The Ski Jumping Problem
(Source: C.ODE.E, Spring 1993)

When a ski jumper leaves the ramp of a ski jump and becomes airborne, the dominant forces that determine the success of the jump are the force due to gravity and the air resistance. The latter can be resolved into a lift force and a drag force. The lift and drag forces can be altered by the jumper’s position over the skis and by the position of the skis relative to the ground. This combination of lift and drag can make the difference between a “good” jump and a short or even dangerous jump.

In this problem, we construct a model that describes a jumper’s path, and examine the dependence of the path on model parameters.
The coordinate system will be as follows:

The origin of the coordinate system is placed at the lip of the ramp where the jumper becomes airborne. Assume that the jumper’s path lies in the $xy$-plane, and that local winds are negligible.

$t$: time, measured in seconds

$x(t)$: horizontal position (ft) at time $t$

$y(t)$: vertical position (ft) at time $t$

$m$: mass of the jumper (including skis and boots)
Position vector: \((x(t), y(t))\)
Velocity vector: \((x'(t), y'(t))\)
Acceleration vector: \((x''(t), y''(t))\)

The force due to gravity is

\[ \vec{F}(t) = (0, -mg) \]

The lift force is perpendicular to the flight path of the jumper, with magnitude proportional to the jumper’s speed:

\[ \vec{L}(t) = \lambda(-y'(t), x'(t)) \]

The drag force opposes the direction of motion, and is proportional to the velocity:

\[ \vec{D}(t) = -\delta(x'(t), y'(t)) \]

Newton’s law of motion leads to the following model equations:

\[
\begin{align*}
mx''(t) &= -\delta x'(t) - \lambda y'(t) \\
my''(t) &= \lambda x'(t) - \delta y'(t) - mg
\end{align*}
\]
Rewrite

\[ mx''(t) = -\delta x'(t) - \lambda y'(t) \]
\[ my''(t) = \lambda x'(t) - \delta y'(t) - mg \]

as

\[ x''(t) = -\frac{\delta}{m} x'(t) - \frac{\lambda}{m} y'(t) = -dx'(t) - ly'(t) \]
\[ y''(t) = \frac{\lambda}{m} x'(t) - \frac{\delta}{m} y'(t) - g = lx'(t) - dy'(t) - g \]

Integrate once to give

\[ x' = -dx - ly + C_1 \]
\[ y' = lx - dy - gt + C_2 \]

Determine \( C_1 \) and \( C_2 \) from the initial conditions, which are \( x(0) = y(0) = 0, x'(0) = V_0, \) and \( y'(0) = 0, \) where \( V_0 \) is the jumper’s velocity upon leaving the ramp.

Obtain

\[ x' = -dx - ly + V_0 \]
\[ y' = lx - dy - gt \]
Suggested numerical experiments

- Assume $V_0 = 78$ ft/s. Compute the jumper’s flight path with $d = 0.02$ and $l = 0.024$. Also with $d = 0.02$, $l = 0.015$, and with $d = 0.018$, $l = 0.05$. Note that even slight changes in $d$ and $l$ can improve the jump distance by a few feet.

- Determine the effect of changing the jumper’s initial velocity (heavier jumpers usually get a better take-off velocity, but less lift – somewhere there is a jumper with optimum weight).

- The drag and lift parameters $d$ and $l$ are determined from the jumper’s weight, profile, clothing, cross-sectional area, and ski position, so that their values usually lie within a fairly small interval. To illustrate what can happen with “unrealistic” values, compute a solution with $V_0 = 78$, $d = 0$ and $l = 0.9$. 