TORSORS OVER THE PUNCTURED AFFINE LINE

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ABSTRACT. We provide a classification of G-torsors over the punctured affine line $\mathrm{Spec}\,(k[t^{\pm 1}])$ where G is a reductive algebraic group defined over a field k of good characteristic. Our classification is in terms of the Galois cohomology of the complete field $k((t))$ with values in G.
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1. Introduction

Let G be a reductive group over a field k. The main objective of our work is to give a description of the isomorphism classes of G-torsors over the punctured affine line $\mathbb{A}_k^\times = \mathrm{Spec}(k[t^{\pm 1}])$ in terms of the Galois cohomology of the complete field $k((t))$ with values in G. The relevance of this characterization is that the latter cohomology set $H^1(k((t)), G)$ can be computed using the methods developed by Bruhat and Tits [BT3].

Many interesting objects defined over a rational function field $k(t)$ can usually be described, or at least be partially understood, through their residues. Examples of this type of behaviour are the famous exact sequences related to Milnor K-theory, Brauer groups and Witt groups. The exact sequence for Milnor's K-groups [Mn, § 2] is given by

$$0 \to K_n^M(k) \to K_M^n(k(t)) \xrightarrow{\partial} \oplus_p \, K_{n-1}^M(k(p)) \to 0$$

where p runs over the closed points of the affine line. Similarly, if k is perfect, we have Faddeev's exact sequence [GS, § 6.4]

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(k(t)) \xrightarrow{\partial} \oplus_p \, H^1(k(p), \mathbb{Q}/\mathbb{Z}) \to 0$$

and the exact sequence for Witt groups

$$0 \to W(k) \to W(k(t)) \xrightarrow{\partial} \oplus_p \, W(k(p)) \to 0$$

in odd characteristic [Mn, § 5].

Brauer groups and Witt groups are closely related to projective linear groups and orthogonal groups respectively. One is thus led to consider the possibility that there may exist analogues of the last two of the above exact sequences for G-torsors over $k(t)$. To study this problem the (traditional) local-global approach leads us to look in detail at the natural map

$$a = \prod a_p \; : \; H^1(k(t), G) \longrightarrow \prod_{p \in \mathbb{P}^1_k} H^1(k(t)_p, G) \tag{1.1}$$

where p runs through the set of closed points of the projective line $\mathbb{P}^1_k$, $k(t)_p$ is the completion of $k(t)$ with respect to the discrete valuation v_p associated to p, and $a_p : H^1(k(t), G) \to H^1(k(t)_p, G)$ is the natural restriction map. Note that, unlike the case of Brauer groups and Witt groups, the set of isomorphism classes of G-torsors over $k(t)$ is not a group. It is therefore unrealistic to expect that a unique short exact sequence with $H^1(k(t), G)$ as a middle term could describe all G-torsors over $k(t)$. The only reasonable hope is to try to describe the fibers of the map a in more or less acceptable terms. This leads to infinitely many exact sequences instead of just one, as we now explain.

We say that an element x of $H^1(k(t), G)$ is *unramified at* p if $a_p(x)$ is in the image of the natural map $H^1(\widehat{O}_p, G) \to H^1(k(t)_p, G)$, where $\widehat{O}_p$ is the ring of integers of $k(t)_p$. Otherwise we say that x is *ramified at* p. In the latter case, and if G/k is quasi-split, using Bruhat-Tits theory we can associate to x an element x_p in $H^1(k(p),H_p)$ where $k(p)$ is the residue of $k(t)_p$, and H_p is a proper subgroup of the $k(p)$-group $G_{k(p)}$ which depends on x. It seems natural to call x_p *the residue of x at p* and call the finite set $S=\{\,p_1,\dots,p_n\}$ where x is ramified *the ramification locus* of x. By Bruhat-Tits theory $H^1(k(t)_p,G)$ is the disjoint union of "anisotropic parts" of $H^1(k(p),H_p)$, namely of those classes that arise from $[\xi]\in H^1(k(p),H_p)$ for which the twisted group ${}_\xi H_p$ is anisotropic. This indicates that we could get a satisfactory description of $H^1(k(t),G)$ in terms of sets $H^1(k(p),H_p)$ if we knew the fibers of the map

$$H^1(k(t),G)_S\longrightarrow\prod_{p\in S}H^1(k(t)_p,G)$$

where $H^1(k(t),G)_S=a^{-1}\Big(\prod_{p\in S}H^1(k(t)_p,G)\times\prod_{p\notin S}H^1(\widehat{O}_p,G)\Big)$.

Note that $H^1(k(t),G)_S$ is the image of the natural mapping

$$H^1(U,G)\to H^1(k(t),G) \tag{1.2}$$

where $U=\mathbb{P}^1_k\setminus\{\,p_1,\dots,p_n\,\}$ (see [H1]). Thus, out of necessity, we are forced to study the image of this map and its fibers.

Assume that characteristic of k is good (see § 5.1 for details). If $S=\emptyset$, then $H^1(k(t),G)_S=H^1(k,G)$ and the fibers of (1.2) are well understood [G1, § I.2]. If $S=\{\,\infty\,\}$ then by a theorem of Raghunathan–Ramanathan (see below) the natural maps $H^1(k,G)\to H^1(\mathbb{A}^1_k,G)$ is bijective and (1.2) is injective. In our paper we consider the next case, namely when S consists of two points $S=\{\,0,\infty\,\}$. It turns out that the map (1.2) is again injective, and we can describe its image in terms of local G-torsors.

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Our work is unequivocally motivated by that of Raghunathan and Ramanathan [RR], where the case of the affine line $\mathbb{A}^1_k=\operatorname{Spec}(k[t])$ is considered. Even though some overlap between the two works is at times evident (extending group schemes to $\mathbb{P}^1_k$, the use Weil restrictions to address representability questions...), there are also substantial differences, notably the use of key results from the theory of multiplicative group schemes developed in [SGA3], as well as techniques pertaining to fundamental domains in buildings due to Soulé (for the affine line) and to Abramenko (in the case of the punctured line). These methods allow us to establish–by different means than those used in [RR]–the existence of maximal tori on certain reductive group schemes. This is one of the crucial points within the main proof. Since our methods work equally well both for the affine and the punctured affine line, we have decided to include a (short) section with a proof of Raghunathan and Ramanathan's original result.

One should observe that the nature of G-torsors over $\mathbb{A}^1_k$ and $\mathbb{A}^\times_k$ are quite different, mostly owing to the fact that, unlike $\mathbb{A}^1_k$, the scheme $\mathbb{A}^\times_k$ is not simply connected (in the algebraic sense). In good characteristic for example, G-torsors over the affine line are always constant, but this need not be the case for G-torsors over the punctured line.

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The main body of the paper is divided into four parts. We begin by giving the statement of the two main theorems. This is followed by a section on "Preliminary results" that provides the conventions, notation and terminology that are used throughout the paper, as well as some general results, some of which are of independent interest, that are used in the proofs of the two main theorems. The proofs of the main theorems themselves are the content of the last two sections. An Appendix with a technical observation about absolutely reduced algebraic groups is included at the end.

2. Statement of the two Main Theorems

Let X be a scheme, and G a group scheme over X. For any scheme Y over X we denote the Y-group scheme $G \times_X Y$ by G_Y. Recall that a *torsor over X under G* is a scheme E over X equipped with a right action of G for which there exists a faithfully flat morphism $Y \to X$, locally of finite presentation, such that $E \times_X Y \simeq G \times_X Y = G_Y$, where G_Y acts on itself by right translation. The set of isomorphisms classes of X-torsors under G is denoted by $H^1(X, G)$.

Let G be a linear algebraic over a field k and X a k-scheme. The structural morphism $X \to \mathrm{Spec}\,(k)$ yields a natural map

$$\eta : H^1(k, G) \longrightarrow H^1(X, G) := H^1(X, G_X).$$

Let E be a torsor over X under G_X. We say that E is *constant* if its isomorphism class belong to the image of the map η, and that E is *geometrically separably trivial* if E becomes trivial after the base change $X \times_k k_s \to X$.

2.1. **Theorem.** [Raghunathan-Ramanathan] *Let G be a linear algebraic k-group whose connected component of the identity is reductive. Every torsor over the affine line $\mathbb{A}^1_k$ under G which is geometrically separably trivial is constant.*

2.2. **Remark.** In [RR] the group G is assumed to be connected and reductive. The generalization we give here is not difficult to obtain from the reductive case. See § 4.3 below.

2.3. **Remark.** According to the bijection (3.3) described in Remark 3.2, the isomorphism classes of geometrically separably trivial $\mathbb{A}^1_k$-torsors under G are classified by $H^1(\mathrm{Gal}(k_s/k), G(\mathbb{A}^1_{k_s}))$. The natural map $\eta : H^1(k, G) \longrightarrow H^1(\mathbb{A}^1_k, G)$ corresponds to the natural map

$$H^1(\mathrm{Gal}(k_s/k), G(k_s)) \longrightarrow H^1(\mathrm{Gal}(k_s/k), G(\mathbb{A}^1_{k_s})) \tag{2.4}$$

arising from the inclusion $G(k_s) \subset G(\mathbb{A}^1_{k_s})$. The map (2.4) is injective because of the existence of rational points on $\mathbb{A}^1_k$. Theorem 2.1 is thus equivalent to (2.4) being bijective. In good characteristic, it is known that every

G-torsor over $\mathbb{A}_k^1$ is geometrically separably trivial, hence constant. In bad characteristic, this is in general not true anymore, even for a semisimple simply connected group (see [G2, § 2.4] for details).

The second part of our work gives a description of torsors over the punctured line $\mathbb{A}_k^{\times} = \mathrm{Spec}(k[t^{\pm 1}])$. It is not true that in this case geometrically separably trivial torsors are constant, even when G is semisimple. The correct parametrization is obtained by looking at the base change corresponding to the completion of the generic fiber. The inclusion $k[t^{\pm 1}] \subset k((t))$ yields a natural map $H^1(\mathbb{A}_k^{\times}, G) \to H^1(k((t)), G)$. Our result shows that, under a certain assumption on the characteristic of the base field, this map is bijective.

2.5. **Theorem.** *Let G be a (connected) reductive algebraic group over k. Assume that the characteristic of k is good for G (see 5.1). Then the natural map $H^1(\mathbb{A}_k^{\times}, G) \to H^1(k((t)), G)$ is bijective.*

2.6. **Remark.** In other words, the Theorem states that there exists a natural bijection between the set of isomorphism classes of $\mathbb{A}_k^{\times}$-torsors under G and the usual Galois cohomology of G over the complete field $k((t))$. Note that the set $H^1(k((t)), G)$ is well understood from the work of Bruhat-Tits [BT3].

3. Preliminary results

This section contains the conventions, notation and terminology that are used throughout the paper. It also contains results that are common to the proofs of the main theorems.

3.1. **Notation and conventions.** Throughout k denotes a field, $\overline{k}$ an algebraic closure of k, and k_s the separable closure of k in $\overline{k}$. Given a scheme X over k and a field extension k'/k we set

$$X_{k'} := X \times_k k'.$$

For convenience we denote X_{k_s} simply by X_s.

Most of our work is related to group schemes over a given base scheme X. For convenience we will sometime refer to these simply as *X-groups.* If $X = \mathrm{Spec}(R)$ we use the terminology X-group and R-group indistinctively. We recall that given an X-group G and a scheme Y over X the Y-group $G \times_X Y$ is denoted by G_Y. That "Y is a scheme over X" is at times abbreviated by simply writing Y/X.

By a *reductive X-group* we will understand a reductive group scheme over X in the sense of [SGA3]. Accordingly, a reductive k-group is a connected reductive group defined over k in the sense of Borel [Bor]. If G is a reductive X-group, then the concepts of *maximal tori, parabolic subgroup, Levi subgroup...* of G are again the ones given by [SGA3].

Let F be a field extension of k. The additive and multiplicative groups over F will be denoted by $\mathrm{G}_{a,F}$ and $\mathrm{G}_{m,F}$ respectively, or simply by G_a and G_m when $F = k$. The separable closure of F, which will always be taken in some algebraic closure $\overline{F}$ of F that contains $\overline{k}$, will be denoted by $F_{\rm sep}$. Thus $k_{\rm sep} = k_s$, but in general one should not confuse $F_{\rm sep}$ with F_s (which by definition equals $F \otimes_k k_s$, and plays no role in our work).

Given a field extension F of k and a torus T over F, there is a natural action of the Galois group $\mathrm{Gal}(F_{\rm sep}/F)$ on the (abstract) group

$$\mathrm{Hom}_{F_{\rm sep}-\mathrm{grp}}(\mathrm{G}_{m,F_{\rm sep}}, T_{F_{\rm sep}}).$$

We call the resulting Galois module *the group of cocharacters of* T, and denote it by $X(T)_*$.

3.2. **Generalities on torsors.** Let X be a scheme, and G a group scheme over X. As it is customary, for any scheme Y over X we denote by p_i, $i = 1, 2$, the corresponding projection $Y \times_X Y \to Y$ on the i-th component and by p_{ij}, $i, j = 1, 2, 3$, the projection $Y \times_X Y \times_X Y \to Y \times_X Y$ on the ij-th component. These projections naturally induce group homomorphisms

$$G(Y) \to G(Y \times_X Y) \quad \text{and} \quad G(Y \times_X Y) \to G(Y \times_X Y \times_X Y)$$

which we still denote by p_i and p_{ij} (instead of the more usual p_i^* and p_{ij}^* notation).

Assume now that Y/X is an fppf cover of X, that is $Y \to X$ is a scheme morphism which is faithfully flat and locally of finite presentation [SGA3, IV.6.4]. (We remind the reader that an fppf cover is a covering morphism [morphism couvrant] for the fppf topology, but not conversely). For such a covering $Y \to X$, we define the corresponding set of cocycles

$$\check{Z}^1(Y/X, G) := \big\{ g \in G(Y \times_X Y) \mid p_{23}(g) p_{12}(g) = p_{13}(g) \big\}$$

and non-abelian cohomology

$$\check{H}^1(Y/X, G) := \check{Z}^1(Y/X, G)/G(Y),$$

where $G(Y)$ acts on $\check{Z}^1(Y/X, G)$ by $g \cdot z = p_2(g)\, z\, p_1(g)^{-1}$ (see [H1, § 1.3] and [M] for details). For convenience we will at times use the notation $\check{Z}^1(X, G)$ to denote the totality of cocycles $\check{Z}^1(Y/X, G)$ with Y as above variable (within a given range, which would always be clear from the context, so as to avoid set-theoretical problems). We now define

$$\check{H}^1(X, G) := \varinjlim_{Y} \check{H}^1(Y/X, G)\,,$$

where the limit is taken over all (equivalence classes of) fppf covers $Y \to X$.

3.1. **Remark.** One defines in an exact analogous fashion $\check{H}^1(X, G)$ whenever G is a sheaf of groups for the fppf topology on X.

Recall that the set of isomorphisms classes of X-torsors under G is denoted by $H^1(X, G)$. Thus $H^1(X, G)$ is a pointed set; its distinguished class, which we denote by 1, is the class of the trivial torsor, namely the scheme G acting on itself by right translation. The subset of $H^1(X, G)$ corresponding to torsors which are trivialized by a given (arbitrary) base change $X' \to X$ is denoted by $H^1(X'/X, G)$. If $X' \to X$ is an fppf cover, $H^1(X'/X, G)$ can be computed by means of cocycles (just as in Galois cohomology), so that $H^1(X'/X, G)$ can be identified with a subset of $\check{H}^1(X'/X, G)$. If G is affine and locally of finite presentation over X then $H^1(X'/X, G) = \check{H}^1(X'/X, G)$, (see [M, Ch. 3, § 4]) and the natural map

$$H^1(X, G) \to \check{H}^1(X, G)$$

is bijective. This will be the situation that we will consider in our paper, and we will indistinctively think of (the isomorphism class) of a torsor as an element of H^1 or the corresponding $\check{H}^1$. Along similar lines we write Z^1 instead of $\check{Z}^1$.

3.2. **Remark.** Assume that G is an algebraic group over k. Following standard practice, we will denote in what follows $H^1(X, G_X)$ simply by $H^1(X, G)$. Because G is of finite type, any X-torsor under G that becomes trivial over X_s already becomes trivial over $X_{k'}$ for some finite Galois extension $k' \subset k_s$ of k (the extension k' depends of course on the given torsor). As a consequence, the natural map

$$\varinjlim_{k'/k \text{ Galois}} H^1(X_{k'}/X, G) \to H^1(X_s/X, G)$$

is bijective. Since $X_{k'}$ is a Galois extension of X whose Galois group is naturally isomorphic to $\mathrm{Gal}(k'/k)$, we have

$$H^1(X_{k'}/X, G) \simeq H^1(\mathrm{Gal}(k'/k), G(X_{k'})).$$

We thus have a bijection

$$H^1(X_s/X, G) \simeq H^1(\mathrm{Gal}(k_s/k), G(X_s))), \tag{3.3}$$

where the H^1 on the right denotes the "usual" Galois cohomology of the profinite group $\mathrm{Gal}(k_s/k)$ acting (continuously) on the (discrete) module $G(X_s)$. The natural map $\eta : H^1(k, G) \longrightarrow H^1(X, G)$ corresponds to the composition of maps in the sequence

$$H^1(\Gamma, G(k_s)) \to H^1(\Gamma, G(X_s)) \simeq H^1(X_s/X, G) \subset H^1(X, G)$$

where $\Gamma = \mathrm{Gal}(k_s/k)$ and the first map arises from the inclusion $G(k_s) \subset G(X_s)$ obtained from the k_s-scheme structure of X_s.

3.3. **Twisting.** Throughout this section X will denote a k-scheme, and G a group scheme over X that *we assume is affine and locally of finite presentation over* X. Let $\mathbf{Aut}(G)$ be the X-group functor of automorphisms of G: For each Y/X

$$\mathbf{Aut}(\mathbf{G})(Y) = \mathrm{Aut}(G_Y).$$

The functor $\mathbf{Aut}(G)$ is always a sheaf of groups for the fppf topology on X, but it need not in general be representable (i.e. a group scheme).

Let $Y \to X$ be an fppf cover. To a cocycle $z \in \check{Z}^1(Y/X, \mathbf{Aut}(G)) \subset \mathrm{Aut}(G_{Y\times_X Y})$) one can associate a twisted group scheme ${}_zG$ over X whose functor of points is given by

$$(3.4) \qquad {}_zG(S) = \{x \in G(Y \times_X S) \mid z^S(p_1^S(x)) = p_2^S(x)\}$$

for any X-scheme S. The notation in (3.4) is as follows. The morphisms $p_i^S : Y \times_X Y \times_X S \to Y \times_X S$ are given by $p_i^S = p_i \times \mathrm{id}_S$. Given $x \in G(Y \times_X S) = \mathrm{Hom}_X(Y \times_X S, G)$ we denote by $p_i^S(x)$ the composition $x \circ p_i^S$. The $p_i^S(x)$ are thus elements of the abstract group $G(Y \times_X Y \times_X S)$. Finally, since z is an automorphism of the $Y \times_X Y$-group $G_{Y\times_X Y}$, it induces an automorphism z^S of the abstract group $G(Y \times_X Y \times_X S)$.

We can reinterpret this definition by saying that the sequence

$$(3.5) \qquad {}_zG(S) \subset G(Y \times_X S) \underset{p_2^S}{\overset{z^S \circ p_1^S}{\rightrightarrows}} G(Y \times_X Y \times_X S)$$

is exact. If $z = 1$ then the twisted group ${}_zG$ is isomorphic to G, and the sequence

$$(3.6) \qquad G(S) \subset G(Y \times_X S) \underset{p_2^S}{\overset{p_1^S}{\rightrightarrows}} G(Y \times_X Y \times_X S)$$

is exact. This allows us to identify $G(S)$ with a subgroup $G(Y \times_X S)$; an identification that we will henceforth use, whenever convenient, without further reference.

Since $z \in \check{Z}^1(Y/X, \mathbf{Aut}(G))$ the group schemes ${}_zG_Y$ and G_Y are isomorphic. This isomorphism can be made explicit at the level of functor of points as we now explain for future reference.

If $i, j, k \in \{1, 2, 3\}$ are different integers and $j < k$ we set

$$z_i = p_{jk}(z) \in \mathrm{Aut}(G_{Y\times_X Y\times_X Y}).$$

Given any scheme S over Y our automorphism z induces naturally an automorphism $z^{S/Y}$ of the (abstract) group $G(Y \times_X Y \times_Y S)$. Similarly the z_i induce automorphisms $z_i^{S/Y}$ of the (abstract) group $G(Y \times_X Y \times_X Y \times_Y S)$. By taking the composite map $S \to Y \to X$ we may also view S as a scheme over X. The canonical isomorphism $Y \times_Y S \simeq S$ yields canonical identifications

$$(3.7) \qquad \begin{gathered} G(S) \simeq G(Y \times_Y S) \\ G(Y \times_X S) \simeq G(Y \times_X Y \times_Y S), \\ G(Y \times_X Y \times_X S) \simeq G(Y \times_X Y \times_X Y \times_Y S). \end{gathered}$$

For $i = 1, 2$ we denote

$$p_{i3} \times \mathrm{id}_S : Y \times_X Y \times_X Y \times_Y S \to Y \times_X Y \times_Y S$$

by $p_{i3}^{S/Y}$. Consider the diagram

(3.8)

$$\begin{array}{ccccc}
{}_zG(S) \hookrightarrow G(Y \times_X S) \simeq G(Y \times_X Y \times_Y S) & \overset{z_3^{S/Y} \circ p_{13}^{S/Y}}{\underset{p_{23}^{S/Y}}{\rightrightarrows}} & G(Y \times_X Y \times_X Y \times_Y S) \\
\downarrow z^{S/Y} & & \downarrow z_1^{S/Y} \\
G(S) \hookrightarrow G(Y \times_X S) \simeq G(Y \times_X Y \times_Y S) & \overset{p_{13}^{S/Y}}{\underset{p_{23}^{S/Y}}{\rightrightarrows}} & G(Y \times_X Y \times_X Y \times_Y S)
\end{array}$$

The top row is the equalizer (3.5) combined with the identifications of (3.7), and the bottom row arises in the same fashion for the trivial cocycle as described in (3.6). Since z is a cocycle we have $z_2 = z_1 z_3$. This yields $z_2^{S/Y} = z_1^{S/Y} z_3^{S/Y}$ which shows that the outermost right square is commutative. That the innermost right square is commutative is easy. This shows that $z^{S/Y}$ induces a group isomorphism ${}_zG(S) \to G(S)$. This isomorphism is functorial on Y-schemes S and thus defines a Y-group scheme isomorphism which we still denote by z:

(3.9) $$z : {}_zG_Y \to G_Y$$

Finally, if $\lambda : {}_zG(X) \to {}_zG(Y)$ denotes the canonical inclusion corresponding to our fppf-cover Y/X, one can easily checks that the diagram

$$\begin{array}{ccccc}
{}_zG(X) & \hookrightarrow & G(Y) & \overset{z \circ p_1}{\underset{p_2}{\rightrightarrows}} & G(Y \times_X Y) \\
\downarrow \lambda & & \downarrow p_1 & & \downarrow p_{12} \\
{}_zG(Y) & \hookrightarrow & G(Y \times_X Y) & \overset{z_3 \circ p_{13}}{\underset{p_{23}}{\rightrightarrows}} & G(Y \times_X Y \times_X Y) \\
\vdots & & \downarrow z & & \downarrow z_1 \\
G(Y) & \hookrightarrow & G(Y \times_X Y) & \overset{p_{13}}{\underset{p_{23}}{\rightrightarrows}} & G(Y \times_X Y \times_X Y)
\end{array}$$

commutes. Here the first row is the equalizer (3.6) for $S = X$, and the last two rows constitute (3.8) in the case $Y = S$.

3.10. **Remark.** Let $Y \to X$ be an fppf cover, and let $z \in \check{Z}^1(Y/X, G) \subset G(Y \times_X Y)$ be a cocycle. The element z defines an (inner) automorphism $\mathrm{int}(z)$ of the group $G_{Y \times_X Y}$. It is clear that $\mathrm{int}(z) \in \check{Z}^1(Y/X, \mathbf{Aut}(G))$. We denote the corresponding twisted group ${}_{\mathrm{int}(z)}G$ by ${}_zG$.

We now turn our attention to maximal tori. We assume that G is a reductive group scheme. For convenience we will denote the twisted group ${}_zG$ by G', and let $\mathcal{T}$ and $\mathcal{T}'$ denote the scheme of maximal tori of G and G' respectively [SGA3, XI.4]. We recall that $\mathcal{T}$ and $\mathcal{T}'$ are affine schemes of finite type over X (*ibid.*). The cocycle z acts naturally on the scheme $\mathcal{T}_{Y \times Y}$ of maximal tori of $G_{Y \times Y}$. We denote (by a slight abuse of notation) the resulting automorphism of $\mathcal{T}_{Y \times Y}$ also by z. In this way we may view z as a cocycle in $\check{Z}^1(Y/X, \mathbf{Aut}(\mathcal{T}))$. By [SGA3, XXIV, Prop.4.2.1], $\mathcal{T}'$ coincides with the twist of $\mathcal{T}$ by z, i.e., we may assume that the functor of points of $\mathcal{T}'$ is given by

$$\mathcal{T}'(S) = \{\, T'' \in \mathcal{T}(Y \times S) \ \mid z(p_1^S(T'')) = p_2^S(T'')\,\}. \tag{3.11}$$

Arguing as above we get the commutative diagram

$$\begin{array}{ccccc}
\mathcal{T}'(X) & \hookrightarrow & \mathcal{T}(Y) & \overset{z\circ p_1}{\underset{p_2}{\rightrightarrows}} & \mathcal{T}(Y \times_X Y) \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_{12}} \\
\mathcal{T}'(Y) & \hookrightarrow & \mathcal{T}(Y \times_X Y) & \overset{z_3\circ p_{13}}{\underset{p_{23}}{\rightrightarrows}} & \mathcal{T}(Y \times_X Y \times_X Y) \\
\vdots & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z_1} \\
\mathcal{T}(Y) & \hookrightarrow & \mathcal{T}(Y \times_X Y) & \overset{p_{13}}{\underset{p_{23}}{\rightrightarrows}} & \mathcal{T}(Y \times_X Y \times_X Y)
\end{array} \tag{3.12}$$

Just as in (3.9), the automorphism z induces a group isomorphism (also denoted by z)

$$z : \mathcal{T}'_Y \to \mathcal{T}_Y \tag{3.13}$$

The following result is natural and very useful.

3.14. **Lemma.** *Let T' be a maximal X-torus of G' (i.e., $T' \in \mathcal{T}'(X)$), and let T'' be the corresponding maximal torus of G_Y given by (3.11). Then $z(T'_Y) = T''$.*

Proof. We have $\lambda(T') = T'_Y$, so we need to show that $z(\lambda(T')) = T''$. Let $T''' \in \mathcal{T}(Y)$ be the image of $\lambda(T')$ under z given by (3.12). The inclusion $\mathcal{T}(Y) \subset \mathcal{T}(Y \times_X Y)$ in the bottom row of (3.12) is given by the projection p_2. Thus $z(p_1(T'')) = p_2(T''')$. By the top row of (3.12) we have $z(p_1(T'')) = p_2(T'')$. It follows that $T'' = T'''$ as desired □

3.15. **Lemma.** *Let G be a reductive group scheme over X, and T a maximal torus of G (assumed to exist). Let Y/X be an fppf cover of X, $z \in \check{Z}^1(Y/X, G)$ a cocycle. If the twisted X-group ${}_zG$ admits a maximal torus T' such that T'_Y and T_Y are conjugate by an element of $G(Y)$ (where ${}_zG_Y$ is identified with G_Y as described in (3.9) above), then*

$$[z] \in \operatorname{Im}[H^1(Y/X, N_G(T)) \longrightarrow H^1(X, G)].$$

Proof. To begin with we recall that $N_G(T)$ is a closed smooth subgroup of G ([SGA3, XI.5 and XIX.1]). As before we denote the twisted group ${}_zG$ by G', and let $\mathcal{T}$ and $\mathcal{T}'$ denote the scheme of maximal tori of G and G' respectively.

According to (3.11) our torus T' of G' corresponds to a torus T'' of G_Y. By Lemma 3.14 T'_Y coincides with T'' via our identification $G'_Y \simeq G_Y$.

Hence $T'' = g^{-1}Tg$ for some $g \in G(Y)$. Let $z' = p_2(g)zp_1(g)^{-1}$. Then $z' \in G(Y \times_X Y)$ is a cocycle equivalent to z. We have

$$zp_1(T'')z^{-1} = p_2(T'')$$

because $T' \in \mathcal{T}'(X)$. This implies that

$$zp_1(g^{-1}T_Y g)z^{-1} = p_2(g^{-1}T_Y g) \quad \text{or}$$

$$zp_1(g)^{-1}p_1(T_Y)p_1(g)z^{-1} = p_2(g)^{-1}p_2(T_Y)p_2(g), \quad \text{or}$$

$$z'p_1(T_Y)(z')^{-1} = p_2(T_Y).$$

Because T is defined over X this last yields

$$z'T_{Y\times_X Y}(z')^{-1} = T_{Y\times_X Y}$$

Thus $z' \in N_G(T)(Y \times_X Y)$. □

3.4. **Reducibility and isotropy.** Let G be a reductive group scheme over a base scheme X. We recall two fundamental notions about G; one global (reducibility), and the other local (isotropy).

We say that G is *reducible* if G admits a proper parabolic subgroup, and *irreducible* otherwise. We denote by $\mathrm{Par}(G)$ the X-scheme of parabolic subgroup schemes of G [SGA3, XXVI.3.5]. This scheme is smooth and projective over X. Since by definition G is a parabolic subgroup of G, to say that G is reducible is to say that $\mathrm{Par}(G)(X) \neq \{G\}$.

3.16. **Remark.** If X is connected, to each parabolic subgroup P of G corresponds a "type" $\mathbf{t} = \mathbf{t}(P)$ which is a subset of the Dynkin diagram of G. Given a type $\mathbf{t}$ the scheme $\mathrm{Par}_{\mathbf{t}}(G)$ of parabolic subgroups of G of type $\mathbf{t}$ is also smooth and proper over X.

We know that if G contains a non-central split subtorus, then G is reducible (*loc. cit.*, 6.3). As we will presently see, the converse is true locally.

Assume that the base scheme X is semilocal and connected. Following [SGA3, XXVI.6.13] we say that G is *isotropic* if (as a reductive X-group) G admits a non-trivial split subtorus. Otherwise we say that G is *anisotropic.* Recall that the radical torus $\mathrm{rad}(G)$ is the unique maximal torus of the centre of G [SGA3, XXII.4.3.6].

3.17. **Proposition.** *Assume that X is semilocal connected.*

(1) [SGA3, XXVI.6.12] *The following are equivalent:*
  - *(a) G is reducible.*
  - *(b) G admits a non-central split subtorus.*

(2) [SGA3, XXVI.6.14] *The following are equivalent:*
  - *(a) G is isotropic.*
  - *(b) G is reducible or* $\mathrm{rad}(G)$ *is isotropic.*

3.5. **Patching group schemes.** A technical tool used by Raghunathan and Ramanathan in their proof falls within the content of the following useful result.

3.18. **Lemma.** *Let X be an algebraic curve over k (i.e. a one-dimensional separated irreducible algebraic scheme over k), and let K denote its function field (i.e. the local ring of its generic point). Let $x \in X$ be a closed point, and assume that the local ring $D = O_{X,x}$ is a discrete valuation ring (for example, if X is smooth). Let $\widehat{D}$ and $\widehat{K}$ denote the corresponding completions.*

Assume we are given a triple (G, F, τ) consisting of:

(i) An affine group scheme G over $U = X - \{x\}$ of finite type.

(ii) An affine and finitely presented group scheme F over $\widehat{D}$.

(iii) A $\widehat{K}$-group scheme isomorphism $\tau : G \times_U \widehat{K} \simeq F \times_{\widehat{D}} \widehat{K}$.

Then there exists a group scheme H, affine and of finite type over X, such that $H \times_X U \simeq G$ and $H \times_X \widehat{D} \simeq F$ and both isomorphisms are compatible with τ. Furthermore, if G and F are smooth, then so is H.

Proof. By [BLR] §6.2 proposition D.4(b) applied to our isomorphism $\tau : (G \times_U K) \times_K \widehat{K} \simeq F \times_{\widehat{D}} \widehat{K}$ there exists a group scheme F_D over D together with isomorphisms

(a) $F_D \times_D \widehat{D} \simeq F$ and

(b) $F_D \times_D K \simeq G \times_U K$.

which are compatible with τ. Note that since $D \to \widehat{D}$ is faithfully flat the descended group F_D is finitely presented.

Fix an affine open neighborhood $\mathrm{Spec}(S)$ of x. Since $D = S_x$ and F_D is finitely presented, there exists $f \in S$ with $f(x) \neq 0$ [i.e. $x \in \mathrm{Spec}(S_f)$] and a finitely presented S_f-group F_f such that $F_f \times_{S_f} D \simeq F_D$.

Choose $g \in S$ such that $\mathrm{Spec}(S_{fg}) \subset \mathrm{Spec}(S) \cap U$. If we set $G_{fg} = G \times_U S_{fg}$ and $F_{fg} = F_f \times_{S_f} S_{fg}$ then (b) above yields an isomorphism of K-groups

$$G_{fg} \times_{S_{fg}} K \simeq F_{fg} \times_{S_{fg}} K.$$

Because both G_{fg} and F_{fg} are of finite type and K is the field of quotients of S_{fg}, there exists some $h \in S$ such that

$$G \times_U S_{fgh} \simeq F_f \times_{S_f} S_{fgh}.$$

Let $V = \mathrm{Spec}(S_{fgh}) \cup \{x\} \subset \mathrm{Spec}(S_f)$. Then the U-group G and the V-group $F_f \times_{S_f} V$ are isomorphic on the overlap $U \cap V = \mathrm{Spec}(S_{fgh})$. We can thus glue these two groups to obtain a group H over X such that $H_U \simeq G$ and $H_{\widehat{D}} \simeq F$.

Let $Y = U \bigsqcup \mathrm{Spec}\, \widehat{D}$. The natural map $Y \to X$ is faithfully flat and quasicompact. The assertions about H being affine and of finite type, and smooth if G and F are smooth, now follow from descent ([EGA4] II Prop. 2.7.1 and IV Cor. 17.7.3(ii)). □

The group H constructed above is said to *correspond* to the given triple (G, F, τ). The following two lemmas easily follow from the above argument and properties of descent in the faithfully flat quasicompact topology.

3.19. **Lemma.** *Assume we are given two triples (G, F, τ) and (G', F', τ') as in the previous Lemma. If $\phi_1 : G' \to G$ and $\phi_2 : F' \to F$ are isomorphisms over U and $\widehat{D}$ respectively such that the diagram*

$$\begin{array}{ccc} G' \times_U \widehat{K} & \xrightarrow{\tau'} & F' \times_{\widehat{D}} \widehat{K} \\ {\scriptstyle \phi_1 \times \mathrm{id}} \big\downarrow & & {\scriptstyle \phi_2 \times \mathrm{id}} \big\downarrow \\ G \times_U \widehat{K} & \xrightarrow{\tau} & F \times_{\widehat{D}} \widehat{K} \end{array}$$

commutes, then the group schemes H and H' over X corresponding to (G, F, τ) and (G', F', τ') are isomorphic.

3.20. **Lemma.** *Assume that we are given a triple (G, F, τ) as in Lemma 3.18. If $G' < G$ and $F' < F$ are closed subgroup schemes such that $\tau(G' \times_U \widehat{K}) = F' \times_{\widehat{D}} \widehat{K}$, then the group scheme H' corresponding to the triple $(G', F', \tau_{|_{G' \times_U \widehat{K}}})$ admits a natural closed immersion into the group scheme H corresponding to (G, F, τ).*

3.21. **Example.** The following example, which gives a procedure for extending certain group schemes over $\mathbb{A}^1_k$ to the projective line $\mathbb{P}^1_k$, is of fundamental importance to the proof of the main theorems.

We maintain the general notation of the previous Lemmata, but look at the particular case when X is the projective line $\mathbb{P}^1_k$ over k, $U = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ is the affine line over k, and $\{x\} = X \setminus U$ is the point at infinity of X. We have $D = \mathcal{O}_{P^1_k, x} = \mathrm{Spec}(k[\frac{1}{t}]_{(\frac{1}{t})})$, $K = k(t) = k(\frac{1}{t})$ where t is our coordinate function on U, $\widehat{D} = k[[\frac{1}{t}]]$ and $\widehat{K} = k((\frac{1}{t}))$.

(a) Let $G = \mathrm{Spec}(k[t][G])$ be a semisimple simply connected group scheme over U. Consider the generic fiber G_K of G and pass to the completion $G_{\widehat{K}}$. Let p be a point of the building $\mathcal{B} = \mathcal{B}_{G_{\widehat{K}}}$ corresponding to $G_{\widehat{K}}$. Recall that $G(\widehat{K})$ acts on $\mathcal{B}$, and that the stabilizer $\mathrm{Stab}_{G(\widehat{K})}(p)$ of p in $G(\widehat{K})$ is the parahoric subgroup of $G(\widehat{K})$ associated to p. This parahoric subgroup, in turn, gives rise to a smooth group scheme F over $\widehat{D}$ whose generic fiber is canonically isomorphic to $G_{\widehat{K}}$ [BT2, §5.1.8], [BT3, §1.7]:

$$\tau : G \times_{k[t]} \widehat{K} \to F \times_{\widehat{D}} \widehat{K}. \tag{3.22}$$

More precisely, Bruhat-Tits theory (*ibid.*) shows that $F = \mathrm{Spec}(\widehat{D}[F])$ where

$$\widehat{D}[F] = \big\{ f \in \widehat{K}[G] \mid f(x) \in \widehat{D}^{sh} \ \text{ for all } \ x \in \mathrm{Stab}_{G(\widehat{K}^{sh})}(p) \big\}. \tag{3.23}$$

where $\widehat{D}^{sh}$ (resp. $\widehat{K}^{sh}$) is the strict henselization of $\widehat{D}$ (resp. $\widehat{K}$).

Moreover $\widehat{D}[F] \otimes_{\widehat{D}} \widehat{K} = \widehat{K}[G]$, and this gives rise to the isomorphism τ of (3.22). The triple (G, F, τ) corresponds then to the diagram

(3.24)
$$\begin{array}{ccccc} & & \widehat{K}[G] & & \\ & \nearrow & & \nwarrow & \\ k[t][G] & & & & \widehat{D}[F] \end{array}$$

Of course, by Lemma 3.18, this data gives rise to a group scheme $H = H(p)$ which is affine and smooth over $\mathbb{P}^1_k$.

(b) For future use we note that the above construction of F (and hence of H) is compatible with any finite Galois field extension k'/k [BT2, §5]. More precisely, if $K' = k'(\frac{1}{t})$ then the extension $\widehat{K}'/\widehat{K}$ is unramified, so it gives rise to a canonical embedding $\mathcal{B}_{G_{\widehat{K}}} \hookrightarrow \mathcal{B}_{G_{\widehat{K}'}}$ of the buildings corresponding to $G_{\widehat{K}}$ and $G_{\widehat{K}'}$ respectively. Let F' denote the group scheme over $\widehat{D}' = k'[[\frac{1}{t}]]$ associated to p viewed now as an element p' of $\mathcal{B}_{G_{\widehat{K}'}}$. Then we have $F \times_{\widehat{D}} \widehat{D}' \simeq F'$ via $\widehat{D'}[F'] = \widehat{D}[F] \otimes_{\widehat{D}} \widehat{D'}$. It follows that $H(p') \simeq H(p) \times_{\mathbb{P}^1_k} \mathbb{P}^1_{k'} \simeq H(p) \times_k k'$.

(c) Assume additionally that G is of the form $G = G_0 \times_k \mathbb{A}^1_k$ where G_0 is a k-split group and that p is contained in the apartment $\mathcal{A} \subset \mathcal{B}$ corresponding to a maximal k-split torus $T_0 \subset G_0$. Then $T_0(\widehat{D}^{sh})$ acts trivially on p and this gives rise to the canonical closed embedding

$$T_{0,\widehat{D}} = T_0 \times_k \widehat{D} \hookrightarrow F$$

of $\widehat{D}$-groups. Since $T_0 \subset G_0$ we also have the canonical closed embedding

$$T_{0,k[t]} = T_0 \times_k k[t] \hookrightarrow G = G_0 \times_k k[t].$$

Both of the above embeddings are compatible with the isomorphism τ of (3.22), i.e.

$$\tau(T_{0,k[t]} \times_{k[t]} \widehat{K}) = T_{0,\widehat{D}} \times_{\widehat{D}} \widehat{K} \subset F \times_{\widehat{D}} \widehat{K}.$$

If we denote by τ' the restriction of τ to $T_{0,k[t]} \times_{k[t]} \widehat{K} \leq G \times_{k[t]} \widehat{K}$, then by Lemma 3.20, the triple $(T_{0,k[t]}, T_{0,\widehat{D}}, \tau')$ corresponds to a group subscheme $T \subset H$ over $\mathbb{P}^1_k$. Moreover, from the diagram

(3.25)
$$\begin{array}{ccccc} & & \widehat{K}[T_{0,k[t]}] = \widehat{K}[T_0] & & \\ & \nearrow & & \nwarrow & \\ k[t][T_{0,k[t]}] & & & & \widehat{D}[T_{0,F}] \\ & \nwarrow & & \nearrow & \\ & & k[T_0] & & \end{array}$$

it follows that $T \simeq T_0 \times_k \mathbb{P}^1_k$. In particular, T is split.

(d) Let $g \in G(k[t]) \subset G(\widehat{K})$. Then g yields an (inner) automorphism $\mathrm{int}(g) \in \mathrm{Aut}(G_{\widehat{K}})$. Let $g^\star : \widehat{K}[G] \to \widehat{K}[G]$ be the comorphism corresponding to $\mathrm{int}(g)$. Set $p' = g(p)$ and let F' be the group scheme over $\widehat{D}$ corresponding to p'. Then we clearly have

$$g^\star : \widehat{D}[F] \to \widehat{D}[F'] = g^\star(\widehat{D}[F]).$$

Thus $g^\star$ gives rise to an isomorphism $\phi : F' \to F$. Set

$$\phi_1 = \mathrm{int}(g) : G = G' \to G,$$

and $\tau' = (\phi \times \mathrm{id}_{\widehat{K}})^{-1} \circ \tau \circ (\mathrm{int}(g) \times \mathrm{id}_{\widehat{K}})$. By Lemma 3.18, the triple (G, F', τ') gives rise to a group scheme H' over $\mathbb{P}^1_k$, and by Lemma 3.19 we have $H' \simeq H$. Tracing through all these constructions and identifications we see that τ' corresponds to the diagram

$$\begin{array}{ccc} & \widehat{K}[G] & \\ \nearrow & & \nwarrow \\ k[t][G] & & \widehat{D}[F'] = g^\star(\widehat{D}[F]) \end{array}$$

3.6. **Weil restriction considerations.** Throughout this section X will denote a geometrically integral projective variety over k, i.e. a geometrically integral closed subscheme in $\mathbb{P}^n$ for some n, and H a group scheme which is affine and of finite type over X.

For a given k-scheme S we denote by h_S the corresponding functor of points:

$$h_S : Sch/k \longrightarrow Sets$$
$$Y \longrightarrow \mathrm{Hom}_k(Y, S).$$

Recall Grothendieck's definition of the Weil restriction of the X-scheme H to k [Gr, exp. 221, Remarque 3.9.c]. This is the functor

$$\prod_{X/k} H \ : \ Sch/k \longrightarrow Sets$$

defined by

$$Y \longrightarrow \mathrm{Hom}_X(X \times_k Y, H)$$

According to *loc. cit.* (see also [H2, page 121]), this functor is representable by an affine k-scheme of finite type, say S/k, which we henceforth assume is fixed in our discussion. Fix an isomorphism

$$\alpha : h_S \to \prod_{X/k} H \tag{3.26}$$

By definition our map α is thus a family of bijections

$$\mathrm{Hom}_k(Y, S) \xrightarrow{\alpha_Y} \mathrm{Hom}_X(Y \times_k X, H) \tag{3.27}$$

which is functorial on k-schemes Y. The identity map $\mathrm{id}_S : S \to S$ defines a morphism

$$\mathrm{ev}_\alpha : S \times_k X \longrightarrow H \tag{3.28}$$

called the *evaluation map.* By Yoneda considerations the bijections of (3.27) are then given by

$$\alpha_Y : f \to \mathrm{ev}_\alpha \circ (f \times \mathrm{id}_X). \tag{3.29}$$

The algebraic k-scheme S can be made into a k-group by transport of structure via α. More precisely, the group structure on $S(Y)$ is as follows: Given $a, b \in S(Y)$ then $ab = c$ where c satisfies

$$\alpha_Y(c) = \alpha_Y(a)\,\alpha_Y(b)$$

where the right-hand side is the multiplication on the group $H(Y \times_k X)$. We denote this group by S^α, or simply by S if no confusion is possible. One checks that ev_α, when viewed as a map from $S^\alpha \times_k X$ to H is in fact a morphism of group schemes over X.

3.30. **Remark.** Assume that the X-group H is obtained by base change from an affine k-group G_0, i.e., that $H = G_0 \times_k X$. Then we may take S to be G_0. Indeed, since X/k is projective the canonical map $\mathcal{O}(Y) \to \mathcal{O}(X \times_k Y)$ is an isomorphism. Thus

$$\mathrm{Hom}_k(Y, G_0) \simeq \mathrm{Hom}_k(k[G_0], \mathcal{O}(Y)) \simeq \mathrm{Hom}_k(k[G_0], \mathcal{O}(Y \times_k X)) \simeq$$

$$\mathrm{Hom}_k(Y \times_k X, G_0) \simeq \mathrm{Hom}_X(Y \times_k X, G_0 \times_k X).$$

Note that the resulting isomorphism $\alpha : h_{G_0} \to \prod_{X/k}(G_0 \times_k X)$ is such that $\mathrm{ev}_\alpha = \mathrm{id}_{G_0 \times_k X}$.

3.31. **Remark.** Let k' be a field extension of k and set $X' = X \times_k k'$ and $H' = H \times_k k'$. Then the Weyl restriction $\prod_{X'/k'} H'$ is represented by $S' = S \times_k k'$. A natural isomorphism

$$\alpha' : h_{S'} \longrightarrow \prod_{X'/k'} H'$$

is obtained from α via the following functorial identifications on schemes Y over k':

$$\begin{aligned}
& h_{S'}(Y) = \mathrm{Hom}_{k'}(Y, S') \simeq \mathrm{Hom}_k(Y, S) = h_S(Y) \\
& \overset{\alpha_Y}{\simeq} \mathrm{Hom}_X(Y \times_k X, H) \\
& \simeq \mathrm{Hom}_{\,k' \times_k X}(Y \times_k X, k' \times_k H) \\
& \simeq \mathrm{Hom}_{k' \times_k X}(Y \times_{k'} (k' \times_k X), k' \times_k H) \\
& = \mathrm{Hom}_{X'}(Y \times_{k'} X', H').
\end{aligned}$$

It follows from this explicit description that the evaluation map commutes with base field extension, namely that

$$\mathrm{ev}_{\alpha'} = \mathrm{ev}_\alpha \times \mathrm{id}_{k'}$$

or, more precisely, that the diagram

$$\begin{array}{ccc} (S \times_k X) \times_k k' & \xrightarrow{ev_\alpha \times id_{k'}} & H \times_k k' \\ \simeq \downarrow & & \downarrow \mathrm{id} \\ S' \times_{k'} X' & \xrightarrow{ev_{\alpha'}} & H \times_k k' \end{array}$$

commutes.

3.32. **Remark.** If $\phi : H_1 \to H_2$ is a morphism of X-groups we have a natural k-group morphism

$$\prod \phi : \prod_{X/k} H_1 \longrightarrow \prod_{X/k} H_2$$

If ϕ is a monomorphism (i.e., if ϕ has trivial kernel) then $\prod \phi$ is also a monomorphism.

Assume our $\prod_{X/k} H_i$ are representable by algebraic k-schemes S_i as explained above, and that we are given natural isomorphisms $\alpha_i : h_{S_i} \simeq \prod_{X/k} H_i$. These yields a morphism of k-groups $h_\phi : S_1^{\alpha_1} \to S_2^{\alpha_2}$ which, at the level of functor of points, is given by

$$h_\phi = \alpha_2^{-1} \circ (\prod \phi) \circ \alpha_1 : h_{S_1} \longrightarrow h_{S_2}$$

Note that if ϕ is a monomorphism then so is h_ϕ.

By Yoneda considerations h_ϕ corresponds to the k-algebra homomorphism $h_\phi(id_{S_1}) : S_1 \to S_2$. Chasing through the definitions we see that the diagram

$$\begin{array}{ccc} S_1 \times_k X & \xrightarrow{ev_{\alpha_1}} & H_1 \\ {\scriptstyle h_\phi(id_{S_1}) \times id_X} \downarrow & & \downarrow \phi \\ S_2 \times_k X & \xrightarrow{ev_{\alpha_2}} & H_2 \end{array}$$

commutes. Of particular interest is the case when $H_1 = T \times_k X$ for some k-group T. As we saw in Remark 3.30, we may take S_1 to be T and the evaluation map $ev_{\alpha_1} : T \times_k X \to T \times_k X$ to be the identity. The above then reads

$$\phi = ev_{\alpha_2} \circ (h_\phi(id_T) \times id_X).$$

3.33. **Remark.** The algebraic k-scheme S need not be reduced. Following the Appendix we associate to S a closed k-subscheme S_r which is absolutely reduced. The k-group structure of S^α carries to S_r, and we denote the resulting algebraic k-group by S_r^α, or simply by S_r when no confusion is possible. If Y is an absolutely reduced scheme over k then every morphism $Y \to S$ factors through the closed immersion $S_r \hookrightarrow S$ (see the Appendix for details). We call the composition

$$S_r \times_k X \hookrightarrow S \times_k X \xrightarrow{ev_\alpha} H$$

the (reduced) evaluation map and denote it by the symbol ev_α^r.

The main result of this section concern the rank of the algebraic k-group S_r.

3.34. **Proposition.** *Let H and S be as above. Assume that $X \times_k k_s$ is connected. Let M_0 be a k-subgroup of multiplicative type of S.*

(1) *The morphism of X-group schemes $M_0 \times_k X \to H$ induced by ev_α is a closed immersion.*
(2) *For all $x \in X$, we have* $\mathrm{rank}(S_r) \leq \mathrm{rank}(H_x)$ *where H_x is the fiber $H \to X$ of x.*

Proof. (1) Since the evaluation map commutes with base change (Remark 3.31) we may assume that $k = k_s$. Let μ denote the kernel of the X-group morphism $ev_\alpha : M_0 \times_k X \to H$ induced by our evaluation map. According to theorem IX.6.8 of [SGA3] μ is a group scheme of multiplicative type and of finite type over X. Furthermore, to establish (1) it suffices to show that μ is trivial.

Since k is separably closed M_0 is a diagonalisable k-group of finite type (prop. X.1.4 of [SGA3]). By Proposition IX.2.11.i of [SGA3], it follows that μ is diagonalisable as well (it is here that we use the assumption that X_s is connected). Hence $\mu = \mu_0 \times_k X$ where μ_0 is a diagonalisable k-group of finite type. But by rigidity of diagonalisable groups ([SGA3], VIII.1.6), we have

$$\mathrm{Hom}_{k-gr}(\mu_0, M_0) \xrightarrow{\sim} \mathrm{Hom}_{X-gr}(\mu_0 \times_k X, M_0 \times_k X).$$

So we are given actually a morphism $\iota_0 : \mu_0 \to M_0$ which is a closed immersion.

Recall that $\prod_{X/k}(\mu_0 \times_k X)$ is represented by μ_0 with the identity map for evaluation map (see Remark 3.30). Similarly for $M_0 \times_k X$ and $S \times_k X$. By Remark 3.32) if we take the Weil restriction for the X-group morphism $ev_\alpha : S \times_k X \to H$, then the corresponding k-group morphism $S \to S$ is the identity map. If we apply these considerations to the composite X-group morphism

$$f : \mu_0 \times_k X \xrightarrow{\iota = \iota_0 \times \mathrm{id}} M_0 \times_k X \to S \times_k X \xrightarrow{ev_\alpha} H \tag{3.35}$$

we obtain

$$f_0 : \mu_0 \xrightarrow{\iota_0} M_0 \hookrightarrow S \xrightarrow{\mathrm{id}} S, \tag{3.36}$$

which is a closed immersion. Since $\mu = \mu_0 \times_k X$ the morphism f of (3.35) is trivial. This forces the closed immersion f_0 of (3.36) to be trivial. Thus $\mu_0 = 1$, and consequently $\mu = 1$ as desired.

(2) Let T be a maximal k-torus of S_r. By (1), the X-group morphism $T \times_k X \to H$ is a closed immersion. It follows that the morphism $T \times_k k(x) \to H_x$ is a closed immersion for all points $x \in X$. Thus $\mathrm{rank}(S_r) = \mathrm{rank}(T) \leq \mathrm{rank}(H_x)$. □

4. Torsors over the affine line

Let E be a geometrically separably trivial $\mathbb{A}^1_k$-torsor under the action of a k-linear algebraic group G whose connected component of the identity is reductive. We must show that E is constant.

We begin our proof by reducing to the case when G is reductive.

4.1. **Lemma.** *Let G be a reductive k-group and T a maximal torus of G. Suppose X is a geometrically irreducible k-scheme for which the canonical map $k_s^\times \to \mathcal{O}_{X_s}(X_s)^\times$ is an isomorphism. Then every geometrically separably trivial X-torsor under $N_G(T)$ is constant.*

Proof. Let $N = N_G(T)$. The bijection (3.3) of Remark 3.2 shows that to a geometrically separably trivial X-torsor E under N corresponds the class of a (continuous) cocycle $u \in Z^1(\mathrm{Gal}(k_s/k), N(X_s))$.

Since T_s is split the underlying scheme structure of N_s is given by $N_s = \bigsqcup_{w\in W} \mathrm{G}_m^l$ where G_m denotes the multiplicative group over k, $l = \mathrm{rk}\,(G)$ and W is the (abstract) Weyl group of (G_s, T_s). Since X is geometrically irreducible we obtain

$$N(X_s) = N_s(X_s) = \mathrm{Hom}(X_s, N_s) =$$

$$\bigsqcup_{w\in W} \mathrm{G}_m^l(X_s) = \bigsqcup_{w\in W} \mathrm{G}_m^l(k_s) = N(k_s).$$

Thus $u \in Z^1(\Gamma, N(k_s))$ and E is constant. □

By considering the case when $X = \mathbb{A}^1_k$ and $G = T$ we obtain a stronger version of Theorem 2.1 for tori.

4.2. **Corollary.** *Let T be a k-torus. Every $\mathbb{A}^1_k$-torsor under T is constant.*

Proof. Since T splits over k_s and $\mathbb{A}^1_k$ has trivial Picard group, every $\mathbb{A}^1_k$-torsor under T is geometrically separably trivial. □

4.3. **Proposition.** *Assume that every geometrically separably trivial $\mathbb{A}^1_k$-torsor under a reductive k-group is constant. If G is an algebraic k-group whose connected component G° is reductive then every geometrically separably trivial $\mathbb{A}^1_k$-torsor under G is also constant.*

Proof. Let E be a G-torsor over $\mathbb{A}^1_k$ which is trivial over $\mathbb{A}^1_s$. Consider the fiber E_0 of E at the origin $0 \in \mathbb{A}^1_k$, and its class $[E_0] \in H^1(k, G)$. We view E_0 as a constant torsor over $\mathbb{A}^1_k$ under G, and consider the twisted group ${}_{E_0}G$ which, for convenience, we will denote by G_0. We have a canonical bijection (the twisting map, see [DG])

$$\eta : H^1(\mathbb{A}^1_k, G_0) \to H^1(\mathbb{A}^1_k, G)$$

which maps the trivial class of $H^1(\mathbb{A}^1_k, G_0)$ to $[E_0]$. Furthermore η maps constant (resp. geometrically separably trivial) torsors into constant (resp. geometrically separably trivial) torsors. By replacing G by G_0 we may thus assume without loss of generality that E_0 is trivial.

Let $C = G/G^\circ$. This is a twisted constant group ([DG] II §5). Consider the commutative diagram

$$\begin{array}{ccccccc}
H^0(\mathbb{A}^1_k, C) & \longrightarrow & H^1(\mathbb{A}^1_k, G^\circ) & \xrightarrow{\phi_1} & H^1(\mathbb{A}^1_k, G) & \xrightarrow{\phi_2} & H^1(\mathbb{A}^1_k, C) \\
\downarrow & & \psi_1 \downarrow & & \psi_2 \downarrow & & \psi_3 \downarrow \\
H^0(\mathbb{A}^1_s, C) & \longrightarrow & H^1(\mathbb{A}^1_s, G^\circ) & \xrightarrow{\gamma} & H^1(\mathbb{A}^1_s, G) & \longrightarrow & H^1(\mathbb{A}^1_s, C)
\end{array}$$

Since by assumption $\psi_2([E]) = 1$, we have $\psi_3(\phi_2([E])) = 1$. By Remark 3.2 we have $\phi_2([E]) \in H^1(\mathrm{Gal}\,(k_s/k), C(\mathbb{A}^1_s)) \subset H^1(\mathbb{A}^1_s, C)$. But since C_s is a constant group $C(\mathbb{A}^1_s) = C(k_s)$. Thus

$$H^1(\mathrm{Gal}\,(k_s/k), C(\mathbb{A}^1_s)) = H^1(\mathrm{Gal}(k_s/k), C(k_s)).$$

In other words, $\phi_2([E])$ is a constant class, hence trivial because of our assumption on the fiber E_0 of E.

Let E° be an $\mathbb{A}^1_k$-torsor under G° such that $[E^\circ] \in H^1(\mathbb{A}^1_k, G^\circ)$ satisfies $\phi_1([E^\circ]) = [E]$. It remains to show that $\psi_1([E^\circ]) = 1$. As before, it suffices to prove that γ has trivial kernel or, equivalently, that $G(\mathbb{A}^1_s) \to C(\mathbb{A}^1_s)$ is surjective. But this is clear. Indeed, we have already observed above that $C(\mathbb{A}^1_s) = C(k_s)$. □

We now turn to the proof of Theorem 2.1. By Proposition 4.3 we may assume that G is reductive. Let ${}_EG$ be the corresponding twisted $\mathbb{A}^1_k$-group. Let $k' \subset k_s$ be a finite Galois extension such that $G_{k'}$ is split and $E_{\mathbb{A}^1_{k'}}$ is trivial. By the Lemmata of §3 applied to $Y = \mathbb{A}^1_{k'}$ and Lemma 4.1 it will suffice to show that the extension k' above can be chosen so that

(a) *There exists a maximal torus T' of ${}_EG$ and a maximal torus T of G such that T'_Y is conjugate to $T_Y = T \times_k Y$ under $G(Y)$.*

Let $\widetilde{G} \to G$ be the simply connected cover of the derived group of G. We can then construct the twisted group ${}_E\widetilde{G}$ by considering the adjoint action of G on $\widetilde{G}$, and this coincides with the simply connected cover of the derived group of the twisted group ${}_EG$ [SGA3]. There exists a natural correspondence between the maximal tori of ${}_E\widetilde{G}$ and those of ${}_EG$ (*ibid.*) This shows that in order to establish that (a) above we may (and henceforth do) assume that G is simply connected.

By choosing a point p in the building $\mathcal{B}$ of ${}_EG_{\widehat{K}}$ we obtain a group $H(p)$ over $\mathbb{P}^1_k$ extending ${}_EG$. Let S be an algebraic k-scheme representing the Weil restriction of $H(p)$ to k, and S_r its corresponding separably reduced version. We have the evaluation maps

$$ev^r_\alpha : S_r \times_k \mathbb{P}^1_k \hookrightarrow S \times_k \mathbb{P}^1_k \xrightarrow{ev_\alpha} H(p).$$

Since $ev_\alpha : S \times_k \mathbb{P}^1_k \to H(p)$ is a morphism over $\mathbb{P}^1_k$ we see that the restriction of ev_α to $S \times_k \mathbb{A}^1_k$ maps into ${}_EG$. This gives the commutative diagram

$$\begin{array}{ccc} S_r \times \mathbb{P}^1_k & \xrightarrow{ev^r_\alpha} & H(p) \\ \cup & & \cup \\ S_r \times \mathbb{A}^1_k & \xrightarrow{ev^r_\alpha} & {}_EG \end{array}$$

By Proposition 3.34.1 we obtain

(b) *If T_r is a maximal torus of S_r (which exists since S_r is an algebraic group over k) then $T' = ev^r_\alpha(T_r \times \mathbb{A}^1_k)$ is a torus of ${}_EG$.*

Since the evaluation map commutes with arbitrary base field change (Remark 3.31) and the construction of S_r commutes with any separable field extension k'/k base change (Appendix, Proposition 6.12) we may replace k by k' when trying to show that the torus T' of (b) satisfies the conditions of (a). Indeed, if $\alpha' : h_{S'} \to \prod_{X'/k'}(H(p) \times_k k')$ is as in Remark 3.31, then the closed immersion

$$(T_r \times_k \mathbb{A}^1_k) \times_k k' \simeq (T_r \times_k k') \times_{k'} \mathbb{A}^1_{k'} \xrightarrow{ev^r_{\alpha'}} {}_EG \times_k k'$$

makes $T' \times_k k' \simeq T' \times_{\mathbb{A}^1_k} \mathbb{A}^1_{k'}$ into a torus of ${}_EG \times_k k' \simeq {}_EG \times_{\mathbb{A}^1_k} \mathbb{A}^1_{k'}$. If this last torus is maximal, then so is our original T' (since the maximality at the level of the geometric fibers is preserved under our base change).

We may therefore assume that ${}_EG = G \times_k \mathbb{A}^1_k = G_{\mathbb{A}^1_k}$. Let T be a maximal split torus of G. By Soulé's theorem [So] there exists $g \in G(k[t])$ such that $q = g(p)$ is a point in the apartment of $\mathcal{B}_{\widehat{K}}$ corresponding to $T_{\widehat{K}}$. As explained in Example 3.21 (d) we have an isomorphism $\tilde{g} : H(p) \to H(q)$ of $\mathbb{P}^1_k$-schemes. The pullback of $\tilde{g}$ along $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ is an inner automorphism of $G_{\mathbb{A}^1_k}$. Thus, by replacing T' by $\tilde{g}(T')$ and by taking Remark 3.21(b) into consideration we may assume without loss of generality that $p = q$,

By Example 3.21 (c) the torus T yields a closed immersion

$$T \times_k \mathbb{P}^1_k \longrightarrow H(q)$$

which by Remark 3.32 is nothing but the restriction of the reduced evaluation map, namely

$$\text{(4.4)} \qquad \mathrm{ev}^r_\alpha : T \times_k \mathbb{P}^1_k \to S_r \times_k \mathbb{P}^1_k \to S \times_k \mathbb{P}^1_k \overset{ev_\alpha}{\to} H(q).$$

By pulling back along $\mathbb{A}^1_k$ we see that the reduced evaluation map induces a closed immersion

$$\text{(4.5)} \qquad ev^r_\alpha : T \times_k \mathbb{A}^1_k \longrightarrow G_{\mathbb{A}^1_k}$$

By Remark 3.30 we know that the functor of global sections of $T \times_k \mathbb{P}^1_k$ is represented by T. Thus from the closed immersion (4.4) we obtain a canonical embedding $T \to S$ (see Remark 3.32) which factors through S_r since T is reduced. The resulting map $T \to S_r$ is an injective morphism of algebraic groups, hence a closed embedding. This allows us to identify T

with a torus of S_r. By Proposition 3.34 this torus is necessarily maximal. We record this important fact for future reference.

(c) *If T_r is a maximal torus of S_r then $T' = ev^r_\alpha(T_r \times \mathbb{A}^1_k)$ is a maximal torus of ${}_EG$. In particular the $\mathbb{A}^1_k$-group ${}_EG$ admits a maximal torus.*

We will see that T' actually satisfies the conditions of (a). This will finish the proof of Theorem 2.1

Let $k' \subset k_s$ be a finite extension of k such that $T_r \times_k k'$ and $T \times_k k'$ are conjugate under an element $s \in S_r(k') \subset S(k')$. We again replace k by k'. Think of s as an $\mathbb{A}^1_k$-point of $S \times_k \mathbb{A}^1_k$ (which we denote by $\tilde{s}$). Then the inner automorphism $\mathrm{int}(\tilde{s})$ of $S \times_k \mathbb{A}^1_k$ is given by $\mathrm{int}(s) \times id$. Consider the element

$$ev_\alpha(\tilde{s}) \in G_{\mathbb{A}^1_k}(\mathbb{A}^1_k) = G(\mathbb{A}^1_k).$$

We have the commutative diagram

$$\begin{array}{ccc} S_r \times_k \mathbb{A}^1_k & \xrightarrow{ev} & G_{\mathbb{A}^1_k} \\ {\scriptstyle \mathrm{int}(\tilde{s})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{int}(ev^r_\alpha(\tilde{s}))} \\ S_r \times_k \mathbb{A}^1_k & \xrightarrow{ev^r_\alpha} & G_{\mathbb{A}^1_k} \end{array}$$

Since $\mathrm{int}(s)(T_r) = T$ this last diagram shows that $T' = \mathrm{ev}^r_\alpha(T_r \times_k \mathbb{A}^1_k)$ is conjugate to $ev_r(T \times_k \mathbb{A}^1_k)$ under the element $\mathrm{int}(ev_\alpha(\tilde{s})) \in G(\mathbb{A}^1_k)$. Since $ev_\alpha(T \times_k \mathbb{A}^1_k)$ is a maximal torus of $G_{\mathbb{A}^1_k}$ the proof of (a), hence also of Theorem 2.1, is complete. □

By reasoning as in Steps (a), (b) and (c) of the above proof we obtain the following important fact.

4.6. **Theorem.** *Let G be a reductive group scheme over $\mathbb{A}^1_k$. Assume that G is "geometrically separably split", i.e. that $G \times_{\mathbb{A}^1_k} \mathbb{A}^1_{k_s}$ is split. Then G has a maximal torus.* □

5. Torsors over the punctured affine line

In what follows we will denote by $X = \mathrm{Spec}\, k[t^{\pm 1}]$ the punctured affine line and by $K = k(t)$ the rational function field of X. We will denote by $\widetilde{K}$ the maximal unramified extension of $\widehat{K} = k((t))$. Recall that $\widetilde{K}$ is the subfield of $k_s((t))$ consisting of those elements $f = \sum_{n \geq N} c_n t^n$ for which the set $\{c_n : n \geq N\}$ belong to a finite separable extension of k. The natural map $\mathrm{Gal}\,(k_s/k) \to \mathrm{Gal}\,(\widetilde{K}/\widehat{K})$ is an isomorphism of profinite groups. We henceforth identify these two groups.

Let $\widehat{O} = k[[t]]$ and $\widetilde{O} = k_s[[t]] \cap \widetilde{K}$. The residue map $\rho : \widetilde{O} \to k_s$ (which is also the specialization map at $t = 0$) induces a group homomorphism $G(\rho) : G(\widetilde{O}) \to G(k_s)$ whose kernel we denote by $G(\widetilde{O})^{(1)}$: These are "the elements of $G(\widetilde{O})$ that are congruent to 1 modulo (t)". We denote $\mathrm{G}_m(\widetilde{O})^{(1)}$ simply by $\widetilde{O}^{(1)}$. Thus $O^{(1)} = \{\sum_{n \geq 0} c_n t^n \in \widetilde{O} : c_0 = 1\}$.

Our goal is to prove Theorem 2.5 by comparing the étale cohomology of X with the Galois cohomology of $k((t))$. As in the case of the affine line, the existence of maximal tori will play a crucial role.

5.1. **Cohomological exponent: Good and bad primes.** Theorem 2.5 for the punctured line has an assumption on the "good characteristic" on the base field. In this section we give the relevant definitions and basic results concerning this point.

5.1. **Lemma.** *Let G be a reductive k-group. There exists a positive integer n with the property that for every field extension F/k and for every maximal torus T of the reductive F-group G_F*

(1) $nH^1(F,T) = 1$.

Moreover, if $X(T)_*$ denotes the $\mathrm{Gal}(F_{\mathrm{sep}}/F)$*-module of cocharacters of T (see 3.1), then*

(2) $nH^1(F,X(T)_*) = 1$.

Proof. (1) Let T be a maximal torus of G_F. Consider a minimal splitting extension L/F of T. Since $H^1(L,T) = 1$ we have $H^1(F,T) = H^1(L/F,T(L))$. In particular, if $m = [L : F]$ then $mH^1(F,T) = 1$. We now show that there is a positive integer n which does not depend on T and F such that m divides n. This would complete the proof of (a).

Recall that G is the almost direct product $G = C \cdot G'$ of its central torus C and derived group G'. This yields $T = C_F \cdot T'$ where T' is the maximal torus of G'_F given by $T' = G'_F \cap T$.

Let l/k be the minimal field extension of k splitting C. The decomposition $T = C_F \cdot T'$ implies that L is the composition $L = L_1 \cdot L_2$ (taken inside some fixed separable closure F_s of F containing k_s) of two fields L_1 and L_2 where $L_1 = l \cdot F$ and L_2 is the minimal splitting extension of the F-torus T'. It is well known that $\mathrm{Gal}\,(L_2/F)$ admits an embedding into the automorphism group $\mathrm{Aut}\,(\Sigma)$ of the root system $\Sigma = \Sigma(G'_F,T')$ of G'_F with respect to T'. It is easy to see that the positive integer $n = [l : k]\cdot|\mathrm{Aut}\,(\Sigma)|$ is then divisible by $m = [L : F]$. Observe that n depends neither on T nor F.

(2) The reasoning is similar to (1). Namely, since $H^1(L,X(T)_*) = 1$ we have $H^1(F,X(T)_*) = H^1(L/F,X(T)_*)$. In particular, $mH^1(F,T) = 1$. Since m divides n the result follows. □

Let G be a reductive k-group. The smallest positive integer satisfying the conditions of Lemma 5.1 is called the *cohomological toral exponent of* G. It will be denoted by $\mathrm{cte}(G)$.

We will use $\mathrm{cte}(G)$ to define the concept of good and bad primes for G. We first define the relevant concepts in the semisimple case. Let G be a semisimple k-group. There exists a unique Chevalley form of G, that is a Chevalley group $H = H_G$ such that $H \otimes_{\mathbb{Z}} k_s \simeq G \otimes_k k_s$. Recall [SGA3, Exp. XXIV.3] that we have a split exact sequence of $\mathbb{Z}$-groups

$$1 \to H_{\mathrm{ad}} \to \mathbf{Aut}(H) \to \mathbf{Out}(H) \to 1, \tag{5.2}$$

where $H_{\rm ad}$ is the adjoint group of H and $\mathbf{Out}(G)$ is the "group of outer automorphisms" of H. The group $\mathbf{Out}(H)$ is a finite constant $\mathbb{Z}$-group whose underlying abstract group is a subgroup of the group of symmetries of the Dynkin diagram of H. The sequence (5.2) comes equipped with a natural section $\mathbf{Out}(H) \to \mathbf{Aut}(H)$ that arises from the fixed choice of "épinglage" used in defining H. After applying the base change $\mathbb{Z} \to k$ and passing to cohomology (5.2) yields the exact sequence of pointed sets

$$H^1(k, H_{\rm ad}) \to \ H^1(k, \mathbf{Aut}(H)) \to \ H^1(k, \mathbf{Out}(H)) \ \to 1. \tag{5.3}$$

Let $z \in Z^1(k, \mathbf{Aut}(H))$ be such that $G \simeq {}_zH$. Let $[z']$ be the image of $[z]$ under the morphism $H^1(k, \mathbf{Aut}(H)) \to H^1(k, \mathbf{Out}(H))$. There is a canonical section $H^1(k, \mathbf{Out}(H)) \to H^1(k, \mathbf{Aut}(H))$ which is obtained from the given section in (5.2). Let $[z'']$ be the image of $[z']$ with respect to this last mapping. Clearly, the twisted group $G_{\rm qs} = {}_{z''}H_k$ is quasi-split. We call $G_{\rm qs}$ the (Chevalley) quasi-split form of G. It has the following characteristic properties:

(a) $G_{\rm qs}$ is a k-form of G, i.e. $G_{\rm qs}$ and G are isomorphic over k_s;

(b) for a field extension F/k the group G_F is an inner form of H_F if and only if $G_{\rm qs} \times_k F$ is split. In particular, the star-action of $\mathrm{Gal}(k_s/k)$ on the Dynkin diagrams of $G_{\rm qs}$ and G is the same.

5.4. **Lemma.** *If P is a parabolic subgroup of G, then $G_{\rm qs}$ contains a parabolic subgroup of type $\mathbf{t}(P)$.*

Proof. Let $\mathbf{t}$ denote the type of P. Since P is k-defined the quotient variety G/P is k-defined as well. In particular $\mathbf{t}$ is stable with respect to the star action of $\mathrm{Gal}(k_s/k)$ on the Dynkin diagram. Let Q be a parabolic subgroup of the k_s-reductive group ${G_{\rm qs}}_s = G_{\rm qs} \times_k k_s$ of type $\mathbf{t}$. Since $\mathbf{t}$ is Galois stable, the variety ${G_{\rm qs}}_s/Q$ is k-defined, hence isomorphic to the variety $\mathrm{Par}_{\mathbf{t}}(G_{\rm qs})$ of parabolic subgroups in $G_{\rm qs}$ of type $\mathbf{t}$ (*c.f.* [MPW, Prop. 1.3]).

Analogously, if B is a Borel subgroup of $G_{\rm qs}$ (which exists, since $G_{\rm qs}$ is quasisplit) the variety $G_{\rm qs}/B$ is isomorphic to the variety $\mathrm{Bor}(G_{\rm qs})$ of Borel subgroups in $G_{\rm qs}$. Without loss of generality we may assume that Q contains B. The canonical morphism $G_{\rm qs}/B \to G_{\rm qs}/Q$ is k-defined. Since $(G_{\rm qs}/B)(k) \neq \emptyset$ we have $\mathrm{Par}_{\mathbf{t}}(G_{\rm qs})(k) = (G_{\rm qs}/Q)(k) \neq \emptyset$ and the Lemma follows.

□

Next we define the concept of quasisplit form in the reductive case. Let G be a reductive k-group, and let C denote its radical torus [SGA3, XXII.4.3.6]. Recall that C is the unique maximal torus of the centre of G. It is not difficult to see that up to isomorphism there exists a unique reductive k-group $G_{\rm qs}$, called the (Chevalley) quasisplit form of G, with the following two properties:

(1) The central tori of G and $G_{\rm qs}$ are isomorphic.

(2) $\mathrm{Der}(G_{\mathrm{qs}}) = \mathrm{Der}(G)_{\mathrm{qs}}$. That is, the quasisplit form of the (semisimple) derived group of G coincides with the derived group of G_{qs}.

We can now state the definition of good and bad primes: Let G be a reductive k-group and G_{qs} its quasisplit form. The prime divisors of the cohomological toral exponent $\mathrm{cte}(G_{\mathrm{qs}})$ of G_{qs} are called *bad primes for G*. Prime numbers which are not bad are called *good.* We say that the characteristic p of the base field k *is good for* G if either $p = 0$ or p is a good prime for G.

5.5. **Remark.** From Steinberg's work we know that every maximal k-torus of G admits a k-embedding into G_{qs}. From the definition it follows that cte(G) divides $\mathrm{cte}(G_{\mathrm{qs}})$. In particular the characteristic of k does not divide cte(G).

5.6. **Remark.** The cohomological toral exponent "depends on the base field". If G is a trialitarian k-group, then 3 divides $\mathrm{cte}(G)$ but not $\mathrm{cte}(G_{k_s})$ (the last assertion follows from the fact that G_{k_s} is a classical group of type D_4).

We now state and prove two stability properties of the set of good primes which will be used while proving the main result on torsors over the punctured line.

5.7. **Lemma.** *Let G be a reductive k-group. Let $P \subset G$ be a parabolic subgroup. If H is a Levi subgroup of P then the good primes for G are also good primes for H. In particular, if the characteristic of k is good for G, then it is also good for H.*

Proof. Let G_{qs} be the quasi-split form of G, and let P_0 be a parabolic subgroup of G_{qs} of the same type as P (see Lemma 5.4). Clearly any Levi subgroup H_0 of P_0 is isomorphic to the quasisplit form H_{qs} of H. Since for any field extension F/k any maximal torus of $H_{0,F}$ is also a maximal torus of $G_{\mathrm{qs},F}$ we see that $\mathrm{cte}(H_0)$ divides $\mathrm{cte}(G_{\mathrm{qs}})$. The result now follows. □

5.8. **Lemma.** *Let $\eta \in Z^1(k, G)$ be a cocycle and ${}_\eta G$ the corresponding twisted group. The set of good primes for G and ${}_\eta G$ coincide.*

Proof. Indeed, the quasisplit forms of G and ${}_\eta G$ are isomorphic, so the result follows by definition. □

5.2. **Existence of maximal tori.** The following result yields the existence of maximal tori that will be used in the proof of Theorem 2.5.

5.9. **Proposition.** *Let G be a reductive group scheme over X. Assume that G is geometrically separably split; that is there exists an isomorphism $f : G_0 \times_{\mathbb{Z}} X_{k'} \xrightarrow{\sim} G \times_X X_{k'}$ where $G_0/\mathbb{Z}$ is a Chevalley reductive group and k'/k is a finite Galois extension.*

Denote by T_0 the standard maximal split torus of the $\mathbb{Z}$-group G_0. Then G admits a maximal X-torus M such that $M \times_X X_s$ is $G(X_s)$-conjugated to $f(T_0 \times_{\mathbb{Z}} X_s)$.

Proof. By [SGA3, XII.4.7.c] there is a one-to-one correspondence between the maximal tori of G, those of its adjoint group G_{ad} and those of the simply connected covering of G_{ad}. We may thus assume without loss of generality that our X-group G is semisimple and simply connected. Following Tits [Ti2], we consider the twin building $\mathcal{B} = \mathcal{B}_+ \times \mathcal{B}_-$ of G_K over the completions of K at 0 and ∞.

Consider a point $p = (p_+, p_-) \in \mathcal{B}$, as well as the two associated parahoric group schemes $F_+/k[[t]]$ and $F_-/k[[\frac{1}{t}]]$ corresponding to p_+ and p_- respectively. The patching process of § 3.5 (applied twice) produces a smooth group scheme $H(p)$ over $\mathbb{P}^1_k$ extending G/X. Let S be the k-group representing the Weil restriction of $H(p)$ to k, and S_r the corresponding absolutely reduced group.

Let $T_r \subset S_r$ be a maximal torus. By Proposition 3.34.1 $ev(T_r \times_k \mathbb{P}_k)$ is a torus of $H(p)$. Pulling back to X we obtain that $M = ev(T_r \times_k X)$is a torus of G. We will show that M has the desired properties. The same reasoning given for the affine shows that we may replace k by k'.

There is a canonical embedding $\mathcal{B} \to \mathcal{B}'$ where $\mathcal{B}'$ is the twin building associated to $G \times_{k((t))} k'((t))$ and $G \times_{k((t))} k'((\frac{1}{t}))$. This allows us to view our chosen point p as an element of the twin building $\mathcal{B}'$. The construction of $H(p)$ is compatible with this identification [see Example 3.21(b)].

We now use the splitting $G_0 \times_k X_{k'} \xrightarrow{\sim} G \times_X X_{k'}$. By Abramenko's result [A, Proposition 5], there exists $g \in G(X_{k'})$ such that $q = g(p)$ lives in the canonical twin apartment corresponding to the torus T_0. Clearly the group schemes $H(q)$ and $H(p) \times_{\mathbb{P}^1} \mathbb{P}^1_{k'}$ are isomorphic, so we may assume that $p = q$. The torus T_0 gives rise to a canonical subtorus $T_0 \times_{\mathbb{Z}} \mathbb{P}^1_{k'} \subset H(p)$, as one can see by applying the reasoning of Example 3.21(c) twice.

The proof can now be finished along the same lines as in the proof given for $\mathbb{A}_k$. □

5.3. **Reformulation of Theorem 2.5.** Henceforth G will denote a reductive k-group where the characteristic of k is good for G.

By Remark 2.1 the isomorphism classes of geometrically separably split torsors over $\mathbb{A}_k^\times$ under G are parametrized by $H^1\big(\mathrm{Gal}(k_s/k), G(k_s[t^{\pm 1}])\big)$. Along similar lines we see that the classes of $H^1(\widehat{K}, G)$ corresponding to torsors that are trivialized by the base change $\widehat{K} \to \widetilde{K}$ are parametrized by $H^1(\mathrm{Gal}(\widetilde{K}/\widehat{K}), G(\widetilde{K})\big)$. Theorem 2.5 can thus be stated as follows:

5.10. **Proposition.** *Under the natural identification of* $\Gamma := \mathrm{Gal}\,(k_s/k)$ *with* $\mathrm{Gal}\,(\widetilde{K}/\widehat{K})$ *we have.*

(1) *Every X-torsor under G is geometrically separably trivial. In particular the canonical map* $H^1(\Gamma, G(k_s[t^{\pm 1}])) \to H^1(X, G)$ *is bijective.*

(2) *The canonical map* $H^1(\Gamma, G(\widetilde{K})) \to H^1(\widehat{K}, G)$ *is bijective: Every $\widehat{K}$-torsor under G is split by the base change $\widehat{K} \to \widetilde{K}$.*

(3) *The assertion of Theorem 2.5 is equivalent to the following: The canonical map*

$$H^1(\Gamma, G(k_s[t^{\pm 1}])) \to H^1(\Gamma, G(\widetilde{K}))$$

is bijective.

We begin with a useful general fact.

5.11. **Lemma.** *Let G be a split reductive group over a field F. Let $z \in Z^1(F, G)$ be a cocycle. There exists a maximal torus $T \subset G$ such that $[z]$ is in the image of the natural map $H^1(F, T) \to H^1(F, G)$.*

Proof. The result essentially follows from arguments in Steinberg's paper [St], which show that the Lemma holds for quasi-split simple group (see the prelude to Theorem 3.1 of [Chr] for details). In particular, the Lemma holds if our G is semisimple and of adjoint type. The reduction to this case is done along standard lines as follows.

Let C be the centre of G and let $G' = G/C$. Let $z' \in Z^1(F, G')$ be the image of z under the canonical map $Z^1(F, G) \to Z^1(F, G')$. Since G' is an adjoint group there exists a maximal torus $T' \subset G'$ such that $[z'] \in \mathrm{Im}[H^1(F, T') \to H^1(F, G')]$. Let $T \subset G$ be the inverse image of T' under $G \to G'$. Since the image of $[z']$ under the map $H^1(F, G') \to H^2(F, C)$ vanishes, the class $[z']$ lifts to a class $[u] \in H^1(F, T)$ as one can see by considering the exact sequence $1 \to C \to T \to T' \to 1$. We now pass to the twisted group ${}_uG$. Under the twisting bijection $H^1(F, G) \to H^1(F, {}_uG)$ the class $[z]$ goes into some class, say $[w]$, whose image under $H^1(F, {}_uG) \to H^1(F, {}_uG')$ is zero. Hence we may assume that $[w] \in H^1(F, {}_uC)$. Note that ${}_uC = C$ is the centre of ${}_uG$ and is contained in ${}_uT = T$. It follows that $[z] = [u + w]$ where the sum $[u + w] = [u] + [w]$ is taken inside the group $H^1(F, T)$. □

We now turn to the proof of Proposition 5.10. We begin by showing that

$$H^1(k_s(t), G) = 1. \tag{5.12}$$

For convenience we denote $k_s(t)$ by F. The main fundamental property of F we are going to use is that it is a field of q-cohomological dimension 1 for all primes q different than the characteristic p of k [Se2, §II.4.2].

Let $z \in Z^1(F, G)$ be a cocycle. Since G_F is split, the previous Lemma reduces the problem to showing that $H^1(F, T) = 1$ for any maximal torus T of G_F.

Let L/F be a minimal Galois extension splitting T. Since T is split over L we have $H^1(L, T) = 1$, hence $H^1(L/F, T(L)) = H^1(F, T)$. Because of the definition of good characteristic the abelian group $H^1(L/K, T(L))$ is the direct sum of its q-Sylow subgroups $H^1(L/F, T(L))_q$ where q runs through the set of primes other than p that divide $n = [L : F]$.

Let Γ be the Galois group of L/F. For each prime q dividing the order of Γ we fix a q-Sylow subgroup $\Gamma_q \subset \Gamma$. We have the tower of fields $F \subset L_q \subset L$

where L_q is the subfield in L corresponding to Γ_q. The standard restriction-corestriction argument shows that the equality $H^1(L/F, T(L))_q = 1$ follows immediately from that of $H^1(L/L_q, T(L)) = 1$, or equivalently, from $H^1(L_q, T) = 1$. To establish this last equality we consider an exact sequence of L_q-tori

$$1 \to T_1 \to P \to T \to 1$$

where P is a permutation torus ([CTS] lemme 3), and its corresponding Galois cohomology sequence. From this it follows that it will suffice to show that the q-torsion part of $H^2(L_q, T_1)$ is trivial. Let $A \subset T_1$ be the q^m-torsion part of T_1 where m is suitably large. Every element in $H^2(L_q, T_1)$ of order a power of q comes from $H^2(L_q, A)$. Since L_q is of q-cohomological dimension 1 we have $H^2(L_q, A) = 1$. This finishes the proof of (5.12).

We now turn to the proof of (1). Let E be a G-torsor over X_s and let ${}_EG$ be the corresponding twisted X_s-group scheme. The idea of the proof is to show that E admits a reduction of structure group to a Borel subgroup B of G. In other words, we want to prove that the isomorphism class of E is in the image of

$$H^1(X_s, B) \to H^1(X_s, G). \tag{5.13}$$

Recall that the class of E is in the image of the map (5.13) if and only if the X_s-group scheme ${}_EG$ has a Borel subgroup.

By (5.12) the generic fiber of ${}_EG$ is split. In particular this generic fiber has a Borel subgroup over $k_s(t)$. Since $X_s = \mathrm{Spec}(k_s[t^{\pm 1}])$ has dimension 1, it follows from a standard argument that ${}_EG$ itself has a Borel subgroup as required. It remains to show that $H^1(X_s, B) = 1$, but this is clear by devissage since $H^1(X_s, \mathrm{G}_a) = 1$ and the Picard group of X_s is trivial.

(2) In analogy to (5.12) we have

$$H^1(\widetilde{k_s(t)}, G) = 1. \tag{5.14}$$

This follows by reasoning as in (1) by taking into consideration that $\widetilde{k_s(t)}$ is of q-cohomological dimension 1 for all primes q other than p (see [Se1] theorem 4.4). The same devissage reasoning used in (1) completes the proof.

(3) This is a direct consequence of (1) and (2). □

5.4. **Reduction of structure group to $N_G(T)$.** Let T be a maximal torus of our reductive k-group G. By combining Lemma 3.15 and Proposition 5.9 we obtain.

5.15. **Lemma.** *The map*

$$H^1\big(\Gamma, N_G(T)(k_s[t^{\pm 1}])\big) \longrightarrow H^1\big(\Gamma, G(k_s[t^{\pm 1}])\big)$$

is surjective. □

We now look in detail at $H^1\big(\Gamma, N_G(T)(k_s[t^{\pm 1}])\big)$. Let $X(T)_*$ be the group of cocharacters of T. Recall that $X(T)_*$ comes equipped with a natural $\mathrm{Gal}(k_s/k)$-module structure (see 3.1). Since the underlying scheme of G_m is

$X = \mathrm{Spec}(k[t^{\pm 1}])$, a cocharacter of T is naturally an element of $T(k_s[t^{\pm 1}])$. This allows us to henceforth identify $X(T)_*$ with a Galois submodule of $T(k_s[t^{\pm 1}])$.

5.16. **Lemma.** *Let $N = N_G(T)$. There exists natural Galois modules isomorphisms:*

(1) $T(k_s) \times X(T)_* \xrightarrow{\sim} T(k_s[t^{\pm 1}])$;

(2) $T(\widetilde{O}) = T(k_s) \times T(\widetilde{O})^{(1)}$;

(3) $T(k_s) \times T(\widetilde{O})^{(1)} \times X(T)_* \xrightarrow{\sim} T(\widetilde{K})$;

(4) $X(T)_* \rtimes N(k_s) \xrightarrow{\sim} N(k_s[t^{\pm 1}])$;

(5) $X(T)_* \times T(\widetilde{O})^{(1)} \rtimes N(k_s) \xrightarrow{\sim} N(\widetilde{K})$.

Proof. (1) T_s corresponds to the ring $k_s[x_1^{\pm 1}, \dots, x_l^{\pm 1}]$ with a given action of Γ. An element $a \in T(k_s[t^{\pm 1}])$ corresponds to a k_s-algebra homomorphism

$$a^* : k_s[x_1^{\pm 1}, \dots, x_l^{\pm 1}] \longrightarrow k_s[t^{\pm 1}]$$

which is entirely defined by the values $a^*(x_i)$. We have $a^*(x_i) = \lambda_i t^{n_i}$. It is clear that there exists $\chi \in X(T)_*$ such that $\chi : x_i \to t^{n_i}$. This shows that $T(k_s[t^{\pm 1}])$ is generated by $T(k_s)$ and $X(T)_*$. That the natural map $T(k_s) \times X(T)_* \to T(k_s[t^{\pm 1}])$ is injective and compatible with the action of Γ is clear.

(2) and (3) Every element of $\widetilde{K}^\times$ can uniquely be written in the form $\lambda f t^n$ with $\lambda \in k_s^\times$, $f \in \mathrm{G}_m(\widetilde{O})^{(1)} = \widetilde{O}^{(1)}$ and $n \in \mathbb{Z}$. Moreover, if the element is in $\widetilde{O}$ then $n = 0$. One now reasons mutatis mutandis as in (1).

(4) We know that

$$N_s = \coprod_{\bar{w} \in W} w T_s$$

where the $w \in N(k_s)$ are representatives of the elements of the Weyl group W. Since $\mathrm{Spec}(k_s[t^{\pm 1}])$ is connected it is then clear that

$$N(k_s[t^{\pm 1}]) = \coprod w T(k_s[t^{\pm 1}]).$$

If $y \in wT(k_s[t^{\pm 1}]) \subset N(k_s[t^{\pm 1}])$ then

$$w^{-1} y \in T(k_s[t^{\pm 1}]) = T(k_s) \times X(T)_*.$$

Since $N(k_s) = \coprod w T(k_s)$ the result follows (after checking compatibility with the action of Γ).

(5) The reasoning is similar to (3) and (4) above. □

5.17. **Lemma.** *Let Γ denote the Galois group $\mathrm{Gal}\,(k_s/k)$. For all $i \geq 1$, we have*

(1) $H^i(\Gamma, T(\widetilde{O})^{(1)}) = 0$;

(2) $H^i(\Gamma, T(k_s)) \xrightarrow{\sim} H^i(\Gamma, T(\widetilde{O}))$;

(3) $H^i(\Gamma, T(k_s[t^{\pm 1}])) \xrightarrow{\sim} H^i(\Gamma, T(\widetilde{K}))$.

Proof. (1) We have a natural isomorphism

$$\widetilde{O}^{(1)} \otimes_{\mathbb{Z}} X(T)_* \simeq T(\widetilde{O})^{(1)} \tag{5.18}$$

This allows us to generalize the classical proof for $T = \mathrm{G}_m$ (e.g. [GS, §6.3]) in an obvious way to yield $H^i(\Gamma, T(\widetilde{O})^{(1)}) = 0$.
(2) Follows from (1) and Lemma 5.16.2
(3) Follows from (2) and Lemma 5.16.1,3 □

5.5. **Proof of surjectivity.** The following result, together with Proposition 5.10.3 and Lemma 5.15, establishes the surjectivity part of Theorem 2.5.

5.19. **Proposition.** *In the commutative diagram*

$$\begin{array}{ccc} H^1(\Gamma, N_G(T)(k_s[t^{\pm 1}])) & \xrightarrow{\beta_{N_G(T)}} & H^1(\Gamma, N_G(T)(\widetilde{K})) \\ \rho_{X_s}\big\downarrow & & \rho_{\widetilde{K}}\big\downarrow \\ H^1(\Gamma, G(k_s[t^{\pm 1}])) & \xrightarrow{\beta_G} & H^1(\Gamma, G(\widetilde{K})) \end{array}$$

obtained by the base change and change of structure group, the maps ρ_{X_s}*,* $\rho_{\widetilde{K}}$ *and* $\beta_{N_G(T)}$ *are surjective (and as a consequence so is* β_G*).*

Proof. That ρ_{X_s} is surjective follows from Lemma 5.15. The same reasoning shows that $\rho_{\widetilde{K}}$ is surjective since we know that for all $z \in Z^1(\Gamma, G(\widetilde{K}))$ the twisted $\widehat{K}$-group ${}_zG_{\widehat{K}}$ contains a maximal $\widehat{K}$-torus which is split by $\widetilde{K}$ [BT2, cor. 5.1.12], and any two such tori are conjugate by an element of $G(\widetilde{K})$.

It remains to show that $\beta_{N_G(T)}$ is surjective. For convenience we will denote $N_G(T)$ simply by N. Let $z \in Z^1(\Gamma, N(\widetilde{K}))$. Then $z = (z_\gamma)_{\gamma\in\Gamma}$ for some $z_\gamma \in N(\widetilde{K})$ which according to Lemma 5.16 can be written in the form

$$z_\gamma = z'_\gamma {z_\gamma}'' {z_\gamma}'''$$

with

$${z_\gamma}' \in N(k_s),\ {z_\gamma}'' \in X(T)_*,\ {z_\gamma}''' \in T(\widetilde{O})^{(1)}.$$

Moreover,

$$z' = ({z_\gamma}')_{\gamma\in\Gamma} \in Z^1(\Gamma, N(k_s)).$$

One can consider the twisted k-group ${}_{z'}T$, and it is well known that the family $z''' = (z'''_\gamma)_{\gamma\in\Gamma}$ is in fact an element of $Z^1(\Gamma, {}_{z'}T(\widetilde{O})^{(1)})$. By Lemma 5.17 $H^1(\Gamma, {}_{z'}T(\widetilde{O})^{(1)}) = 1$. Choose $x \in T(\widetilde{O})^{(1)}$ such that

$$x^{-1} z'''_\gamma\, {z'_\gamma}\, {}^\gamma x (z'_\gamma)^{-1} = 1.$$

An immediate calculation shows that

$$x^{-1} z_\gamma {}^\gamma x \in X(T)_* \rtimes N(k_s).$$

By Lemma 5.16(4) it now follows that $[z]$ is in the image of our map

$$\beta_N : H^1(\Gamma, N(k_s[t^{\pm 1}])) \longrightarrow H^1(\Gamma, N(\widetilde{K}))$$

as desired. □

5.6. **Proof of Injectivity.** The proof that the natural map

$$H^1\big(\Gamma, G(k_s[t^{\pm 1}])\big) \to H^1\big(\Gamma, G(\widetilde{K})\big).$$

is injective is the most delicate part of the argument. For convenience we will divide the reasoning into several steps. By Lemma 5.15 the injectivity result we want to prove can be reformulated as follows:

5.20. **Proposition.** *Let $z_1 = (a_\sigma)_{\sigma\in\Gamma}$ and $z_2 = (b_\sigma)_{\sigma\in\Gamma}$ be two cocycles in $Z^1(\Gamma, N(k_s[t^{\pm 1}]))$ whose image in $H^1\big(\Gamma, G(\widetilde{K})\big)$ coincide. Then z_1 and z_2 have the same image in $H^1\big(\Gamma, G(k_s[t^{\pm 1}])\big)$.*

The rest of this section is devoted to the proof of this result.

5.6.1. *Linear and translation parts of a cocycle.* We begin with a straightforward application of Lemma 5.16 that will be used in the main proof.

5.21. **Lemma.** *There is a unique decomposition $a_\sigma = a'_\sigma(t)a''_\sigma$ and $b_\sigma = b'_\sigma(t)b''_\sigma$ where $a'_\sigma(t)$, $b'_\sigma(t) \in X(T)_*$ and a''_σ, $b''_\sigma \in N(k_s)$.* □

We will call the families $z'_1 = (a'_\sigma(t))$ and $z'_2 = (b'_\sigma(t))$ [resp. $z''_1 = (a''_\sigma)$ and $z''_2 = (b''_\sigma)$] the *translation* [resp. *linear*] *parts* of the cocycles z_1 and z_2. Since $X(T)_*$ is a normal subgroup of $N(k_s[t^{\pm 1}])$, one can easily check that the z''_1 is a cocycle in $Z^1(\Gamma, N(k_s))$. If we identify $({}_{z''_1}T)(k_s[t^{\pm 1}]) = T(k_s[t^{\pm 1}])$ then one checks that the translation parts z'_1 is a cocycle with coefficients in the twisted tori ${}_{z''_1}T$. Similar considerations apply to z_2.

5.22. **Remark.** The linear part is constant, in the sense that it takes values in $N(k_s)$. The use of the parameter t within the notation $a'_\sigma(t)$ of the translation part of a_σ is used to emphasize that a_σ is being thought as a morphism from $\mathrm{Spec}(k_s[t^{\pm 1}])$ to T. This point will become relevant later on when we deal with the action of G on buildings.

5.6.2. *Equality of linear parts.* The first step of the proof of injectivity is to show that the classes of the linear parts of z_1 and z_2 coincide. The proof is mainly based on the following theorem of Bruhat-Tits.

5.23. **Theorem.** (Bruhat-Tits, [BT3, 3.15]) *Let H be a reductive group defined over a field l. Then the canonical map $H^1(l, H) \to H^1(l((t)), H)$ is injective.* □

A much easier statement of similar flavour is the following.

5.24. **Lemma.** *Let H be a reductive algebraic group over a field l. Then the following are equivalent:*

1. *H is irreducible (resp. anisotropic) over l;*
2. *H is irreducible (resp. anisotropic) over $l(t)$;*
3. *H is irreducible (resp. anisotropic) over $l((t))$.*

Proof. We have only to prove that (1) $\Longrightarrow$ (3) in both the irreducible and anisotropic cases.

Assume first that H is irreducible over l. If $H_{l((t))}$ is reducible then it has a parabolic subgroup of type $\mathbf{t}$ different than H (see Remark 3.16), and therefore $\mathrm{Par}_{\mathbf{t}}(H)\big(l((t))\big) = \mathrm{Par}_{\mathbf{t}}(H_{l((t))})\big(l((t))\big) \neq \emptyset$. Since $\mathrm{Par}_{\mathbf{t}}(H)$ is proper, we have $\mathrm{Par}_{\mathbf{t}}(H)\big(l[[t]]\big) \neq \emptyset$ by the valuative criterion of properness. But then $\mathrm{Par}_{\mathbf{t}}(H)(l) \neq \emptyset$, which is a contradiction. Thus $H_{l((t))}$ is irreducible.

Now we assume that H is anisotropic, namely that H is irreducible and its radical torus C is anisotropic (Proposition 3.17.2). This same Proposition, together with the first step, reduces the proof to showing that $C_{l((t))}$ is anisotropic. The torus C is the twist of G_m^r by a continuous morphism $\phi : \mathrm{Gal}(l_s/l) \to \mathrm{GL}_r(\mathbb{Z})$. To say that C is anisotropic is equivalent to $(\mathbb{Z}^r)^\phi = 0$. Since the map $\mathrm{Gal}(l((t))_s/l((t))) \to \mathrm{Gal}(l_s/l)$ is split, it follows that $(\mathbb{Z}^r)^{\phi_{l((t))}} = 0$ where $\phi_{l((t))} : \mathrm{Gal}(l((t))_s/l((t))) \to \mathrm{Gal}(l_s/l) \to \mathrm{GL}_r(\mathbb{Z})$ is the composite map. Thus the torus $C \times_l l((t))$ is anisotropic. □

By our hypothesis on the good characteristic of the base field k there exists a positive integer n not divisible by $\mathrm{char}(k)$ with the property that $nH^1(F, X(T)_*) = 0$ for all field extension F/k.

5.25. **Lemma.** *The following diagram commutes*

$$\begin{array}{ccccc} H^1(k[t^{\pm 1}], G) & \longrightarrow & H^1(k[t^{\pm \frac{1}{n}}], G) & \longrightarrow & H^1(k((t^{\frac{1}{n}})), G) \\ & \searrow^{ev_*} & \uparrow & & \\ & & H^1(k, G) & & \end{array}$$

where ev_* stands for the evaluation at 1.

Proof. Let $z \in Z^1(\Gamma, G(k_s[t]))$ be a cocycle. By Lemma 5.15, we may assume that $z \in Z^1(\Gamma, N(k_s[t]))$. Let z' and z'' be the translation and linear parts of z. Clearly we have $ev_*(z) = z''$. Twisting by z'' we may additionally assume for the proof of the lemma that $z'' = 1$. Then the image of $z = z'$ in $H^1(k[t^{\pm \frac{1}{n}}], T)$ is divisible by n, hence trivial by our choice of n. □

5.26. **Lemma.** *The classes of the linear parts of the cocycle z_1 and z_2 are equal in $H^1(k, G)$.*

Proof. This follows immediately from Theorem 5.23 and Lemma 5.25. □

As a by-product we may assume for the proof of injectivity that the classes of the linear parts of z_1 and z_2 are trivial in $H^1(k, G)$. Indeed this follows from the following simple lemma.

5.27. **Lemma.** *Let G be a reductive k-group and let $\eta \in Z^1(k, G)$ be an arbitrary cocycle. If the injectivity assertion of Theorem 2.5 holds for the twisted k-group ${}_\eta G$ then it holds for G.*

Proof. Twisting by η induces a commutative diagram

$$\begin{array}{ccc} H^1(X,G) & \longrightarrow & H^1(X,{}_\eta G) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ H^1(\widehat{K},G) & \longrightarrow & H^1(\widehat{K},{}_\eta G) \end{array}$$

where the horizontal arrows are the twisting bijections. It follows that f is injective if and only if g is injective. □

5.6.3. *Reduction to k-loop torsors.* Before we can finish the proof of injectivity we need one more result related to a type of torsors, called "k-loop torsors" in [GP3], that arise in connections with infinite dimensional Lie theory (see [GP1], [GP2] and [Pi] for details). According to the reformulation given in Proposition 5.20 we may assume that our cocycles z_1 and z_2 are of the form $z_1 = (a_\sigma)_{\sigma\in\Gamma'}$ and $z_2 = (b_\sigma)_{\sigma\in\Gamma'}$ where $\Gamma' = \mathrm{Gal}(k'/k)$ is the Galois group of a finite extension $k \subset k' \subset k_s$, and both the a_σ and the b_σ belong to $N(k'[t^{\pm1}])$. After further extending k' if necessary, we may also assume that k' contains a primitive n-root of unity (recall that n is prime to the characteristic of k) and splits our fixed maximal torus $T \subset G$. Consider the chain of finite field extensions

$$k(t) \subset k'(t) \subset L = k'(t')$$

where $t' = t^{1/n}$. The extensions $L/k'(t)$ and $L/k(t)$ are Galois. Let $\Lambda = \mathrm{Gal}\,(L/k(t))$ and $\Delta = \mathrm{Gal}\,(L/k'(t))$. Clearly we have $\Lambda = \Delta \rtimes \Gamma'$ where $\Gamma' = \mathrm{Gal}(k'/k)$. The natural mappings $\Lambda \to \Gamma'$, $G(k') \to G(k'(t'))$ and $G(k'[t^{\pm1}]) \to G(k'[(t')^{\pm1}])$ induce a diagram

$$\begin{array}{ccc} & & H^1(\Lambda, G(k')) \\ & & {\scriptstyle g}\big\downarrow \\ H^1(\Gamma', G(k'[t^{\pm1}])) & \xrightarrow{\ f\ } & H^1(\Lambda, G(k'[(t')^{\pm1}])) \end{array}$$

5.28. **Lemma.** *$f([z_1])$ and $f([z_2])$ are in the image of g. Furthermore, $f([z_1]$ and $f([z_2])$ are equivalent to cocycles $u_1 = (c_{1,\lambda})$ and $u_2 = (c_{2,\lambda})$ with coefficients in $G(k')$ such that $c_{1,\lambda} = c_{2,\lambda} = 1$ for every $\lambda \in \mathrm{Gal}\,(k'/k)$.*

Proof. Recall that $z_1 = (a_\sigma)$ where $\sigma \in \Gamma'$ and $a_\sigma \in N(k'[t^{\pm1}]) \subset G(k'[t^{\pm1}])$. Recall also that $a_\sigma = a'_\sigma(t)a''_\sigma$ where $z'_1 = (a'_\sigma(t))$ and $z''_1 = (a''_\sigma)$ are the translation and linear parts of z_1. As we showed before we may assume that the linear part of z_1 is trivial. Since $\Lambda = \Delta \rtimes \Gamma'$, every $\lambda \in \Lambda$ can be written uniquely in the form $\lambda = \delta\sigma$ with $\delta \in \Delta$ and $\sigma \in \Gamma'$. With this notation, and according to Remark 5.22, $f([z_1])$ is given by a family $(a'_\lambda(t'))$ where

$$a'_\lambda(t') = a_\sigma(t) = a_\sigma((t')^n) = a_\sigma(t')^n.$$

Let $u(t) \in X(T)_* \subset T(k'[t^{\pm1}])$ be such that

$$a_\sigma(t)^n = u(t)^{1-\sigma}. \tag{5.29}$$

We claim that the cocycle $(u(t')^{-1}a'_\lambda(t')u(t')^\lambda)$ takes values in $G(k')$. Indeed, let $\lambda = \delta\sigma$. Given that $\delta(t') = \zeta t'$ where ζ is an nth root of unity we have

$$(u(t'))^\delta = vu(t')$$

where $v \in T(k')$ has entries consisting of nth roots of unity [where we identify $T_{k'}$ with ${\mathrm{G}_m}^{\mathrm{rk}(G)}$]. Furthermore, since σ acts trivially on t'

$$u(t')^\lambda = (u(t')^\delta)^\sigma = (vu(t'))^\sigma = v^\sigma u(t')^\sigma$$

and equation (5.29) implies

$$a_\sigma(t')^n = u(t')^{1-\sigma}.$$

Thus we have

$$u(t')^{-1}a'_\lambda(t')u(t')^\lambda = u(t')^{-1}a_\sigma(t')^n u(t')^\lambda =$$
$$u(t')^{-1}(u(t')u(t')^{-\sigma})(v^\sigma u(t')^\sigma) = v^\sigma$$

This shows that the cocycle $(u(t')^{-1}a'_\lambda(t')u(t')^\lambda)$ takes values in $G(k')$ and has the required property. Since the class of this cocycle equals $f([z_1])]$ the proof of the Lemma for z_1 is complete. Analogously for z_2. □

5.6.4. *Reduction to the case where* ${}_{u_1}(G_{\widehat{K}})$ *is irreducible.* Let Q be a minimal $\widehat{K}$-parabolic subgroup of ${}_{u_1}(G_{\widehat{K}})$. Since u_1 is trivial over $\widehat{K}(t^{\frac{1}{n}})$ we have

$$({}_{u_1}G) \times_X \widehat{K}(t^{\frac{1}{n}}) \cong G \times_k \widehat{K}(t^{\frac{1}{n}}).$$

Then by Lemma 5.24, Q is in the conjugacy class of a k–parabolic subgroup, say P, of G. Hence

$${}_{u_1}(G_X/P_X)(\widehat{K}) \neq \emptyset.$$

We have

$${}_{u_1}(G_X/P_X)(\widehat{K}) = \Big\{ x \in (G/P)(k'((t^{\frac{1}{n}}))) \mid u_{1,\lambda}.\lambda(x) = x \ \ \forall \lambda \in \Lambda \Big\} \neq \emptyset.$$

By properness, we have $(G/P)(k'[[t^{\frac{1}{n}}]]) = (G/P)(k'((t^{\frac{1}{n}})))$. Taking into consideration the fact that $u_{1,\lambda} \in G(k')$ and that the residue map $k'[[t']] \to k'$ commutes with the action of the Galois group Λ, we get

$$\Big\{ y \in (G/P)(k') \mid u_{1,\lambda} \cdot \lambda(y) = y \ \ \forall \lambda \in \Lambda \Big\} \neq \emptyset.$$

Let y_0 be a point from the above set. Since $u_{1,\lambda} = 1$ for all $\lambda \in \mathrm{Gal}(k'/k)$, it follows $y_0 \in (G/P)(k)$. Hence y_0 is of the form $y_0 = gP$ for some $g \in G(k)$. The equalities $u_{1,\lambda} \cdot \lambda(y_0) = y_0$ can be read off as $u_{1,\lambda} \in gPg^{-1}$ for every $\lambda \in \Lambda$, or equivalently $g^{-1}u_{1,\lambda}g^\lambda \in P$. Thus we have proved that $[u_1] \in \mathrm{Im}\,[H^1(X,P) \to H^1(X,G)]$. Arguing analogously we get that $[u_2]$ is in the same image.

Let H be a Levi subgroup of P. Since X is affine, the map $H^1(X,H) \to H^1(X,P)$ is bijective [SGA3, XXVI.2.3] so we may additionally assume that

$[u_1]$, $[u_2] \in H^1(X,H)$. To complete the reduction to the irreducible case it will suffice to prove that the images of $[u_1]$ and $[u_2]$ under $H^1(X,H) \to H^1(\widehat{K},H)$ are equal (for then if injectivity fails for G, it also fails for H, and we can assume from the outset that G was chosen of smallest possible dimension so that injectivity fails). For this in turn it suffices to show that the composition (of natural maps)

$$H^1(\widehat{K},H) \to H^1(\widehat{K},P) \to H^1(\widehat{K},G)$$

is injective. This follows from the following two results.

5.30. **Lemma.** *The natural mapping $H^1(\widehat{K},H) \to H^1(\widehat{K},P)$ is bijective.*

Proof. This is a special case of [SGA3, XXVI.2.3]. □

5.31. **Lemma.** *The natural mapping $H^1(\widehat{K},P) \to H^1(\widehat{K},G)$ is injective.*

Proof. Let f denote the map under consideration. Let μ_1, $\mu_2 \in Z^1(\widehat{K},P)$ be such that $f([\mu_1]) = f([\mu_2])$. After twisting by μ_2 we may assume that $\mu_2 = 1$ and $f([\mu_1]) = 1$. Consider the exact sequence

$$G(\widehat{K}) \xrightarrow{g} (G/P)(\widehat{K}) \longrightarrow H^1(\widehat{K},P) \xrightarrow{f} H^1(\widehat{K},G).$$

By Borel-Tits theorem [BT, th. 4.13], the map g is surjective. Hence $\operatorname{Ker} f = 1$ and therefore $[\mu_1] = 1$ as desired. One may also quote [SGA3, XXVI.5.10] □

Before concluding the proof of injectivity we need one more final reduction.

5.32. **Lemma.** *Let $C = \operatorname{rad}(G)$ the radical torus of G and consider its maximal split subtorus C_d. If the natural map $H^1(X,G/C_d) \to H^1(\widehat{K},G/C_d)$ is injective, then the natural map $H^1(X,G) \to H^1(\widehat{K},G)$ is also injective.*

Proof. Since C_d is central the fibers of the natural map
$H^1(X,G) \to H^1(X,G/C_d)$ arise as quotients of $H^1(X,C_d) = 1$, so our map is injective. □

5.6.5. *Proof of injectivity.* We finally come to the proof of Proposition 5.20. By Lemma 5.32 we may assume that the connected centre of our reductive group G is an anisotropic k-torus, and that, furthermore, the twisted X-groups ${}_{z_1}G_X$ and ${}_{z_2}G_X$ are irreducible over $\widehat{K}$. Since the radical tori of the ${}_{z_i}G_{\widehat{K}}$ are isomorphic to $\operatorname{rad}(G) \times_k \widehat{K}$, it follows from Proposition 3.17.2 that the $\widehat{K}$-groups ${}_{z_1}G_{\widehat{K}}$ and ${}_{z_2}G_{\widehat{K}}$ are anisotropic.

By Lemma 5.28 we may also assume that $z_1 = (a_\lambda)_{\lambda\in\Lambda}$ and $z_2 = (b_\lambda)_{\lambda\in\Lambda}$ are cocycles in $Z^1(\Lambda, G(k'))$ where Λ is the Galois group of the extension $L = k'(t')$ of $k(t)$ described in 5.6.3. We will finish the proof by showing that there exists $g \in G(k')$ such that $a_\lambda = g^{-1} b_\lambda\, g^\lambda$ for all $\lambda \in \Lambda$.

To this end we consider the two extended Bruhat-Tits buildings [BT2, §4.2.16] $\mathcal{B}_{\widehat{K}}$ and $\mathcal{B}_{\widehat{L}}$ of G over $\widehat{K}$ and $\widehat{L} = k'((t'))$ respectively, as well as the apartment $\mathcal{A}_{\widehat{L}}$ corresponding to $T_{\widehat{L}}$. Recall that $\mathcal{A}_{\widehat{L}} = X(T)_* \otimes_{\mathbb{Z}} \mathbb{R}$ and denote by o its origin. The group $G(\widehat{L})$ acts on $\mathcal{B}_{\widehat{L}}$. We also have a canonical action of $\Lambda = \mathrm{Gal}\,(L/K) \simeq \mathrm{Gal}\,(\widehat{L}/\widehat{K})$ on $\mathcal{B}_{\widehat{L}}$ and a canonical identification $\mathcal{B}_{\widehat{K}} = \mathcal{B}_{\widehat{L}}^{\Lambda}$. We also observe that the action of $G(\widehat{L})$ on $\mathcal{B}_{\widehat{L}}$ is Λ-equivariant. By construction, the point o is fixed by Λ. Furthermore the $k'[[t']]$-parahoric group scheme attached to o is nothing but the Chevalley group scheme $G \times_k k'[[t']]$ [BT2, 3.2.13]. Hence the the stabilizer of o in $G(\widehat{L})$ is $G(k'[[t']])$.

The two cocycles z_1 and z_2 give rise to two twisted actions of Λ on $\mathcal{B}_{\widehat{L}}$, namely $\lambda_1(x) = a_\lambda(\lambda(x))$ and $\lambda_2(x) = b_\lambda(\lambda(x))$ for all $x \in \mathcal{B}_{\widehat{L}}$. The invariant subsets in $\mathcal{B}_{\widehat{L}}$ with respect to these two twisted actions of Λ are the buildings of the twisted $\widehat{K}$-groups ${}_{z_1}G_{\widehat{K}}$ and ${}_{z_2}G_{\widehat{K}}$. Since these twisted $\widehat{K}$-groups are anisotropic, by the Bruhat-Tits-Rousseau's theorem ([Ro] and [Pr]) the fixed point set for each of the two actions described above consists of a single point. Since z_1 and z_2 take values in $G(k') \subset G(k'[[t']])$ and $G(k'[[t']])$ is the stabilizer of o, these fixed points are necessary the origin o.

Since z_1 and z_2 are cocycles in $Z^1(\Lambda, G(k')) \subset Z^1(\mathrm{Gal}(L/k(t)), G(L))$ which are equivalent over $\widehat{K}$, there exists $\widehat{g} \in G(\widehat{L})$ such that

$$a_\lambda = \widehat{g}^{-1}\, b_\lambda\, \widehat{g}^{\lambda}. \tag{5.33}$$

It is easy to see that $\widehat{g}^{-1}o$ is invariant with respect to the second twisted action of Λ. Hence $\widehat{g}^{-1}o = o$, which shows that $\widehat{g} \in G(k'[[t']])$. We now "evaluate (5.33) at $t' = 0$", namely we apply the base change given by the residue map $k'[[t']] \to k'$. Since this evaluation map commutes with the action of the Galois group Λ, for $g = \widehat{g}(0) \in G(k')$ we finally obtain $a_\lambda = g^{-1}\, b_\lambda\, g^{\lambda}$ as desired. □

6. Appendix: The absolutely reduced subscheme attached to a scheme

Throughout the appendix k denotes a field, $\overline{k}$ an algebraic closure of k and k_s the separable closure of k is $\overline{k}$. The nilradical of a (commutative unital) ring A will be denoted by $\mathfrak{n}(A)$. We fix a basis $(\alpha_i)_{i \in I}$ of $\bar{k}$ viewed as a k-space. The category of commutative associative and unital algebras over k will be denoted by k-alg.

Let A be an object of k-alg. For convenience we set $A \otimes_k \bar{k} = \overline{A}$. For a subspace V of A we let $\overline{V} = V \otimes_k \bar{k} \subset \overline{A}$. We will make repeated use of the following elementary fact.

6.1. **Lemma.** *Let V be a subspace of A. Assume $x = \sum x_i \otimes \alpha_i$ belongs to $\overline{V} \subset \overline{A}$. Then $x_i \in V$ for all i.*

Proof. Extend a k-basis of V to a basis of A. □

Following [RR] we consider the following ideal of A:

$$\overline{\mathfrak{n}}(A) = \bigcap_{\substack{I \lhd A \\ \overline{I} \supset \mathfrak{n}(\overline{A})}} I,$$

namely the intersection of all ideals I of A for which $\overline{I} = I \otimes_k \bar{k}$ contain the nilradical of $\overline{A}$. Note that $\overline{I} = I\overline{A}$.

Recall that given $x \in \overline{A}$ we can uniquely write $x = \sum x_i \otimes \alpha_i$ where $x_i \in A$.

6.2. **Lemma.** $\overline{\mathfrak{n}}(A) = \langle\, x_i \mid x \in \mathfrak{n}(\overline{A})\,\rangle$.

Proof. For convenience let us denote the ideal $\langle\, x_i \mid x \in \mathfrak{n}(\overline{A})\,\rangle$ by J. It is clear from the definition that $\mathfrak{n}(\overline{A}) \subset J \otimes_k \bar{k}$. Thus, it will suffices to show that $J \subset I$ whenever I is an ideal of A for which $I \otimes_k \bar{k} \supset \mathfrak{n}(\overline{A})$. But this is an immediate consequence of Lemma 6.1. □

6.3. **Remark.** It follows from the Lemma (but also from elementary linear algebra considerations) that J does not depend of the choice of basis $(\alpha_i)_{i\in I}$.

6.4. **Corollary.** *For A to be absolutely reduced it is necessary and sufficient that $\overline{\mathfrak{n}}(A) = 0$.*

6.5. **Remark.** It is incorrectly asserted in [RR] that the ring $A/\overline{\mathfrak{n}}(A)$ is absolutely reduced, namely that $A/\overline{\mathfrak{n}}(A) \otimes_k \bar{k}$ is reduced. The following counterexample is due to A. Merkurjev.

6.6. **Example.** Let k be a separably closed field of characteristic 2 such that $\bar{k} \neq k$. Consider the polynomial $f(x,y) = x^4 + ay^2 \in k[x,y]$ where $a \in k$ is not a square. It is easy to see that f is irreducible over k and hence $A = k[x,y]/(f(x,y))$ is a domain. However in $\bar{k}[x,y]$ we have $f(x,y) = (x^2+by)^2$, where $b = \sqrt{a}$, whence $\mathfrak{n}(\overline{A}) = (\bar{x}^2 + b\bar{y})$. From Lemma 6.2 it follows that $\overline{\mathfrak{n}}(A) = \langle \bar{x}^2, \bar{y}\rangle$. This implies that the ring

$$A/\overline{\mathfrak{n}}(A) \simeq k[x,y]/\langle x^2, y\rangle \simeq k[x]/(x^2)$$

is not reduced.

6.7. **Proposition.** *Let $\psi : A \to B$ be a morphism in k-alg. If B is absolutely reduced then ψ factors through $A/\overline{\mathfrak{n}}(A)$.*

Proof. Since $\mathfrak{n}(\overline{B}) = 0$ we have $\mathfrak{n}(\overline{A}) \subset \ker(\psi \otimes 1) = \ker(\psi) \otimes \bar{k}$. By taking Lemma 6.2 into consideration, this yields $\overline{\mathfrak{n}}(A) \subset \ker\psi$ as desired. □

6.8. **Lemma.** *Assume A is a Hopf algebra. Then $\overline{\mathfrak{n}}(A)$ is a Hopf ideal of A.*

Proof. Fix a k-basis $\{a_s\}_{s\in S}$ of A where $S = S_1 \bigsqcup S_2$ and $\{a_s\}_{s\in S_1}$ is a basis of $\overline{\mathfrak{n}}(A)$. We have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes_k A \\ \downarrow & & \downarrow \\ \overline{A} & \xrightarrow[\overline{\Delta}]{} & \overline{A} \otimes_{\bar{k}} \overline{A} \end{array}$$

and the canonical isomorphism

$$\psi : \ \overline{A} \otimes_{\bar{k}} \overline{A} \to (A \otimes_k A) \otimes_k \bar{k}.$$

Let $x \in \mathfrak{n}(\overline{A})$ and write $x = \sum x_i \otimes \alpha_i$. By Lemma 6.2 it will suffice to show that

$$\Delta(x_i) \in \overline{\mathfrak{n}}(A) \otimes A + A \otimes \overline{\mathfrak{n}}(A),$$

namely that if we write

$$\Delta(x_i) = \sum_{s,t\in S} c_{st}^{(i)} \, a_s \otimes a_t$$

then $c_{st}^{(i)} = 0$ whenever both s and t belong to S_2.

Recall that by Lemma 6.2 we have $\mathfrak{n}(\overline{A}) \subset \overline{\mathfrak{n}}(A) \otimes_k \bar{k}$. Since $\mathfrak{n}(\overline{A})$ is a Hopf ideal of $\overline{A}$

$$\overline{\Delta}(x) \in \mathfrak{n}(\overline{A}) \otimes_{\bar{k}} \overline{A} + \overline{A} \otimes_{\bar{k}} \mathfrak{n}(\overline{A}).$$

Thus

$$\psi(\overline{\Delta}(x)) \in (\overline{\mathfrak{n}}(A) \otimes_k A) \otimes_k \bar{k} + (A \otimes_k \overline{\mathfrak{n}}(A)) \otimes_k \bar{k}. \tag{6.9}$$

On the other hand

$$\begin{aligned} \psi(\overline{\Delta}(x)) &= \psi\left(\overline{\Delta}\left(\sum_i x_i \otimes \alpha_i\right)\right) \\ &= \sum_i \psi(\overline{\Delta}(x_i \otimes \alpha_i)) \\ &= \sum_i \left(\sum_{s,t} c_{s,t}^{(i)} \, a_s \otimes a_t\right) \otimes \alpha_i \\ &= \sum_{i,s,t} a_s \otimes a_t \otimes c_{st}^{(i)} \alpha_i. \end{aligned}$$

By (6.9) we have $c_{s,t}^{(i)} = 0$ whenever both s and t belong to S_2. $\square$

Define a sequence $j_0(A) \subset j_1(A) \subset ...$ of ideals of A inductively as follows:

$j_0(A) = \overline{\mathfrak{n}}(A)$
$j_{i+1}(A) = \{x \in A : x + j_i(A) \in \overline{\mathfrak{n}}(A/j_i(A))\}.$

6.10. **Proposition.** *Let* $j(A) = \bigcup_{i\geq 0} j_i(A)$.

(i) *If* A *is a Hopf algebra then* $j(A)$ *is a Hopf ideal of* A.

(ii) *If* A *is noetherian then* $A/j(A)$ *is absolutely reduced.*

(iii) *If* $\psi : A \to B$ *is a morphism in* k*-alg and* B *is absolutely reduced, then* ψ *factors through* $A/j(A)$.

Proof. The first assertion follows by induction on i with the aid of Lemma 6.8. By Corollary 6.4 we see that if $j_i(A) = j_{i+1}(A)$ then $A/j_i(A)$ is absolutely reduced. This establishes (ii). Finally if ψ is as in (iii), then by Proposition 6.7 ψ factors through $A/\overline{\mathfrak{n}}(A)$, and one concludes by the inductive definition of $j(A)$. □

We assume for the remainder of this section that the k-algebra A is of finite type, and denote $A/j(A)$ by A_r. If k'/k is a field extension we have a natural inclusion $\overline{\mathfrak{n}}(A\otimes_k k') \subset \overline{\mathfrak{n}}(A)\otimes_k k'$. This yields $j(A\otimes_k k') \subset j(A)\otimes_k k'$, hence a canonical surjective k'-algebra homomorphism

$$\chi_{k'} : (A \otimes_k k')_r \longrightarrow A_r \otimes_k k'$$

It will be convenient to reformulate the result under consideration in terms of k-schemes. Set $X = \mathrm{Spec}(A)$ and $X_r = \mathrm{Spec}(A_r)$. Then $\chi_{k'}$ corresponds to the closed immersion

$$\mathrm{Spec}(\chi_{k'}) : \mathrm{Spec}(A_r \otimes_k k') \longrightarrow \mathrm{Spec}((A \otimes_k k')_r). \tag{6.11}$$

6.12. **Proposition.** *If* $k' \subset k_s$ *then* $\chi_{k'}$ *is an isomorphism.*

Proof. We first prove the useful

6.13. **Lemma.** *Let* $\overline{X(k_s)}$ *denote the closure of* $X(k_s) \subset X(\bar{k})$ *with respect to the Zariski topology of* $X(\bar{k})$. *Then* $X_r(\bar{k}) = \overline{X(k_s)}$.

Proof. Since X_r is absolutely reduced the inclusion $\overline{X_r(k_s)} \subset X_r(\bar{k})$ is an equality [Bor, AG 13.3]. Thus

$$X_r(\bar{k}) = \overline{X_r(k_s)} \subset \overline{X(k_s)}.$$

For the reverse inclusion it will suffice to show that $X(k_s) \subset X_r(\bar{k})$ since $X_r(\bar{k})$ is a closed subset of $X(\bar{k})$. Now $X(k_s) = \mathrm{Hom}\,(\mathrm{Spec}(k_s), X)$. Since $\mathrm{Spec}(k_s)$ is absolutely irreducible any morphism $\phi : \mathrm{Spec}(k_s) \to X$ factors through X_r. Thus the inclusion $X_r(k_s) \subset X(k_s)$ is in fact an equality. □

We now finish the proof of the Proposition. Assume that $k' \subset k_s$. Then to show that $\chi_{k'}$ is an isomorphism we may pass to $\bar{k}$. This corresponds to the closed immersion

$$\mathrm{Spec}(A_r \otimes_k \bar{k}) \hookrightarrow \mathrm{Spec}\left((A \otimes_k k')_r \otimes_{k'} \bar{k}\right). \tag{6.14}$$

By the last Lemma both of these affine varieties have the same $\bar{k}$-points, so their underlying topological subspaces agree ([DG] I §3 prop. 6.8). This means that $A_r \otimes_k \bar{k}$ is defined by an ideal inside the nilradical of $(A \otimes_k k')_r \otimes_{k'} \bar{k}$. Since this last $\bar{k}$-algebra is reduced (Proposition 6.10) the result follows. □

A more functorial approach to the type of problem we have considered in this Appendix can be found in B. Conrad's recent preprint [Crd].

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