delta(f)(x).\end{aligned}$$

So,

$$P_r(f)(x) = \sum_{k=0}^\infty r^k Y_k(f)(x) = (1-r)^{1+\delta}\sum_{k=0}^\infty A_k^\delta r^k \sigma_k^\delta(f)(x).$$

This combined with (3.3) yields

$$P^+(f)(x) \leq \sigma_*^\delta(f)(x), \quad \text{a.e. } x \in \mathbf{S}^{d-1}$$

and hence (3.2).

The proof of the inverse part of the theorem is essentially contained in [7]. In fact, by the proof of Lemma 4.2 of [7], it follows that for an H^p-atom supported in $B(y,r)$, we have

$$\sigma_*^\delta(a)(x) \le \begin{cases} C_{p,\delta} r^{-(d-1)/p+d/2+\delta}|x-y|^{-(d/2+\delta)}, & 0<|x-y|\le \pi/2, \\ C_{p,\delta} r^{-(d-1)/p+d/2+\delta}|x+y|^{-(d/2+\delta)}, & 0<|x+y|\le \pi/2, \\ C_{p,\delta} r^{-(d-1)/p}, & x\in \mathbf{S}^{d-1}, \end{cases}$$

which implies

$$\begin{aligned} \|\sigma_*^\delta(a)\|_{L^p}^p \ &\le \ C_{p,\delta}\Bigg(r^{-(d-1)} \int_{[0,r]\cup[\pi-r,\pi]} \sin^{d-2}\theta\, d\theta + r^{-(d-1)+p(d/2+\delta)} \\ &\qquad\qquad \int_r^{\pi-r} (\sin\theta)^{-p(d/2+\delta)+d-2}\, d\theta\Bigg) \\ &\le \ C_{p,\delta}. \end{aligned}$$

The inverse inequality

$$\|\sigma_*^\delta(f)\|_{L^p} \le C_{p,\delta}\|f\|_{H^p}$$

then follows by the atomic decomposition theorem. ∎

Proof of Theorem 4. Let $N, L \in \mathbf{Z}_+$. We need to estimate $\sigma_N^{\ell+4}(\sigma_L^\delta(f))(x)$. Without loss of generality, we may assume $N, L \ge 4\ell$. We consider the following two cases:

Case 1. $0\le N\le L$.
In this case, we set

$$\mu_k = \begin{cases} \dfrac{A_{N-k}^{\ell+4} A_{L-k}^{\delta}}{A_N^{\ell+4} A_L^{\delta}}, & \text{if } 0\le k\le N, \\ 0, & \text{if } k\ge N+1. \end{cases}$$

Then, clearly,

$$\sigma_N^{\ell+4}(\sigma_L^\delta(f)) = \sum_{k=0}^{N} \mu_k Y_k(f),$$

and straightforward computation shows that, for $0\le k\le N$,

$$|\triangle^{\ell+1}\mu_k| \le \begin{cases} C_\delta \dfrac{1}{N^{\ell+1}}, & \text{if } 0\le N\le L/2, \\ C_\delta \dfrac{(L-k+1)^{\delta+3}}{L^{\delta+\ell+4}}, & \text{if } L/2\le N\le L. \end{cases}$$

So, using Abel's transform $\ell+1$ times yields

$$|\sigma_N^{\ell+4}\sigma_L^\delta(f)(x)| \le C_\delta \sigma_*^\ell(f)(x). \tag{3.4}$$

Case 2. $N \geq L+1$.
In this case, we let

$$a_k = \begin{cases} \dfrac{A_{N-k}^{\ell+4}}{A_N^{\ell+4}} - \dfrac{A_{N-L}^{\ell+4}}{A_N^{\ell+4}}, & \text{if } 0 \leq k \leq N, \\ 0, & \text{if } k \geq N+1, \end{cases}$$

$$b_k = \begin{cases} \dfrac{A_{L-k}^{\delta}}{A_L^{\delta}}, & \text{if } 0 \leq k \leq L, \\ 0 & \text{if } k \geq L+1. \end{cases}$$

Then we have

$$\sigma_N^{\ell+4}(\sigma_L^\delta(f))(x) = \sum_{k=0}^{L} a_k b_k Y_k(f)(x) + \frac{A_{N-L}^{\ell+4}}{A_N^{\ell+4}} \sigma_L^\delta(f)(x). \tag{3.5}$$

For $\delta > 0$, it is easy to verify the following estimates:

$$|a_k| \leq C_\delta \frac{|L-k+1|}{N}, \tag{3.6}$$

$$|\triangle^i a_k| \leq C_\delta \left(\frac{1}{N}\right)^i, \qquad i = 1, \dots, \ell+1, \tag{3.7}$$

$$|\triangle^i b_k| \leq C_\delta \frac{(L-k+1)^{\delta-i}}{(L+1)^\delta}, \qquad i = 0, 1, \dots, \ell+1. \tag{3.8}$$

We also note that if $\delta > 0$ is an integer, then

$$|\triangle^\ell b_k| = |\triangle^{1+\delta} b_k| = \begin{cases} O(1/L^\delta), & \text{if } L-\ell \leq k \leq L, \\ 0, & \text{if } 0 \leq k \leq L-\ell-1, \end{cases} \tag{3.9}$$

and

$$|\triangle^{\ell+1} b_k| = |\triangle^{\delta+2} b_k| = \begin{cases} O(1/L^\delta), & \text{if } L-\ell \leq k \leq L, \\ 0, & \text{if } 0 \leq k \leq L-\ell-1. \end{cases} \tag{3.10}$$

If $\delta > 0$ is not an integer, then using (3.6)–(3.8), we have, for $0 \leq k \leq L$,

$$|\triangle^{\ell+1}(a_k b_k)| \leq C_\delta \frac{(L-k+1)^{\delta-[\delta]-1}}{(L+1)^{\delta+1}};$$

if $\delta > 0$ is an integer, then using (3.6)–(3.10), we obtain

$$|\triangle^{\ell+1}(a_k b_k)| \leq \begin{cases} C_\delta L^{-\ell-1}, & \text{if } 0 \leq k \leq L-2\ell, \\ C_\delta L^{-\ell}, & \text{if } L-2\ell < k \leq L. \end{cases}$$

Hence, in either case, we have

$$\sum_{k=0}^{L} |\triangle^{\ell+1}(a_k b_k)| k^\ell \leq C_\delta.$$

The inequality

$$|\sigma_N^{\ell+4}(\sigma_L^{\delta}(f))(x)| \leq C_\delta(\sigma_*^{\ell}(f)(x) + |\sigma_L^{\delta}(f)(x)|) \tag{3.11}$$

then follows from (3.5).

Now a combination of (3.4) and (3.11) gives (3.1) and completes the proof. ■

Finally, we end this section with the following:

Corollary 6. *Let $\{\mu_k\}_{k=0}^{\infty}$ be a sequence of complex numbers, $0 < p \leq 1$, $\delta(p) := (d-1)/p - d/2$ and $\ell = [\delta(p)] + 1$. Suppose the following conditions are satisfied*:

(i) $\sup_k |\mu_k| \leq M < \infty$;

(ii) $\sum_{k=0}^{\infty} |\triangle^{\ell+1}\mu_k|(k+1)^{\ell} \leq M$.

Then

$$\left\| \sum_{k=0}^{\infty} \mu_k Y_k(f) \right\|_{H^p} \leq CM\|f\|_{H^p},$$

where $C > 0$ is independent of M, $\{\mu_k\}$, and f.

This corollary is probably well known. However, we would like to give an alternative proof here using Theorem 3.

Proof. Let

$$T(f) := \sum_{k=0}^{\infty} \mu_k Y_k(f).$$

Then by Theorem 3, it suffices to prove

$$\sigma_*^{\ell+2}(Tf) \leq CM\sigma_*^{\ell}(f). \tag{3.12}$$

Applying Abel's transform $\ell + 1$ times gives

$$\sigma_N^{\ell+2}(Tf) = \sum_{k=0}^{N} \triangle^{\ell+1}\left(\frac{A_{N-k}^{\ell+2}}{A_N^{\ell+2}}\mu_k\right) A_k^{\ell}\sigma_k^{\ell}(f)(x), \tag{3.13}$$

where we define $A_j^{\ell+2} = 0$ for $j < 0$. On the other hand, according to conditions (i) and (ii), one can easily verify that, for all $v = 0, 1, \ldots, \ell$,

$$\sum_{k=0}^{\infty} |\triangle^{v+1}\mu_k| k^v \leq CM.$$

Thus

$$\begin{aligned} \sum_{k=0}^{N} \left| \triangle^{\ell+1}\left(\frac{A_{N-k}^{\ell+2}}{A_N^{\ell+2}}\mu_k\right)\right| A_k^{\ell} \;&\leq\; C \sum_{v=0}^{\ell+1}\sum_{k=0}^{N} \left| \triangle^{\ell+1-v}\left(\frac{A_{N-k-v}^{\ell+2}}{A_N^{\ell+2}}\triangle^{v}\mu_k\right)\right| (k+1)^{\ell} \\ &\leq\; CM. \end{aligned} \tag{3.14}$$

Now combining (3.13) with (3.15), we get (3.12) and complete the proof. ■

4. Proofs of the Main Results

We begin with the proof of Theorem 1.

Proof of Theorem 1. By Corollary 5, we have

$$\|\sigma_k^\lambda(f)\|_{H^1} \le C_d(\|f\|_{H^1} + \|\sigma_k^\lambda(f)\|_{L^1}).$$

Therefore, it will suffice to prove

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\lambda(f)\|_{L^1}}{k} \le C_d\|f\|_{H^1}. \tag{4.1}$$

For the proof of (4.1), we define

$$E_k^\lambda(f)(x) := \gamma_k\int_{\mathbf{S}^{d-1}} f(y)P_k^{(d-3/2,(d-3)/2)}(xy)\,d\sigma(y), \tag{4.2}$$

where $P_k^{(\alpha,\beta)}$ denotes the Jacobi polynomial as defined in [12],

$$\gamma_k = \frac{\Gamma(d/2)\Gamma(k+1)\Gamma(k+d-1)}{(4\pi)^{(d-1)/2}\Gamma\left(k+\frac{d}{2}\right)\Gamma\left(k+\frac{d-1}{2}\right)} \sim k^{1/2}. \tag{4.3}$$

The operator E_k^λ was introduced by Wang [13] in the investigation of the pointwise convergence of the Cesàro operator σ_k^λ. It is known that (see [14, (3.1.10)] or [4, Lemma 2.3])

$$\sigma_k^\lambda(f)(x) = \beta_k E_k^\lambda(f)(x) + T_k^\lambda(f)(x),$$

where

$$\begin{aligned}
\beta_k &= \frac{\Gamma(k+\frac{3}{2}d-2)\Gamma(2k+\frac{3}{2}d-1)}{\Gamma(k+d-1)\Gamma(2k+2d-2)} = O(1),\\
T_k^\lambda(f) :&= \sum_{v=1}^{\infty} b(k,v)\sigma_k^{\lambda+v}(f),\\
|b(k,v)| &\le C_d v^{-(3/2)d}.
\end{aligned}$$

So the proof of (4.1) is reduced to the proof of

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|E_k^\lambda(a)\|_{L^1}}{k} \le C_d, \tag{4.4}$$

where a is an H^1-atom supported in $B(z,r)$ with $0 < r < 0.1$.

To prove (4.4), we need the following:

Lemma 4.1. *With the same notations as above, we have*

$$\sum_{k=1}^{N}\left[\int_{0\le xz\le\cos 9r}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\right]^2\le C_d r^{-1}\log^2\frac{1}{r}, \tag{4.5}$$

and

$$\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\le C_d kr\log\frac{1}{r}. \tag{4.6}$$

For the moment, we take this lemma for granted and proceed with the proof. Since

$$\begin{aligned}\int_{xz\ge\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x) &\le C_d r^{(d-1)/2}\left(\int_{B(z,9r)}|E_k^{\lambda}(a)|^2\,d\sigma(x)\right)^{1/2}\\ &\le C_d r^{(d-1)/2}\|a\|_2\le C_d\end{aligned}$$

and

$$\int_{xz\le 0}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\le C_d,$$

it suffices to show

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\le C_d. \tag{4.7}$$

To prove (4.7), we consider the following two cases:

Case 1. $r^{-1}\le N$.
In this case, we have, by (4.6),

$$\sum_{k=1}^{[r^{-1}]}\frac{1}{k}\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\le C_d\sum_{k=1}^{[r^{-1}]}r\log\frac{1}{r}\le C_d\log\frac{1}{r}\le C_d\log N,$$

and by (4.5),

$$\begin{aligned}&\sum_{k=[r^{-1}]+1}^{N}\frac{1}{k}\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\\ &\qquad\le C_d r^{1/2}\left(\sum_{k=[r^{-1}]+1}^{N}\left|\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\right|^2\right)^{1/2}\\ &\qquad\le C_d\log\frac{1}{r}\le C_d\log N.\end{aligned}$$

Case 2. $N<r^{-1}$.
In this case, we have, by (4.6),

$$\sum_{k=1}^{N}\frac{1}{k}\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\le C_d\sum_{k=1}^{N}r\log\frac{1}{r}\le C_d Nr\log\frac{1}{r}\le C_d\log N.$$

The last inequality follows since the function $\log x/x$ is decreasing over (e, ∞).

So, in both cases, we prove (4.4) and hence (4.1). ■

Now the proof of Theorem 1 is reduced to the proof of Lemma 4.1. To prove this lemma, we define, for $\theta \in \mathbf{R}$, the average operator S_θ by

$$S_\theta(f)(x) := \int_{\{y\in\mathbf{S}^{d-1}:x\cdot y=0\}} f(x\cos\theta + y\sin\theta)\,d\gamma(y), \qquad x \in \mathbf{S}^{d-1},$$

with $d\gamma$ being the Lebesgue measure on $\{y \in \mathbf{S}^{d-1} : x \cdot y = 0\}$ normalized by

$$\gamma\{y \in \mathbf{S}^{d-1} : x \cdot y = 0\} = 1.$$

We need the following:

Lemma 4.2. *Let a be an H^1-atom supported in $B(z, r)$ for some $z \in \mathbf{S}^{d-1}$ and $r \in (0, 0.1)$. Let $x \in \mathbf{S}^{d-1}$ such that $9r \le t := \arccos xz \le \pi/2$. For $\theta \in (0, \pi)$, put*

$$g_x(\theta) = S_\theta(a)(x)\sin^{d-2}\theta.$$

Then we have:

(i) $\operatorname{supp} g_x(\cdot) \subset [t - r, t + r]$.
(ii) $\int_0^\pi g_x(\theta)\,d\theta = 0$.
(iii) $|g_x(\theta)| \le C_d r^{-1}$.

Proof. Parts (i) and (ii) are obvious. To prove (iii), we write

$$\Sigma_{x,\theta} := \{y \in \mathbf{S}^{d-1} : xy = \cos\theta\},$$

and by $d\gamma_{x,\theta}$ we denote the usual Lebesgue measure on $\Sigma_{x,\theta}$ normalized by $\gamma_{x,\theta}(\Sigma_{x,\theta}) = \sin^{d-2}\theta$. We first note that, for $\theta \in [t - r, t + r]$,

$$\gamma_{x,\theta}(\Sigma_{x,\theta} \cap B(z, r)) \le C_d r^{d-2}. \tag{4.8}$$

For the moment, we take this for granted and proceed with the proof. By the definition, we have

$$S_\theta(a)(x) := \frac{1}{\sin^{d-2}\theta} \int_{\Sigma_{x,\theta}\cap B(z,r)} a(y)\,d\gamma_{x,\theta}(y),$$

which, by (4.8), implies

$$|g_x(\theta)| \le C_d\theta^{-(d-2)} r^{-(d-1)} r^{d-2}\theta^{d-2} \le C_d r^{-1}.$$

So, it remains to prove (4.8). Let $u \in \Sigma_{x,\theta} \cap B(z, r)$. Suppose $u = x\cos\theta + \xi\sin\theta$, $z = x\cos t + \xi_1 \sin t$, with $\xi, \xi_1 \in \Sigma_{x,\pi/2}$. Then, for $\theta \in [t - r, t + r]$,

$$|u - z|^2 = 4\sin^2\frac{t-\theta}{2} + 2\sin\theta\sin(t)(1 - \xi_1\xi) \le 4\sin^2\frac{r}{2}.$$

So,

$$1 - \xi\xi_1 \leq \frac{4\sin^2 r/2}{2\sin\theta \sin t} \leq \frac{\pi^2}{4}\frac{r^2}{\theta^2}$$

and

$$\Sigma_{x,\theta} \cap B(z,r) \subset \left\{ x\cos\theta + \xi\sin\theta : 1 - \xi_1\xi \leq \frac{\pi^2}{4}\frac{r^2}{\theta^2} \right\}.$$

Equation (4.8) then follows since

$$\gamma_{x,\theta}\left\{ x\cos\theta + \xi\sin\theta : 1 - \xi_1\xi \leq \frac{\pi^2}{4}\frac{r^2}{\theta^2} \right\} \sim (\sin^{d-2}\theta)\left(\frac{r}{\theta}\right)^{d-2} \sim r^{d-2}.$$

This completes the proof. ■

Proof of Lemma 4.1. We first prove (4.5). Let

$$\varphi_k(\theta) = c_k P_k^{(d-3/2,(d-3)/2)}(\cos\theta)\sin^{d-1}\frac{\theta}{2}\cos^{(d-2)/2}\frac{\theta}{2},$$

where

$$\begin{aligned} c_k &= \left(\int_0^\pi |P_k^{(d-3/2,(d-3)/2)}(\cos\theta)|^2 \sin^{2d-2}\frac{\theta}{2}\cos^{d-2}\frac{\theta}{2}\,d\theta \right)^{-1/2} \\ &= \left(\frac{(2k+\frac{3}{2}d-2)\Gamma(k+1)\Gamma(k+\frac{3}{2}d-2)}{\Gamma(k+d-\frac{1}{2})\Gamma(k+\frac{d-1}{2})} \right)^{1/2}. \end{aligned}$$

Then $\{\varphi_k\}_{k=0}^\infty$ forms a complete orthonormal system over $(0,\pi)$, and by (4.2) and (4.3),

$$E_k^\lambda(a)(x) = \beta_k \int_0^\pi \frac{S_\theta(a)(x)}{\sin\theta/2}\cos^\lambda\theta/2\varphi_k(\theta)\,d\theta, \tag{4.9}$$

where

$$\beta_k = \left(\frac{(2k+\frac{3}{2}d-2)\Gamma(k+\frac{3}{2}d-2)\Gamma\left(k+\dfrac{d-1}{2}\right)}{\Gamma(k+d-\frac{1}{2})\Gamma(k+1)} \right)^{1/2} \frac{(4\pi)^{(d-1)/2}\Gamma\left(k+\dfrac{d}{2}\right)}{\Gamma\left(\dfrac{d}{2}\right)\Gamma(k+d-1)} \leq C_d.$$

It follows that

$$\sum_{k=1}^N |E_k^\lambda(a)(x)|^2 \leq C_d \int_0^\pi \frac{|S_\theta(a)(x)|^2}{\sin^2\theta/2}\cos^{2\lambda}\frac{\theta}{2}\,d\theta \leq C_d r^{-1}(\arccos xz)^{-2d+2}, \tag{4.10}$$

where the last inequality is a consequence of Lemma 4.2. Using (4.10) and Hölder's inequality, we deduce

$$\begin{aligned}
&\sum_{k=1}^{N}\left|\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x)\right|^2\\
&\quad\le\sum_{k=1}^{N}\left(\int_{0\le xz\le\cos(9r)}(\arccos xz)^{-(d-1)}\,d\sigma(x)\right)\\
&\quad\times\left(\int_{0\le xz\le\cos(9r)}(\arccos xz)^{d-1}|E_k^{\lambda}(a)(x)|^2\,d\sigma(x)\right)\\
&\quad\le C_d\log\frac{1}{r}\int_{0\le xz\le\cos(9r)}\left((\arccos xz)^{d-1}\sum_{k=1}^{N}|E_k^{\lambda}(a)(x)|^2\right)d\sigma(x)\\
&\quad\le C_d r^{-1}\log\frac{1}{r}\int_{9r}^{\pi/2}t^{d-1}t^{-2d+2}t^{d-2}\,dt\le C_d r^{-1}\left(\log\frac{1}{r}\right)^2,
\end{aligned}$$

which gives (4.5).

Finally, we prove (4.6). For simplicity, we write $t=\arccos xz$. Then, by Lemma 4.2, we have, for $9r\le t\le\pi/2$,

$$\begin{aligned}
E_k^{\lambda}(a)(x) &= O(1)k^{1/2}\int_{t-r}^{t+r}(S_\theta(a)(x)\sin^{d-2}\theta)P_k^{(d-3/2,(d-3)/2)}(\cos\theta)\,d\theta\\
&= O(1)k^{1/2}\int_{t-r}^{t+r}g_x(\theta)(P_k^{(d-3/2}, (d-3)/2)(\cos\theta)\\
&\qquad\qquad - P_k^{(d-3/2,(d-3)/2)}(\cos(t+r)))\,d\theta,
\end{aligned}$$

where $g_x(\theta)=S_\theta(a)(x)\sin^{d-2}\theta$.

Since, for $0<\theta\le\pi/2$,

$$\left|\frac{d}{du}P_k^{(d-3/2,(d-3)/2)}(u)\Big|_{u=\cos\theta}\right|=\tfrac{1}{2}(k+\tfrac{3}{2}d-2)|P_{k-1}^{(d-1/2,(d-1)/2)}(\cos\theta)|\le C_d(k+1)^{1/2}\theta^{-d},$$

it follows by Lemma 4.2(iii) that

$$|E_k^{\lambda}(a)(x)|\le C_d krt^{-d+1}\int_{t-r}^{t+r}|g_x(\theta)|\,d\theta\le C_d krt^{-d+1}.$$

Hence

$$\begin{aligned}
\int_{0\le xz\le\cos(9r)}|E_k^{\lambda}(a)(x)|\,d\sigma(x) &\le C_d kr\int_{9r}^{\pi/2}t^{d-2}t^{-d+1}\,dt\\
&\le C_d kr\log\frac{1}{r},
\end{aligned}$$

which gives (4.6) and completes the proof. ■

This completes the proof of Theorem 1. ■

Now we turn to the proof of Corollary 2. We need two lemmas.

Given $r > 0$, we define the *rth-order derivative* $f^{(r)}$, in a distributional sense, by

$$f^{(r)} \sim \sum_{k=1}^{\infty} (k(k+d-2))^{r/2} Y_k(f).$$

Lemma 4.3 (Bernstein's Inequality). *For $0 < p \le 1$, $r > 0$, and every spherical polynomial T_N of degree less than or equal to N,*

$$\|T_N^{(r)}\|_{H^p} \le C N^r \|T_N\|_{H^p},$$

where $C > 0$ is independent of N and T_N.

Lemma 4.4. *Suppose $r > 0$ and η is a C^∞-function on $\mathbf{R}$ with the properties that $\eta(x) = 1$ for $0 \le |x| \le 1$ and $\eta(x) = 0$ for $|x| > 2$. For $t > 0$, define*

$$V_t(f) := \sum_{k=0}^{\infty} \eta(tk) Y_k(f).$$

Then, for $f \in H^p(\mathbf{S}^{d-1})$, $0 < p \le 1$,

$$\sup_{t>0} \|V_t(f)\|_{H^p} \le C \|f\|_{H^p}.$$

Lemma 4.3 can be obtained by standard methods (see the proof of Theorem 3.2 in [8]), while Lemma 4.4 is a simple consequence of Corollary 6. We omit the details.

Proof of Corollary 2. The lower estimate is obvious. To prove the upper estimate, we suppose $2^{2^m} \le N < 2^{2^{m+1}}$ and without loss of generality, we may assume $\int_{\mathbf{S}^{d-1}} f(x)\, d\sigma(x) = 0$. Let η and V_t be as defined in Lemma 4.4. For simplicity, we set

$$g_j = V_{2^{-2^{j-2}}}(f), \qquad j \ge 2.$$

Then we have

$$\begin{aligned}
\sum_{j=20}^{N} \frac{1}{j} \|f - \sigma_j^\lambda(f)\|_{H^1} \;\le\; & \sum_{j=3}^{m+1} \sum_{k=2^{2^{j-1}}+1}^{2^{2^j}} \frac{1}{k} \|f - g_j\|_{H^1} \\
& + \sum_{j=3}^{m+1} \sum_{k=2^{2^{j-1}}+1}^{2^{2^j}} \frac{1}{k} \|\sigma_k^\lambda(f - g_j)\|_{H^1} \\
& + \sum_{j=3}^{m+1} \sum_{k=2^{2^{j-1}}+1}^{2^{2^j}} \frac{1}{k} \|\sigma_k^\lambda(g_j) - g_j\|_{H^1} \\
=:\; & I + J + L.
\end{aligned}$$

For the first sum, we have

$$\begin{aligned} I &\le C\sum_{j=3}^{m+1}\sum_{k=2^{2^{j-1}}+1}^{2^{2^j}}\frac{1}{k}E_{2^{2^{j-2}}}(f,H^1)\\ &\le C\sum_{j=3}^{m+1}\sum_{k=2^{2^{j-3}}+1}^{2^{2^{j-2}}}\frac{1}{k}E_k(f,H^1)\\ &\le C\sum_{j=1}^{N}\frac{1}{j}E_j(f,H^1). \end{aligned}$$

For the second sum, using Theorem 1, we have

$$\begin{aligned} J &\le C\sum_{j=3}^{m+1}2^j\|f-g_j\|_{H^1}\\ &\le C\sum_{j=3}^{m+1}\sum_{k=2^{2^{j-3}}+1}^{2^{2^{j-2}}}\frac{1}{k}E_{2^{2^{j-2}}}(f,H^1)\\ &\le C\sum_{j=1}^{N}\frac{1}{j}E_j(f,H^1). \end{aligned}$$

To estimate the third sum, we first claim that, for $2^{2^{j-1}}+1\le k\le 2^{2^j}$,

$$\|\sigma_k^\lambda(g_j)-g_j\|_{H^1}\le Ck^{-1}\|g_j'\|_{H^1}. \tag{4.11}$$

For the moment we take this last inequality for granted and proceed with the proof. Using Bernstein's inequality (Lemma 4.3), we deduce that, for $2^{2^{j-1}}+1\le k\le 2^{2^j}$,

$$\begin{aligned} k^{-1}\|g_j'\|_{H^1} &= k^{-1}\|(V_{2^{-2^{j-2}}}(f))'\|_{H^1}\\ &\le Ck^{-1}\sum_{n=0}^{2^{j-2}}2^n\|V_{2^{-(n-1)}}(f)-V_{2^{-n}}(f)\|_{H^1}\\ &\le Ck^{-1}\sum_{n=0}^{2^{j-2}}2^nE_{2^{n-1}}(f,H^1), \end{aligned}$$

where $E_{2^{-1}}(f,H^1)=E_0(f,H^1)$. This combined with (4.11) gives

$$\begin{aligned} L &\le C\sum_{j=3}^{m+1}\sum_{k=2^{2^{j-1}}+1}^{2^{2^j}}\frac{1}{k^2}\sum_{n=0}^{2^{j-2}}2^nE_{2^{n-1}}(f,H^1)\\ &\le C\sum_{n=0}^{2^{m-1}}E_{2^{n-1}}(f,H^1)\\ &\le C\sum_{n=2}^{2^{m-1}}\sum_{k=2^{n-2}+1}^{2^{n-1}}\frac{E_k(f,H^1)}{k}+CE_0(f,H^1)\\ &\le C\sum_{j=0}^{N}\frac{E_j(f,H^1)}{j+1}. \end{aligned}$$

Finally, noticing that, for $0 \le j \le 20$,

$$\|f - \sigma_k^\lambda(f)\|_{H^1} \le C\|f\|_{H^1} \le C' E_0(f, H^1),$$

we obtain the desired upper estimate.

Now it remains to prove (4.11). To this end, let $\xi \in C^\infty(\mathbf{R})$ such that $\xi(x) = 1$ for $0 \le |x| \le \frac{1}{2}$ and $\xi(x) = 0$ for $|x| \ge \frac{3}{4}$. Since, for $j \ge 3$ and $2^{2^{j-1}} + 1 \le k \le 2^{2^j}$,

$$g_j = V_{2^{-2^{j-2}}}(f) \in \mathbf{P}_{2\cdot 2^{2^{j-2}}} \subset \mathbf{P}_{[k/2]}$$

it follows that

$$\lim_{u\to\infty} \|\sigma_u^\lambda(g_j) - g_j\|_{C(\mathbf{S}^{d-1})} = 0,$$

and hence

$$\sigma_k^\lambda(g_j) - g_j = \sum_{u=k}^{\infty}(\sigma_u^\lambda(g_j) - \sigma_{u+1}^\lambda(g_j)) \tag{4.12}$$

$$= -\lambda \sum_{u=k}^{\infty} \frac{1}{(u+1+\lambda)(u+1)} \sum_{v=0}^{[\frac{3}{4}k]} \frac{A_{u-v}^\lambda}{A_u^\lambda} \frac{(u+1)v}{u-v+1} \xi\left(\frac{v}{k}\right) Y_v(g_j).$$

For simplicity, put

$$a_v = \begin{cases} \xi\left(\dfrac{v}{k}\right) \dfrac{A_{u-v}^\lambda}{A_u^\lambda} \dfrac{u+1}{u+1-v} \dfrac{v}{(v(v+2\lambda))^{\frac{1}{2}}}, & \text{if } 1 \le v \le \frac{3}{4}k, \\ 0, & \text{if } v > \frac{3}{4}k. \end{cases}$$

Then a straightforward computation shows that, for $0 \le v \le \frac{3}{4}k \le \frac{3}{4}u$,

$$|\triangle^\ell a_v| \le C(k^{-\ell} + (v+1)^{-\ell-1}), \qquad \ell = 0, \ldots, d+1,$$

where $\triangle^\ell$ is as defined in Section 3. So, by Corollary 6, it follows that, for $u \ge k$,

$$\left\| \sum_{v=0}^{[\frac{3}{4}k]} \frac{A_{u-v}^\lambda}{A_u^\lambda} \frac{(u+1)v}{u-v+1} \xi\left(\frac{v}{k}\right) Y_v(g_j) \right\|_{H^1} = \left\| \sum_{v=0}^{[\frac{3}{4}k]} a_v Y_v(g_j') \right\|_{H^1} \le C\|g_j'\|_{H^1},$$

which combined with (4.12) gives (4.11).

This completes the proof of Corollary 2. ∎

5. Strong Approximation in $H^p(\mathbf{S}^{d-1})$, $0 < p < 1$

In this section, we shall state the results for $H^p(\mathbf{S}^{d-1})$, $0 < p < 1$. It turns out that the proofs in this case are much simpler.

Theorem 7. *For* $0 < p < 1$, $\delta = \delta(p) := (d-1)/p - d/2$, *and* $f \in H^p(\mathbf{S}^{d-1})$, *we have*

$$\frac{1}{\log N} \sum_{k=1}^{N} \frac{\|\sigma_k^\delta(f)\|_{H^p}^p}{k} \le C\|f\|_{H^p}^p.$$

Proof. By Corollary 5 and the atomic decomposition theorem, it suffices to prove that, for a p-atom a,

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\delta(a)\|_{L^p}^p}{k}\le C_p.$$

The proof of this last inequality is contained in [5] and is essentially a consequence of the following estimates of $\sigma_k^\delta(a)$, which were obtained in the proof of Lemma 4.2 of [7]:

$$|\sigma_k^\delta(a)(x)|\le C\min\{r^{-(d-1)/p},(kr)^{s-(d-1)(1/p-1)}(\sin\theta)^{-(d-1)/p}\},$$

where a is a p-atom supported in a spherical cap $B(y,r)$, $\theta=\arccos xy$ and $s=0$ or $[(d-1)(1/p-1)]+1$. ■

As a consequence of Theorem 7, we have

Corollary 8. *For $0<p<1$, $\delta=\delta(p):=(d-1)/p-d/2$, and $f\in H^p(\mathbf{S}^{d-1})$, we have*

$$\sum_{j=0}^{N}\frac{1}{j+1}\|\sigma_j^\delta(f)-f\|_{H^p}^p\approx\sum_{j=0}^{N}\frac{1}{j+1}E_j^p(f,H^p),$$

with the constants of equivalence being independent of f and N.

The proof of this last corollary is almost identical to that of Corollary 2. We omit the details.

6. Concluding Remarks

Remark 6.1. Though we prove the results only for Cesàro means in the preceding sections, the same method works equally well for generalized Riesz means.

For $\delta>-1$ and $\alpha>0$, the *generalized Riesz mean* $R_k^{\delta,\alpha}$ is defined by

$$R_k^{\delta,\alpha}(f)(x):=\sum_{j=0}^{k}\left(1-\left(\frac{j}{k+1}\right)^\alpha\right)^\delta Y_j(f)(x).$$

Given $r>0$, we define the *rth-order K-functional* on $H^p(\mathbf{S}^{d-1})$, $0<p\le1$, by

$$K_r(f,t)_{H^p}:=\inf\{\|f-g\|_{H^p}+t^r\|g^{(r)}\|_{H^p}:g,g^{(r)}\in H^p\},\qquad t>0.$$

We have the following results:

Theorem 9. *For $\alpha>0$, $0<p\le1$, $\delta=\delta(p):=(d-1)/p-d/2$ and $f\in H^p(\mathbf{S}^{d-1})$, we have*

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|R_k^{\delta,\alpha}(f)\|_{H^p}^p}{k}\le C_p\|f\|_{H^p}^p.$$

Corollary 10. *For* $\alpha > 0, 0 < p \le 1$, $\delta = \delta(p) := (d-1)/p - d/2$, *and* $f \in H^p(\mathbf{S}^{d-1})$, *we have*

$$\sum_{j=0}^{N} \frac{1}{j+1} \|R_j^{\delta,\alpha}(f) - f\|_{H^p}^p \approx \sum_{j=0}^{N} \frac{1}{j+1} E_j^p(f, H^p).$$

Corollary 11. *For* $\alpha > 0, 0 < p \le 1$, $\delta = \delta(p) := (d-1)/p - d/2$, *and* $f \in H^p(\mathbf{S}^{d-1})$, *we have*

$$\frac{1}{\log N} \sum_{k=1}^{N} \frac{\|R_k^{\delta,\alpha}(f) - f\|_{H^p}^p}{k} \le C K_1^p\left(f, \left(\frac{1}{\log N}\right)^{1/p}\right)_{H^p}.$$

We point out that this last corollary is a simple consequence of Corollary 10 and some standard realization results on K-functionals (see [9]).

Remark 6.2. Let X be a compact rank one symmetric space (besides the sphere $\mathbf{S}^d$, these spaces are: the real projective space $P^d(R)$; the complex projective space $P^d(C)$; the quaternionic projective space $P^d(H)$; and the Cayley projective plane $P^{16}(Cayley)$). To each distribution f on X we can associate a spherical harmonic expansion $\sum_{k=0}^{\infty} f_k$, i.e., the expansion of f in the series of eigenvectors of the Laplace–Beltrami operator of X. If $x \in X$ and $0 \le r < 1$, the Poisson integral of f is defined by

$$f_r(x) = \sum_{k=0}^{\infty} r^k f_k(x).$$

Let

$$P^+ f(x) := \sup_{0 \le r < 1} |f_r(x)|.$$

The Hardy space $H^p(X)$, $0 < p \le 1$, is then defined by the condition

$$\|f\|_{H^p} = \left(\int_X |P^+ f(x)|^p \, dx\right)^{1/p} < \infty.$$

Everything has been proved in the preceding sections for the Hardy spaces on $\mathbf{S}^d$ can be extended to these Hardy spaces. Most of the proofs go through with hardly any change.

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