Nonlinear pricing in imperfect financial markets

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International summer school in mathematical finance on
"Informational and Imperfect Financial Markets",
University of Alberta in Edmonton, Canada, June 25-30, 2016

Abstract

The aim of the lectures is to study pricing and hedging issues for various options (European, American, game options) in the case of imperfections on the market. These imperfections are taken into account via the *nonlinearity of the wealth dynamics*.

In this setting, the pricing system is expressed as a nonlinear g-expectation/g-evaluation induced by a nonlinear BSDE.

We shall address in particular superhedging issues for American and game options in this context and their links with generalized optimal stopping problems and Dynkin games.

Overview

1. The perfect market model (linear case)
   Linear pricing system and linear BSDEs
2. Market with imperfections
   Nonlinear pricing system and nonlinear BSDEs (g-evaluation).
   Links with dynamic risk measures
3. Nonlinear pricing of American Contingent claims -
   Introduction to reflected BSDEs. Links with optimal stopping.
4. Game options in an imperfect market. Links with generalized
   Dynkin games and doubly reflected nonlinear BSDEs.
5. Market model with default

I. Linear pricing theory

Market model

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}))$ be a filtered probability space, where $(\mathcal{F}_t)_{t\geq 0}$ is generated by the Wiener process W and completed. Let $T > 0$ be fixed.

Consider a complete financial market with one risk free and one risky investment possibility, with prices S^0, S^1 per unit given by, respectively

$$(1) \qquad \begin{cases} dS^0_t = S^0_t r_t dt \; ; \; S^0_0 = 1 \\ dS^1_t = S^1_t[\mu_t dt + \sigma_t dW_t] \; ; \; S^1_0 > 0; \end{cases}$$

where r_t, μ_t, σ_t are bounded predictable processes. Assume $\sigma_t \neq 0$ and σ_t^{-1} bounded.

Definition (Strategy)

A strategy is given by a couple of predictable processes

- V_t the value of the wealth of the investor (or the value of his portfolio)
- φ_t the portfolio process equal to the amount of total wealth invested in the risky asset at time t.

A strategy is **self-financing** if

$$\begin{aligned} dV_t &= (V_t - \varphi_t)\frac{dS_t^0}{S_t^0} + \varphi_t \frac{dS_t^1}{S_t^1} \\ &= (V_t - \varphi_t) r dt + \varphi_t (\mu_t \ dt + \sigma_t dW_t] \\ &= (rV_t + \theta_t \sigma_t \varphi_t) dt + \sigma_t \varphi_t dW_t \end{aligned}$$

where

$$\theta_t := \frac{\mu_t - r_t}{\sigma_t}$$

is the risk premium process.

We consider a European contingent claim with maturity $T > 0$ and payoff ξ which is $\mathcal{F}_T$-measurable. The problem is to find a self-financing strategy (V_t, φ_t) which reaches ξ at time T.

- Classical approach (without BSDE theory) : (Karatzas-Shreve...)
  - Discounting and change of probability measure.
  - integrability conditions under $\mathbb{Q}$ (unique martingale measure)
- BSDE approach
  - No discounting and no change of probability measure.
  - integrability conditions (for ξ and φ) under $\mathbb{P}$ (reference probability)

Linear pricing theory
Classical approach

Let $\mathbb{Q}$ be the unique martingale probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \mid_{\mathcal{F}_T} = \exp(-\int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T \theta_s^2 ds).$$

From Girsanov theorem, under $\mathbb{Q}$, $\tilde{W}_t := W_t + \int_0^t \theta_s ds$ is a $\mathcal{F}$-Brownian motion and by Itô, $\tilde{S} := e^{-\int_0^t r_s ds} S_t$ is a martingale under $\mathbb{Q}$.

$$d\tilde{S} = \tilde{S}\sigma_t d\tilde{W}_t.$$

We consider the discounted wealth $\tilde{V} := e^{-\int_0^t r_s ds} V_t$. We get

$$d\tilde{V} = \tilde{\varphi}\sigma_t d\tilde{W}_t,$$

where $\tilde{\varphi} := e^{-\int_0^t r_s ds}\varphi_t$ is the discounted portfolio.

The pricing/hedging problem is to construct a couple (V_t, φ_t) (value of portfolio, portfolio) such that

$$\begin{cases} dV_t = (r_t V_t + \theta_t \varphi_t \sigma_t)dt + \varphi_t \sigma_t dW_t \; ; \; 0 \le t \le T. \\ V_T = \xi \;\text{ a.s.} \end{cases}$$

that is to find $(\tilde{V}_t, \tilde{\varphi}_t)$

$$(2) \qquad \begin{cases} d\tilde{V}_t = \tilde{\varphi}\sigma_t d\tilde{W}_t \; ; \; 0 \le t \le T. \\ \tilde{V}_T = \tilde{\xi} \;\text{ a.s. } \quad \text{where } \tilde{\xi} = e^{-\int_0^T} \xi. \end{cases}$$

In order to have uniqueness we need integrability conditions. Here the integrability conditions are under $\mathbb{Q}$. It is required that $\mathbb{E}_{\mathbb{Q}}(\xi^2) < +\infty$ (or equivalently, $\mathbb{E}_{\mathbb{Q}}(\tilde{\xi^2}) < +\infty$ since r_t is bounded), and we look for φ in $\mathbb{H}^2_{\mathbb{Q}}$ (set of predictable processes, square integrable under $\mathbb{Q}$, that is $\mathbb{E}_{\mathbb{Q}}(\int_0^T \varphi_s^2 ds < \infty)$ (or equivalently, $\mathbb{E}_{\mathbb{Q}}(\int_0^T \tilde{\varphi}_s^2 ds < \infty)$, since σ_t, σ_t^{-1} bounded).

Theorem

If $\xi \in L^2_{\mathbb{Q}}(\mathcal{F}_T)$, $\exists!(\tilde{V}, \tilde{\varphi}) \in \mathbb{H}^2_{\mathbb{Q}} \times \mathbb{H}^2_{Q}$ *satisfying* (2).

Proof. Take

$$\tilde{V}_t = \mathbb{E}_{\mathbb{Q}}(\tilde{\xi} \mid \mathcal{F}_t).$$

By the theorem of representation of martingales under $\mathbb{Q}$ with respect to Brownian motion $\tilde{W}$,

$$\exists!(\Psi_t) \in \mathbb{H}^2_{\mathbb{Q}}, \text{ s.t. } \tilde{V}_t = \tilde{V}_0 + \int_0^t \Psi_s d\tilde{W}_s.$$

Set $\tilde{\varphi}_t = \Psi_t \sigma_t^{-1}$. We obtain $d\tilde{V}_t = \tilde{\varphi}\sigma_t d\tilde{W}_t$; $\tilde{V}_T = \tilde{\xi}$.
We get $\varphi_t = e^{\int_0^t r_s ds}\tilde{\varphi}_t$ and $V_t = e^{\int_0^t r_s ds}\tilde{V}_t$, i.e.,under $\mathbb{P}$:

$$\begin{aligned} V_t &= \mathbb{E}_{\mathbb{P}}[e^{-\int_t^T r_s ds} e^{-\int_t^T \theta_s dW_s - \int_t^T \theta_s^2 ds} \xi \mid \mathcal{F}_t] \\ &= \mathbb{E}_{\mathbb{P}}[\Gamma_{t,T}\xi \mid \mathcal{F}_t]\ ;\ 0 \le t \le T \ \text{ a .s.} \end{aligned}$$

where the process Γ, called deflator or change of numeraire, is given by

$$d\Gamma_{t,s} = \Gamma_{t,s}[-r_s ds - \theta_s dW_s]\ ; \Gamma_{t,t} = 1, \quad \text{i.e.}$$

$$\Gamma_{t,s} = \exp(-\int_t^s \theta_u dW_u + \int_t^s \{-r_u - \frac{1}{2}\theta_u^2\})du.$$

Linear pricing theory

BSDE approach

The problem can be formulated in terms of a BSDE problem. Set

$$Z_t = \varphi_t \sigma_t. \tag{3}$$

Suppose $\mathbb{E}_{\mathbb{P}}[\xi^2] < \infty$ (in $L^2_{\mathbb{P}}(\mathcal{F}_T)$). We want to find a couple of square integrable (with respect to $\mathbb{P}$) predictable processes (V, Z) such that

$$\begin{cases} dV_t = (r_t V_t + \theta_t Z_t)dt + Z_t dW_t \; ; \; 0 \leq t \leq T. \\ V_T = \xi \quad \text{a.s.} \end{cases} \tag{4}$$

Equation (4) is an example of a (linear) BSDE in the pair (V_t, Z_t) of unknown processes. If we can solve this equation, then the replicating portfolio φ_t is given by $Z_t \sigma_t^{-1}$ and V_t is the hedging price of the claim at time t.

Note that, in contrast to ordinary SDEs, Equation (4) *has two unknown processes and the terminal value V_T of V is given, not the initial value.*

Both "classical" and "BSDE" approaches are possible for the linear case, but in the case of imperfections (nonlinear wealth) , the classical approach can no more be used.

II. Imperfect financial markets

An example leading to nonlinear wealth

Different borrowing and lending interest rates, denoted respectively by R_t and r_t, adapted and bounded with $R_t \geq r_t$.

Amount borrowed at t: $(V_t - \varphi_t)^-$ (at interest rate R_t)
Amount lended at t: $(V_t - \varphi_t)^+$ (invested at bond rate r_t)
We have:

$$V_t - \varphi_t = (V_t - \varphi_t)^+ - (V_t - \varphi_t)^-$$

Self- financing property gives:

$$dV_t = (V_t - \varphi_t)^+ r_t dt - (V_t - \varphi_t)^- R_t dt + \varphi_t \frac{dS_t^1}{S_t^1}.$$

Using $(V_t - \varphi_t)^+ = (V_t - \varphi_t) + (V_t - \varphi_t)^-$, we get, by easy computations

$$dV_t = r_t V_t dt - (R_t - r_t)(V_t - \varphi_t)^- dt + \varphi_t \sigma_t \theta_t dt + \varphi_t \sigma_t dW_t$$

Consider a European option with payoff $\xi \in L^2(\mathcal{F}_T)$. Pricing and hedging this claim by constructing a replicating portfolio lead to the following problem : Find $(V, \varphi) \in \mathbb{H}^2 \times \mathbb{H}^2$ such that

$$\begin{cases} dV_t = (r_t V_t - (R_t - r_t)(V_t - \varphi_t)^-)dt + \varphi_t \sigma_t \theta_t dt + \varphi_t \sigma_t dW_t; \\ V_T = \xi. \end{cases}$$

This is an example of "nonlinear" BSDE (coefficient of dt is nonlinear in V and ϕ).

Does a solution exists ?

Nonlinear BSDEs

- $(\Omega, \mathcal{F}, P)$
- $W = (W_t)_{t \geq 0}$: d-dim Brownian motion; $W_t = (W_t^1, ..., W_t^d)^*$
- $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$ augmented with the P-null sets.

$T > 0$: terminal time

- $\mathbb{H}^2(\mathbb{R}) = \{(\phi_t)_{0 \leq t \leq T}$, real-valued predictable processes s.t. $\mathbb{E}(\int_0^T \phi_t^2 dt) < \infty\}$. We denote $\|\phi\|_{\mathbb{H}^2} = \mathbb{E}(\int_0^T \phi_t^2 dt)$.
- $\mathbb{H}^2(\mathbb{R}^d) = \{(\phi_t)_{0 \leq t \leq T}$, predictable processes s.t. $\mathbb{E}(\int_0^T |\phi_t|^2 dt) < \infty\}$.

Definition (Standard coefficients)

- $\xi \in L^2(\mathcal{F}_T)$ (the terminal condition)
- g (the driver) : $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to g(\omega, t, y, z)$ such that
  - (i) g is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ - measurable
  - (ii) $E \int_0^T g(t, 0, 0)^2 dt < \infty$
  - (iii) g is uniformly Lipschitz w.r.t. y, z, i.e.
    $\exists C > 0$ s.t. $dt \otimes dP$- a.s., $\forall y, y' \in \mathbb{R}$, $\forall z, z' \in \mathbb{R}^d$,

$$g(\omega, t, y, z) - g(\omega, t, y', z') \leq C(|y - y'| + |z - z'|).$$

Theorem (Existence and uniqueness for BSDEs)

There exists a unique solution $(Y, Z) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$ *of the BSDE associated to standard coefficients* (ξ, g), *that is satisfying:*

$$
\begin{cases} -dY_t = g(t, Y_t, Z_t)dt - Z_t^* dW_t \;\; 0 \le t \le T \\ Y_T = \xi. \end{cases} \tag{5}
$$

This result can be extended when the terminal time T is replaced by a stopping time $S \in \mathcal{T}_{0,T}$.

Proof

Suppose g does not depend on y and z, i.e. $g(\omega, t, y, z) = g(t, \omega) \in \mathbb{H}^2(\mathbb{R})$

$$(6) \qquad -dY_t = g(t)dt - Z_t^* dW_t; \;\; Y_T = \xi.$$

- *Uniqueness.* Integrating (6) between t and T, we get

  $$Y_t = \xi + \int_t^T g(s)ds - \int_t^T Z_s^* dW_s$$

  Taking $\mathbb{E}(\cdot \mid \mathcal{F}_t)$ we get

  $$(7) \qquad Y_t = \mathbb{E}[\xi + \int_t^T g(s)ds \mid \mathcal{F}_t].$$

- *Existence.* Consider the candidate Y given by (7). Then $Y_t + \int_0^t g(s)ds = \mathbb{E}[\xi + \int_0^T g(s)ds \mid \mathcal{F}_t]$. It is a martingale. By the martingale representation thm, there exists a unique Z in $\mathbb{H}^2$ s. t. $Y_t + \int_0^t g(s)ds = Y_0 + \int_0^t Z_s^* dW_s$. and we get $dY_t + g(t)dt = Z_t^* dW_t$ and $Y_T = \xi$. (Y, Z) is thus a solution in $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$ of the BSDE.

Moreover we have

$$\mathbb{E}[\sup_{0 \le t \le T} Y_t^2] < +\infty$$

i.e. $Y \in \mathcal{S}^2$.

Hint: Use Burkholder-Davis-Gundy inequalities.

Proof of existence and uniqueness of the solution of the BSDE: We define a mapping Φ as follows. Given $(U, V) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$, let

$$\Phi(U, V) = (Y, Z)$$

where (Y, Z) is the solution of the BSDE associated with driver $g(s) = g(\omega, s, U_s(\omega), V_s(\omega))$ and terminal condition ξ (which is well defined.)
We can show that Φ is a contraction (for the norm $\|\phi\|_\beta^2 := \mathbb{E}[\int_0^T e^{\beta s} \phi_s^2 ds]$ with β well chosen large enough) and thus admits an unique fixed point, which corresponds to the solution of BSDE (5). $\square$

Detailed proof on the blackboard (see also Pham's book).

Linear BSDEs

Theorem

Let α, β be bounded predictable processes, $\xi \in L^2_P(\mathcal{F}_T)$ and ϕ predictable with $\mathbb{E}_{\mathbb{P}}[\int_0^T \phi_t^2 dt] < \infty$. Then there exists a unique solution (Y, Z) of square integrable predictable processes of the linear BSDE

$$-dY_t = [\phi_t + \alpha_t Y_t + \beta_t Z_t]dt - Z_t dW_t \; ; \; 0 \leq t \leq T; \quad Y_T = \xi$$

given by

$$Y_t = \mathbb{E}[\Gamma_{t,T}\xi + \int_t^T \Gamma_{t,s}\phi_s ds \mid \mathcal{F}_t] \; ; \; 0 \leq t \leq T \;\; a.s.$$

where the process Γ, usually called adjoint process, is given by

$$d\Gamma_{t,s} = \Gamma_{t,s}[\alpha_s ds + \beta_s dW_s] \; ; \Gamma_{t,t} = 1, \quad i.e.$$

$$\Gamma_{t,s} = \exp(\int_t^s \beta_u dW_u + \int_t^s \{\alpha_u - \frac{1}{2}\beta_u^2\})du.$$

Sketch of Proof:

- Apply Itô formula to $Y_s \Gamma_{t,s}$
- we obtain $Y_s \Gamma_{t,s} + \int_t^s \Gamma_{t,r} \phi_r dr$ is a martingale. Hence,

$$Y_t = \mathbb{E}[\Gamma_{t,T} \xi + \int_t^T \Gamma_{t,s} \phi_s ds \mid \mathcal{F}_t] \; ; \; 0 \leq t \leq T \;\text{ a .s.}$$

$\square$

Remark: Note that if $\xi \geq 0$ and $\phi \geq 0$, then $Y_t \geq 0$.

Application to the replication portfolio problem in the linear case:

$$\begin{cases} -dY_t = (-r_t Y_t - \theta_t Z_t)dt - Z_t dW_t \; ; \; 0 \le t \le T. \\ Y_T = \xi \;\text{ a.s.} \end{cases}$$

In this case, we have $\phi_t = 0$, $\alpha_t = -r_t$, $\beta_t = -\theta_t$. This leads to the classical quasi explicit expression for the wealth Y_t (value of replicating portfolio):

$$Y_t = \mathbb{E}[\Gamma_{t,T}\xi \mid \mathcal{F}_t],$$

where $\Gamma_{t,s}$ is the *deflator* process, started at t, such that

$$\Gamma_{t,s} = \exp\{-\int_t^s r_u du\} \exp\{-\int_t^s \theta_u dW_u - \frac{1}{2}\int_t^s \theta_u^2 du\}.$$

This defines a *linear* price system $Y : L_P^2 \to \mathbb{H}_P^2$; $\xi \mapsto Y(\xi)$, which is increasing and corresponds to the classical free-arbitrage price system. $Y(\xi)$ is called the *hedging price* of ξ.

Note that in particular if $\xi \ge 0$ a.s. then the price $Y_t \ge 0$.

Linear BSDEs were first introduced by Bismut 1973 as the adjoint equation associated with the stochastic Pontryagin maximum principle for stochastic control.

Theorem (Comparison theorem for BSDEs)

Let (g^1, ξ^1), (g^2, ξ^2) *be standard coefficients and* (Y^1, Z^1), (Y^2, Z^2) *be the solutions of the associated BSDEs. Assume*

- $\xi^1 \geq \xi^2$
- $g^1(t, Y_t^2, Z_t^2) \geq g^2(t, Y_t^2, Z_t^2)$

Then $Y_t^1 \geq Y_t^2$, $0 \leq t \leq T$ *a.s.*

Proof: Set $\bar{Y} = Y^1 - Y^2$, $\bar{Z} = Z^1 - Z^2$ and take $d = 1$ to simplify.

By difference of the 2 BSDEs,

$$\begin{cases} -d\bar{Y}_t = \{g^1(t, Y_t^1, Z_t^1) - g^2(t, Y_t^2, Z_t^2)\}dt - \bar{Z}_t dW_t \\ \bar{Y}_T = \xi^1 - \ \xi^2. \end{cases}$$

Now

$$\begin{aligned} & g^1(t, Y_t^1, Z_t^1) - g^2(t, Y_t^2, Z_t^2) = \\ & \{g^1(t, Y_t^1, Z_t^1) - g^1(t, Y_t^2, Z_t^1)\} + \{g^1(t, Y_t^2, Z_t^1) - g^1(t, Y_t^2, Z_t^2)\} \\ & + \{g^1(t, Y_t^2, Z_t^2) - g^2(t, Y_t^2, Z_t^2)\} \\ & = \Delta_y g^1(t)(Y_t^1 - Y_t^2) + \Delta_z g^1(t)(Z_t^1 - Z_t^2) + \phi_t \end{aligned}$$

where

$$\Delta_y g^1(t) := \frac{g^1(t, Y_t^1, Z_t^1) - g^1(t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2} \text{ if } Y_t^1 - Y_t^2 \neq 0;\ 0 \text{ otherwise}$$

$$\Delta_z g^1(t) := \frac{g^1(t, Y_t^2, Z_t^1) - g^1(t, Y_t^2, Z_t^2)}{Z_t^1 - Z_t^2} \text{ if } Z_t^1 - Z_t^2 \neq 0;\ 0 \text{ otherwise}$$

$$\phi_t := g^1(t, Y_t^2, Z_t^2) - g^2(t, Y_t^2, Z_t^2) \geq 0$$

So

$$
\begin{cases} -d\bar{Y}_t = \{\Delta_y g^1(t)\bar{Y}_t + \Delta_z g^1(t)\bar{Z}_t + \phi_t\}dt - \bar{Z}_t dW_t \\ \bar{Y}_T = \xi^1 - \xi^2 \end{cases}
$$

This is a linear BSDE. Note that $\Delta_y g^1(t)$ and $\Delta_z g^1(t)$ are bounded since g is Lipschitz.

Since $\phi_t \geq 0$ and $\xi^1 - \xi^2 \geq 0$ we get that $\bar{Y}_t \geq 0$, $0 \leq t \leq T$ a.s.
$\square$

Theorem (Strict Comparison theorem for BSDEs)

Let (g^1, ξ^1), (g^2, ξ^2) be standard coefficients and (Y^1, Z^1), (Y^2, Z^2) be the solutions of the associated BSDEs. Assume

- $\xi^1 \geq \xi^2$ *a.s.*
- $g^1(s, Y^2_s, Z^2_s) \geq g^2(s, Y^2_s, Z^2_s)$ *a.s* $0 \leq s \leq T$
- $Y^1_t = Y^2_t$ *on* $A \in \mathcal{F}_t$ *(for fixed* t *in* $[0, T]$*)*

Then $\xi^1 = \xi^2$ *on* A *and* $g^1(s, Y^2_s, Z^2_s) = g^2(s, Y^2_s, Z^2_s)$ *a.s on* $A \times [t, T]$

Corollary (Sufficient condition of positivity)

If $\xi \geq 0$ a .s. and $g(t,0,0) \geq 0$ a.s., then $Y_t \geq 0$, $0 \leq t \leq T$ a.s.

Proof. We apply the comparison theorem to $(g^1 = g, \xi^1 = \xi)$, $(g^2 = 0, \xi^2 = 0)$.

Solution associated to (g^1, ξ^1) : $(Y^1, Z^1) = (Y, Z)$.
Solution associated to $(0,0)$: $(Y^2, Z^2) = (0,0)$.

$\square$

Application to the example of higher interest rate for borrowing

Consider an option with payoff $\xi \in L^2(\mathcal{F}_T)$ in this market. Recall that the pricing/hedginp problem of European option with payoff ξ leads to the BSDE: find (V, φ) such that

$$dV_t = (r_t V_t - (R_t - r_t)(V_t - \varphi_t)^-)dt + \varphi_t \sigma_t \theta dt + \varphi_t \sigma_t dW_t;\, V_T = \xi.$$

or equivalently, setting $Z_t = \varphi_t \sigma_t$,

$$-dV_t = g(t, V_t, Z_t)dt - Z_t dW_t,\ V_T = \xi.$$

with

$$g(t, y, z) = -r_t y + (R_t - r_t)(y - \frac{z}{\sigma})^- + \theta_t z$$

By the thm of existence and uniqueness, there exists a unique solution (V, Z) of this BSDE in $\mathbb{H}^2 \times \mathbb{H}^2$ with nonlinear driver g.
Note that $g(t, 0, 0) = 0$ in this example.
By the corollary, we obtain that if $\xi \geq 0$ a.s, then
$V_t \geq 0,\ \ 0 \leq t \leq T$ a.s.

Other examples of imperfections leading to nonlinear dynamics:

$$-dV_t = g(t, V_t, \varphi_t \sigma_t)dt - \varphi_t \sigma_t dW_t$$

- Large investor with trading strategy φ_t affecting the market prices: $r_t(\omega) = \bar{r}(t, \omega, \varphi_t)$ and similarly for σ, θ.

$$g(t, V_t, \varphi_t \sigma_t) = -\bar{r}(t, \varphi_t) V_t - \varphi_t \, (\bar{\theta}\bar{\sigma})(t, \varphi_t).$$

- Taxes on the profits on risky investments.

$$g(t, V_t, \varphi_t \sigma_t) = -(r_t V_t + \varphi_t \theta_t \sigma_t) + \rho \varphi_t^+.$$

Here, $\rho \in]0, 1[$ represents an instantaneous tax coefficient on the gains made from the investments in the risky assets .

Imperfect financial markets

More generally, when there are market imperfections captured by the nonlinearity of the wealth process

$$-dV_t = g(t, V_t, \sigma_t \varphi_t)dt - \sigma_t \varphi_t dW_t;$$

with driver $g(t, y, z)$ uniformly Lipschitz with respect to y and z, the theory of BSDEs implies that given a claim $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution (Y, φ) with Y and φ in $\mathbb{H}^2(\mathbb{R})$ (i.e. predictable and square integrable) such that

$$-dY_t = g(t, Y_t, \sigma_t \varphi_t)dt - \sigma_t \varphi_t dW_t; \quad Y_T = \xi.$$

Y_t is the price of the option at time t and φ_t is the hedging portfolio.

Nonlinear pricing system $\mathcal{E}^g$

The operator $\mathcal{E}^g : (\xi, T) \mapsto Y_\cdot(\xi, T)$, where $(Y(\xi, T), Z(\xi, T))$ is solution of the associated BSDE with coefficients (ξ, g) and terminal time T was first introduced in El-Karoui-Quenez '96 and called *nonlinear pricing system*, and later called *g-conditional expectation* in Peng'2004.

This notion can be extended to the case where T is replaced by a stopping time $\tau \in \mathcal{T}_0$ and ξ by a random variable $\eta \in L^2(\mathcal{F}_\tau)$.

Definition

The g-conditional expectation $\mathcal{E}^g$ is defined for all **stopping time** $\tau \in \mathcal{T}_0$ and all $\eta \in L^2(\mathcal{F}_\tau)$ by

$$\mathcal{E}^g_{t,\tau}(\eta) := Y_t(\eta, \tau); \quad 0 \leq t \leq T,$$

where $(Y(\eta, \tau), Z(\eta, \tau))$ is the solution of the BSDE associated with driver g, terminal time τ and terminal condition η.

In the case of a perfect market, $g(t, y, z) = -r_t y - \theta_t z$ (linear) and the pricing system is linear, i.e.

$$\forall \alpha, \beta \in \mathbb{R}, \mathcal{E}^g_{t,T}(\alpha\xi^1 + \beta\xi^2) = \alpha\mathcal{E}^g_{t,T}(\xi^1) + \beta\mathcal{E}^g_{t,T}(\xi^2)$$

Properties of the nonlinear pricing system (or g-evaluation)

- The price system is **increasing**: if $\xi^1 \geq \xi^2$ a.s. then $\mathcal{E}_{t,T}(\xi^1) \geq \mathcal{E}_{t,T}(\xi^2)$ a.s.
  (By comparison thm)

- if $g(t,0,0) = 0$ a.s. on $[0,T] \times \Omega$, then $\mathcal{E}_{t,T}(0) = 0 \; \forall t$.
  (i.e. the price of the null option is 0)
  (By uniqueness of the solution: $(0,0)$ is solution and it is the only one)
  Moreover the price system is positive, i.e. if $\xi \geq 0$ a.s. then $\mathcal{E}_{t,T}(\xi) \geq 0$.
  (By comparison thm)

- If $g(t,0,0) = 0$ a.s. then the pricing system satifies the **non arbitrage property** , i.e. if for t, $A \in \mathcal{F}_t$ given, if
  - if $\xi_1 \geq \xi^2$ a.s. on A
  - and $\mathcal{E}_{t,T}(\xi^1) = \mathcal{E}_{t,T}(\xi^2)$ a.s. on $A \in \mathcal{F}_t$

  Then $\xi_1 = \xi^2$ a.s. on A
  Proof: Apply strict comparison thm with $g^1 = g^2 = g$.

- if g is convex w.r.t. y, z, then the price system is convex, i.e. $\forall \alpha \in [0,1]$, $\forall \xi^1, \xi^2 \in L^2(\mathcal{F}_T)$,

$$\mathcal{E}_{t,T}(\alpha\xi^1 + (1-\alpha)\xi^2) \leq \alpha\mathcal{E}_{t,T}(\xi^1) + (1-\alpha)\mathcal{E}_{t,T}(\xi^2)$$

(By comparison thm)

- The price system is **time-consistent**, i.e. if $S \leq T$, then

$$\mathcal{E}_{t,T}(\xi) = \mathcal{E}_{t,S}(\mathcal{E}_{S,T}(\xi)) \quad 0 \leq t \leq S$$

  This corresponds to the flow property of BSDEs
  (also holds if S is a stopping time $\leq T$).

  In other words, $\forall s \in [0, T]$, $\forall \xi \in L^2(\mathcal{F}_s)$, the g-evaluation $\mathcal{E}^g_{t,s}(\xi)$ is an $\mathcal{E}^g$-martingale.

  A continuous adapted process X_t in $\mathcal{S}^2$ is said to be an $\mathcal{E}$-martingale if $\mathcal{E}_{\sigma,\tau}(X_\tau) = X_\sigma$ a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.

A brief introduction to risk measures

Static framework

We assume that uncertainty is described by a measurable space $(\Omega, \mathcal{F})$ of possible scenarios, and that risky positions belong to a linear space $\mathcal{X}$.

Let $\mathcal{A} \subset \mathcal{X}$: Subset of "acceptable" positions. We require that $\mathcal{A}$ contains all constants and that

$$X \in \mathcal{A} \text{ whenever } X \geq Y \text{ for some } Y \in \mathcal{A}.$$

We define the monetary risk measure by

$$\rho(X) : \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}$$

The functional ρ satisfies:

1. $\rho(X + m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and all constants m. (*Translation invariance*)
2. If $X_1 \leq X_2$ then $\rho(X_1) \geq \rho(X_2)$. (*Monotonicity*)

Interpretation: $\rho(X)$ is the amount to be added to X to make it "acceptable", since by translation invariance, $\rho(X + \rho(X)) = 0$.

Example: *Value at Risk* (VaR)
Suppose that a probability measure is given on the set Ω of all scenarios.

A position is acceptable for the VaR if the probability of loss $P[X < 0]$ is below some level β:

$$VaR(X) := \inf\{m \in \mathbb{R} \mid P[X + m < 0] \leq \beta\}$$

It is thus a β-quantile of the distribution of X under P.

VaR is widely used in practice but it is not convex; however convexity is a good property for a risk measure since it means that diversification should not increase the risk.

This has motivated an axiomatic approach for a general theory of monetary risk measures called

- "coherent" ($\mathcal{A}$ is a cone),
  Arzner et al. '99
- then extended to the convex case ($\mathcal{A}$ convex)
  For all $\lambda \in [0,1]$ and all $X_1, X_2 \in \mathcal{X}$.

$$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$$

(Föllmer & Schied '02, Frittelli & Rosazza-Gianin '02)

Example: *the entropic risk measure*

$$\rho(X) = \frac{1}{\gamma} \ln E_P[\exp(-\gamma X)] = \sup_{Q \in \mathcal{Q}} \{E_Q[-X] - \frac{1}{\gamma} H(Q \mid P)\}$$

where $H(Q \mid P)$ is the relative entropy of Q with respect to P defined by:

$$H(Q \mid P) = \begin{cases} E_Q[\ln \frac{dQ}{dP}] \text{ if } Q \text{ admits a density w.r.t. } P \\ +\infty \text{ otherwise} \end{cases}$$

Dual Representation:

Convex dualiy theory implies that convex risk measures are typically of the form:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \{E_Q[-X] - \alpha(Q)\}$$

where $\mathcal{Q}$ is a set of probability measures and α is a penalty fonction which can take the value $+\infty$.

The penalty term vanishes in the special case of coherent risk measures.

Interpretation in terms of *model uncertainty* : "Worst case"

Dynamic risk measures and relations to BSDEs

Extension to the dynamic case.
Additional axioms are required to get *time-consistency* ;
$\rightarrow$ Natural link with BSDEs.

In the Brownian case, cf e.g. P. Barrieu - N. El Karoui 2004).

in the case with jumps , see M.C. Quenez, A.S. SPA'13

Dynamic risk measures associated to BSDEs

Let g be a standard Lipschitz driver. For each $T > 0$ and $\xi \in L^2(\mathcal{F}_T)$, set

$$\rho_t(\xi, T) := -\mathcal{E}^g_{t,T}(\xi), \quad 0 \leq t \leq T,$$

If T represents a given maturity and ξ a financial position at time T, then $\rho_t(\xi, T)$ is interpreted as the risk measure of ξ at time t.

The functional $\rho : (\xi, T) \mapsto \rho_\cdot(\xi, T)$ defines then a *dynamic risk measure* induced by the BSDE with driver g.

- $\rho_t(\cdot, T)$ is **monotonous** (nonincreasing with respect to financial position ξ:) if $\xi^1 \geq \xi^2$ then $\rho_t(\xi^1, T) \leq \rho_t(\xi^2, T)$ a.s.
- ρ satisfies the **nonarbitrage** property: If $\xi^1 \geq \xi^2$ and if $\rho_t(\xi^1, T) = \rho_t(\xi^2, T)$ a.s. on an event $A \in \mathcal{F}_t$, then $\xi^1 = \xi^2$ a.s. on A.
- If g is concave w.r.t. y, z, then $\rho_t(\cdot, T)$ is convex.
- If g only depends on z (not on y), then ρ is invariant by translation (*monetary risk measure*), i.e. $\rho_t(\xi + \xi', T) = \rho_t(\xi, T) - \xi', \ \forall \xi \in L^2(\mathcal{F}_T), \xi' \in L^2(\mathcal{F}_t)$.

Example: *the dynamic entropic risk measure*

$$\rho_t(\xi, T) := \frac{1}{\gamma} \ln \mathbb{E}[\exp(-\gamma \xi) \mid \mathcal{F}_t]$$

is associated to the BSDE with driver $g(t, y, z) := \frac{1}{2}\gamma z^2$.

Recursive utility

(Duffie & Epstein (1992), Epstein & Zin (1989), Kreps & Porteus (1978)).

Let $g(t,y,c)$ be an $\mathcal{F}_t$-adapted process. Assume that $c \to g(t,y,c)$ is concave for all t,y. The *recursive utility* process of a given consumption process $c(\cdot) \geq 0$ is defined as the solution Y of the equation

$$Y_t = \mathbb{E}\left[\int_t^T g(s, Y_s, c_s))ds \mid \mathcal{F}_t\right] \; ; \; 0 \leq t \leq T. \tag{8}$$

This equation is equivalent to the following BSDE in (Y,Z):

$$\begin{cases} -dY_t = g(t, Y_t, c_t)dt - Z_t dB_t \\ Y_T = 0. \end{cases} \tag{9}$$

The Markovian case - Links of BSDEs with PDEs

Let $(t, x) \in [0, T] \times \mathbf{R}^p$. Let $(S_s^{t,x}, t \leq s \leq T)$ the solution of the forward SDE:

$$\begin{cases} dS_s = b(s, S_s)ds + \sigma(s, S_s)dW_s, \quad t \leq s \leq T \\ S_t = x \end{cases}$$

where $b : [0, T] \times \mathbf{R}^p \to \mathbf{R}^p$ and $\sigma : [0, T] \times \mathbf{R}^p \to \mathbf{R}^{p \times d}$ are Borelian Lipschitz w.r.t. x.
Let $(Y_s^{t,x}, Z_s^{t,x})$ be the solution of the BSDE

$$(10) \qquad \begin{cases} -dY_s = g(t, S_s^{t,x}, Y_s, Z_s)dt - Z_s^* dW_s \quad t \leq s \leq T \\ Y_T = \Psi(S_T^{t,x}), \end{cases}$$

where $\Psi : \mathbf{R}^p \to \mathbf{R}$, g is Lipschitz w.r.t. (y, z) with constant C (uniformly wrt (t, s)), and b, σ, g, Ψ have linear growth.

$Y_t^{t,x}$ is a deterministic function of (t, x), denoted by $u(t, x)$.

Theorem (Generalized Feyman-Kac representation)

Let v a function of class $C^{1,2}$ such that

$$|v(t,x)| + |\sigma(t,x)^*\partial_x v(t,x)| \leq C(1+|x|),$$

solution of the PDE

$$\begin{cases} \partial_t v(t,x) + \mathcal{L}v(t,x) + g(t,x,v(t,x),\sigma(t,x)^*\partial_x v(t,x)) = 0 \\ v(T,x) = \Psi(x) \end{cases}$$

where $\partial_x v$ denotes the gradient of v and $\mathcal{L}_{t,x}$ is the infinitesimal generator of S_s:

$$\mathcal{L}_{t,x} := \sum_i b_i(t,x)\partial_{x_i} + \sum_{i,j} a_{i,j}(t,x)\partial^2_{x_i x_j}$$

with $a_{ij} = \frac{1}{2}(\sigma\sigma^*)_{ij}$. Then

$$v(t,x) = Y_t^{t,x}$$

where $(Y^{t,x}, Z^{t,x})$ is the solution of the BSDE (10)

Moreover

$$Y_s^{t,x} = v(s, S_s^{t,x}) \;\; s \geq t$$
$$Z_s^{t,x} = \sigma(s, S_s^{t,x})\partial_x v(s, S_s^{t,x}) \;\; s \geq t.$$

Proof. Apply Itô to $v(s, S_s^{t,x})$. We get :

$$dv(s, S_s^{t,x}) = \{\partial_t v(s, S_s) + \mathcal{L}v(s, S_s)\}ds + \partial_x v(s, S_s)^* \sigma(s, S_s)dW_s$$

Since v satisfies the PDE, the term in ds above is equal to $-g(t, S_s, v(s, S_s), \; \sigma(s, S_s)^* \partial_x v(s, S_s))$.

Moreover $v(T, S_T) = \Psi(S_T)$ (terminal condition).
So $(v(s, S_s^{t,x}), \; \sigma^*(s, S_s^{t,x})\partial_x v(s, S_s^{t,x}))$ is solution of the BSDE (10). By uniqueness it is equal a.s. to (Y_s, Z_s). □

reference: Pardoux-Peng: BSDEs and quasi-linear PDEs 1992

This is a "verification" theorem.

We have also a theorem which states that the function $u(t,x) := Y_t^{t,x}$ is a **viscosity** solution of the PDE. (El Karoui-Peng-Quenez 97).

III. Nonlinear pricing of American Contingent claims

Recall that an American option gives the right to the owner to exercise the option at any time before maturity.

Consider the imperfect market $\mathcal{M}^g$, when under nonlinear constraints, the strategy (wealth V, portfolio φ) satisfies:

$$-dV_t = g(t, V_t, \sigma_t^* \varphi_t) dt - \sigma_t^* \varphi_t dW_t.$$

where g is nonlinear (Lipschitz).

Consider an American option associated with horizon $T > 0$ and a payoff given by a continuous process $(\xi_t, 0 \le t \le T)$.
At time 0, it consists in the selection of a stopping time $\nu \in \mathcal{T}$ leading to the payment of the payoff ξ_ν from the seller to the buyer.

We have seen that for an European option with payoff ξ and maturity T, its initial *hedging price* is given by $\mathcal{E}^g_{0,T}(\xi)$.

A "natural" price for the American option is the quantity

$$\sup_{\nu \in \mathcal{T}} \mathcal{E}^g_{0,\nu}(\xi_\nu), \tag{11}$$

which we call the *g-value* of the American option (*first introduced in NEK-Quenez'96 and called "fair value"*).

Main results:

1. **The g-value $\sup_{\nu \in \mathcal{T}} \mathcal{E}^g_{0,\nu}(\xi_\nu)$ of the American option with payoff ξ coincides with the solution of an associated reflected BSDE with barrier ξ.**
2. **It is equal to the superhedging price u_0 of the American option**, (i.e. the minimal initial capital which enables the seller to invest in a portfolio which will cover his liability to pay to the buyer no matter the exercise time of the buyer).

Mathematical tools: theory of (nonlinear) **reflected** BSDEs.

Reflected BSDEs

Reflected BSDEs have been introduced by El Karoui et al. (1997).

The solution of such a BSDE is forced to stay above a given process, called obstacle (or barrier). An increasing process is introduced to push the solution upwards in a minimal way so that it remains above the obstacle.

Reflected BSDEs

Given:

- a standard driver $g(t, y, z)$
- an obstacle $(\xi_t)_{,0\leq t\leq T}$, continuous on $[0, T[$, adapted, belonging to $\mathcal{S}^2$, i.e. $\mathbb{E}(\sup_t |\xi_t|^2) < \infty$ and satisfying $\lim_{t\to T, t<T} \xi_t \leq \xi_T$.

A solution of the reflected BSDE associated with driver g and obstacle $\xi_.$ consists in a triplet (Y, Z, K) in $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R})$ such that

(12)

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)dt + dK_t - Z_t dW_t; \\ Y_T = \xi_T \\ Y_t \geq \xi_t,\ 0 \leq t \leq T \text{ a.s.}, \\ K \text{ is a nondecreasing continuous predictable process with } K_0 = 0 \text{ and} \\ \int_0^T (Y_t - \xi_t)dK_t = 0\ \ (\text{ i.e. } t \mapsto K_t \text{ increases only on}\{t, Y_t = \xi_t\}) \end{cases}$$

The process K is the minimal push which allows the solution to stay above the obstacle.

Reflected BSDEs

The special case of a driver process

Suppose g does not depend on y, z, k, that is $g(\omega, t, y, z) = g(\omega, t)$, where g is a process in $\mathbb{H}^2$.

Theorem (Link with classical optimal stopping problems)

If (Y, Z, K) satisfies the reflected BSDE associated with driver $g = g(\omega, t)$ and obstacle ξ, then

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[\xi_\tau + \int_t^\tau g(s)ds \mid \mathcal{F}_t] \quad a.s.$$

and the supremum is attained at

$$D_t = \inf\{s \geq t; Y_s = \xi_s\}.$$

Proof. *on blackboard* $\square$

Theorem (Existence and Uniqueness)

The reflected BSDE associated with driver $g = g(\omega, t)$ and obstacle ξ admits a unique solution (Y, Z, K).

Proof.

- Uniqueness.
  It comes from the previous characterization theorem.
- Existence.
  The proof is based on Doob-Meyer decomposition for super martingales, the theorem of martingale representation and the optimal stopping theory. (*Proof on the blackboard*).

$\square$

Theorem (Existence and Uniqueness)

Let $g(t, y, z)$ and $\xi_.$ be a couple of standard parameters. Then, the associated reflected BSDE admits a unique solution (Y, Z, K).

Proof. Define a mapping Φ as follows. Given $(U, V) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$, let $(Y, Z) = \Phi(U, V)$ where (Y, Z, K), with

$$K_t = -Y_t + Y_0 - \int_0^t g(s, U_s, V_s)\, ds + \int_0^t Z_s^* dW_s,\ 0 \leq t \leq T,$$

is the solution of the Reflected BSDE associated with driver $g(s) = g(s, U_s, V_s)$ and obstacle ξ (which is well defined from previous proposition)
Prove that Φ is a contraction in the Banach space $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^d)$ equipped with the norm $\|\cdot\|_\beta$ (with β well chosen) and thus admits an unique fixed point, which corresponds to the solution of RBSDE (12).
Detailed proof on the blackboard. $\square$

Theorem (Characterization)

Let (Y, Z, K) be the solution of the reflected BSDE. Then

$$Y_t = \operatorname*{ess\ sup}_{\tau \in \mathcal{T}_{t,T}} \mathcal{E}^g_{t,\tau}(\xi_\tau)$$

and the supremum is attained at

$$D_t := \inf\{s \geq t;\, Y_s = \xi_s\},$$

that is

$$Y_t = \mathcal{E}_{t,D_t}(\xi_{D_t}).$$

Proof. Let

$$D_t := \inf\{s \geq t; Y_s = \xi_s\}.$$

On $[t, D_t]$, the solution $Y_t > \xi_t$, so (K_t) remains constant and $(Y, Z) \equiv$ the solution of the BSDE with driver g, terminal time D_t and terminal condition $\xi_{D_t}(= Y_{D_t}$ by continuity of Y and $\xi)$. In other terms,

$$Y_t = \mathcal{E}_{t,D_t}(\xi_{D_t}).$$

It remains to prove that for each $\tau \in \mathcal{T}_{t,T}$, we have: $Y_t \geq \mathcal{E}_{t,\tau}(\xi_\tau)$. Set $X'_t := \mathcal{E}_{t,\tau}(\xi_\tau)$; It satisfies the BSDE

$$-dX'_s = g(s, X'_s, \pi'_s)ds - \pi'^*_s dW_s, t \leq s \leq \tau \ \ ; X'_\tau = \xi_\tau$$

Note that on $[t, \tau]$, the pair (Y, Z) satisfies

$$-dY_s = g(s, Y_s, Z_s)ds + dK_s - Z^*_s dW_s; \ \ Y_\tau = Y_\tau.$$

In other terms, (Y, Z) is a solution of the BSDE associated with terminal time τ, terminal condition Y_τ and driver $g(s, \cdot, \cdot) + \frac{dK_s}{ds}$. But we have : $g(s, y, z)ds + dK_s \geq g(s, y, z)$ and $Y\tau \geq \xi_\tau$. The comparison thm for BSDEs implies $Y_t \geq \mathcal{E}_{t,\tau}(\xi_\tau)$. $\square$

Theorem (Comparison theorem for reflected BSDEs)

Let (f^1, ξ^1), (f^2, ξ^2) *be standard coefficients and* (Y^1, Z^1, K^1), (Y^2, Z^2, K^2) *be the solutions of the associated reflected BSDEs. Assume*

- $\xi^1_t \geq \xi^2_t$, $0 \leq T$, *a.s.*
- $f^1(t, y, z \geq f^2(t, y, z) \quad \forall t, y, z$

Then $Y^1_. \geq Y^2_.$, $\mathbb{P}$*-a.s.*

Proof. By the comparison thm for BSDEs, for all stopping time $\tau \geq t$, we have:

$$\mathcal{E}^1_{t,\tau}(\xi^1_\tau) \geq \mathcal{E}^2_{t,\tau}(\xi^2_\tau).$$

Taking the supremum over τ, we get:

$$Y^1_t \geq Y^2_t.$$

□

Theorem (Optimality criterium)

We have the following equivalence:
(i) τ^* *is t-optimal , i.e.*

$$Y_t = \underset{\tau \in \mathcal{T}_{t,T}}{ess\ sup}\, \mathcal{E}^g_{t,\tau}(\xi_\tau) = \mathcal{E}_{t,\tau^*}(\xi_{\tau^*})$$

(ii) $dK_s = 0$, $t \leq s \leq \tau^*$ *and* $Y^*_\tau = \xi_{\tau^*}$.
In other terms $(Y_s, t \leq s \leq \tau^*)$ *is solution of the BSDE associated with the terminal condition* ξ_{τ^*}.

Proof. (ii) $\Rightarrow$ (i): clear
(i) $\Rightarrow$ (ii) application of the strict comparison thm (detailed proof on the blackboard) $\square$

Application to American option pricing

Under nonlinear constraints, the wealth is supposed to satisfy:

$$-dV_t = g(t, V_t, \sigma_t^* \pi_t)dt - \sigma_t^* \pi_t dW_t.$$

If the exercise time $\tau \in \mathcal{T}_{t,T}$ is fixed, then the American option is a European option with maturity τ and payoff ξ_τ. The price of this European option is given by $\mathcal{E}_{t,\tau}(\xi_\tau)$.

Definition

The g-value of the American option at time t is defined by the $\mathcal{F}_t$-measurable random variable:

$$Y_t = \underset{\tau \in \mathcal{T}_{t,T}}{ess\ sup}\, \mathcal{E}^g_{t,\tau}(\xi_\tau) \quad \text{a.s.}$$

Using the previous theory we have that Y is the solution of the reflected BSDE assciated to driver g and obstacle ξ.
More precisely,

Proposition: $\exists$ Z in $\mathbb{H}(\mathbb{R}^d)$, $\exists$ K continuous, increasing process, such that

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)dt + dK_t - Z_t^* dW_t \\ Y_T = \xi_T \\ Y_t \geq \xi_t \;\; 0 \leq t \leq T \quad \text{a.s.} \\ \int_0^T (Y_s - \xi_s) dK_s = 0 \end{cases}$$

Moreover, $D_t := \inf\{s \geq t; Y_s = \xi_s\}$ is optimal i.e. $Y_t = \mathcal{E}_{t,D_t}(\xi_{D_t})$.

Theorem

The g-value $Y_0 := \sup_{\nu \in \mathcal{T}} \mathcal{E}^g_{0,\nu}(\xi_\nu)$ of the American option with payoff ξ satisfies

$$Y_0 = u_0$$

where u_0 is the superhedging price of the American option defined as the lowest price which allows the seller to be superhedged, i.e.

$$u_0 = \inf \mathcal{H}$$

where

$$\mathcal{H} = \{x \in \mathbf{R},\ \exists \varphi \in \mathbb{H}^2 \ s.t\ .\ V_t^{x,\varphi} \geq \xi_t \ a.s.\ 0 \leq t \leq T\}$$

Proof:

1. $Y_0 \leq u_0$.

Let $x \in \mathcal{H}$. Let us show that $Y_0 \leq x$.
For all $\tau \in \mathcal{T}_0$, we have: $\xi_\tau \leq V_\tau^{x,\varphi}$.
So by comparison thm, $\mathcal{E}_{0,\tau}(\xi_\tau) \leq \mathcal{E}_{0,\tau}(V_\tau^{x,\varphi}) = V_0^{x,\varphi} = x$.
Taking supremum over τ, we get $Y_0 = \sup_{\tau \in \mathcal{T}_0} \mathcal{E}_{0,\tau}(\xi_\tau) \leq x$.

2. $u_0 \leq Y_0$

Consider $V_t^{Y_0,\varphi}$ the wealth associated to portfolio $\varphi_t = (\sigma_t^*)^{-1} Z_t$ and initial value Y_0. It satisfies:

$$V_t = Y_0 - \int_0^t g(s, Y_s, Z_s) ds + \int_0^t Z_s^* dW_s$$

and we have

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s) ds + \int_0^t Z_s^* dW_s - K_t.$$

We can prove by using a comparison argument for forward differential equations , that

$$V_t^{Y_0,\varphi} \geq Y_t.$$

Now

$$Y_t \geq \xi_t,$$

So $V_t^{Y_0,\varphi} \geq \xi_t$, for all $0 \leq t \leq T$.
So Y_0 belongs to $\mathcal{H}$.

$\square$

Relation with PDEs (Variational inequalities)

We now focus on the Markovian case.
Let b and σ be continuous mappings, globally Lipschitz. For each $(t,x) \in [0,T] \times \mathbb{R}$, let $\{X_s^{t,x}, t \leq s \leq T\}$ the solution of

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r$$

We consider the Reflected BSDE associated driver $f(s, X_s^{t,x}, .)$ and obstacle $\xi^{t,x}$ of the following form:

$$\begin{cases} \xi_s^{t,x} := h(s, X_s^{t,x}), \;\; s < T \\ \xi_T^{t,x} := g(X_T^{t,x}) \end{cases}$$

where $g \in \mathcal{C}(\mathbb{R})$ has at most polynomial growth at infinity, $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is jointly continuous in t and x and there exist $p \in \mathbb{N}$ and $C > 0$, such that $|h(t,x)| \leq C(1 + |x|^p), \forall t, x$ and $h(T,x) \leq g(x), \;\; \forall x$.

For each $(t,x) \in [0,T] \times \mathbb{R}$, there exists a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{S}^2$ of adapted processes, which solves the Reflected BSDE:

(13)

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \displaystyle\int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} \\ \qquad - \displaystyle\int_s^T Z_r^{t,x} dW_r \\ Y_s^{t,x} \geq \xi_s^{t,x}, 0 \leq s \leq T \quad \text{a.s.} \ , \\ K^{t,x} \text{ is a nondecreasing, continuous predictable process with} \\ \qquad\qquad K_t^{t,x} = 0 \text{ and such that} \\ \displaystyle\int_t^T (Y_s^{t,x} - \xi_s^{t,x}) dK_s^{t,x} = 0 \text{ a.s.} \end{cases}$$

Theorem

The function u, defined by

$$u(t,x) := Y_t^{t,x}, \quad t \in [0,T],\ x \in \mathbb{R}$$

is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the following PDE with obstacle

(14)

$$\begin{cases} \min(u(t,x) - h(t,x), \\ -\frac{\partial u}{\partial t}(t,x) - Lu(t,x) - f(t,x,u(t,x),(\sigma\frac{\partial u}{\partial x})(t,x)) = 0, (t,x) \in [0,T) \\ u(T,x) = g(x), x \in \mathbb{R} \end{cases}$$

where

$$L\phi(x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2\phi}{\partial x^2}(x) + b(x)\frac{\partial\phi}{\partial x}(x).$$

Definition

• A continuous function u is said to be a *viscosity subsolution* of (14) if $u(T,x) \leq g(x), x \in \mathbb{R}$, and if for any point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for any $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its minimum at (t_0, x_0), we have

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\ &-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t,x))(t_0,x_0)) \leq 0.\end{aligned}$$

In other words, if $u(t_0, x_0) > h(t_0, x_0)$,

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t,x))(t_0,x_0)) \leq 0.$$

• A continuous function u is said to be a *viscosity supersolution* of (14) if $u(T,x) \geq g(x), x \in \mathbb{R}$, and if for any point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for any $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its maximum at (t_0, x_0), we have

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\ &-\frac{\partial}{\partial t}\phi(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0)) \geq 0.\end{aligned}$$

Under some additional hypothesis on f, there exists a unique solution of the obstacle problem (27) in the class of continuous functions with polynomial growth.

IV. Nonlinear pricing of Game options
Definition

- **Game options** are derivative contracts that can be terminated by both counterparties at any time before maturity T (introduced by Kifer '2000).
- extend the setup of American options by allowing the seller to cancel the contract.

More precisely

- If the buyer exercises at time τ, he gets ξ_τ from the seller,
- If the seller cancels at σ before τ, then he pays ζ_σ to the buyer.

We assume $\zeta_t - \xi_t \geq 0$ for all t and this difference is interpreted as a penalty for the seller for cancellation of the contract.

The game option consists for the seller to select a cancellation time $\sigma \in \mathcal{T}$ and for the buyer to choose an exercise time $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the amount $I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}$.

Definition (Superhedging price)

It is the minimal initial wealth which enables the seller to choose a cancellation time σ and a portfolio which cover his liability to pay to the buyer up to σ no matter the exercise time chosen by the buyer (Kifer).

Definition: For each initial wealth x, a *super-hedge* against the game option is a pair (σ, φ) of a stopping time σ and a portfolio φ s.t. $V^{x,\varphi}_{t\wedge\sigma} \geq I(t, \sigma)$, $0 \leq t \leq T$ a.s. (or equivalently $V^{x,\varphi}_t \geq \xi_t$, $0 \leq t \leq \sigma$ a.s. and $V^{x,\varphi}_\sigma \geq \zeta_\sigma$ a.s.).

Let $\mathcal{A}(x)$ be the *set of all super-hedges associated with initial wealth* x.
The *superhedging price* u_0 is then defined as the infimum of initial wealths such that there exists a hedge against the game option, that is

$$u_0 = \inf\{x \in \mathbf{R},\ \exists(\sigma, \varphi) \in \mathcal{A}(x)\}.$$

In the case of perfect markets, Kifer shows in the CRR discrete-time model and in the Black-Scholes model that $u_0 =$ value function of a *Dynkin game*:

$$u_0 = \sup_\tau \inf_\sigma \mathbb{E}_{\mathbb{Q}}[\tilde{\xi}_\tau \mathbf{1}_{\tau \le \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}] = \inf_\sigma \sup_\tau \mathbb{E}_{\mathbb{Q}}[\tilde{\xi}_\tau \mathbf{1}_{\tau \le \sigma} + \tilde{\zeta}_\sigma \mathbf{1}_{\tau > \sigma}],$$

where $\tilde{\xi}_t$ and $\tilde{\zeta}_t$ are the discounted values of ξ_t and ζ_t.

Here, $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation under the unique martingale probability measure $\mathbb{Q}$ of the market model.

Nonlinear pricing of Game options

Consider the imperfect market $\mathcal{M}^g$, when under nonlinear constraints, the wealth satisfies:

$$-dV_t = g(t, V_t, \sigma_t^* \varphi_t)dt - \sigma_t^* \varphi_t dW_t.$$

where g is nonlinear (Lipschitz).

Suppose that the seller has chosen his cancellation time σ.
Then, the game option reduces to an American option with payoff $I(., \sigma)$, whose superhedging initial price is given by $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)]$.
It is thus natural to define the *g-value of the game option* as

$$\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}(I(\tau, \sigma)).$$

Questions :

$$? \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau\wedge\sigma}(I(\tau,\sigma)) = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau\wedge\sigma}(I(\tau,\sigma))$$

In other words, does this generalized Dynkin game has a value?

? Links with superhedging price of game options

The classical approach based on a change of probability measure and an actualization procedure (see Hamadène) cannot be adapted to the nonlinear case.

Mathematical tools: theory of (nonlinear) *doubly reflected* BSDEs and links with *generalized* Dynkin games.
Dumitrescu-Quenez-Sulem 2014

Doubly reflected BSDEs

We define Doubly reflected BSDEs (DRBSDEs), for which the solution is constrained to stay between two given processes called barriers $\xi \leq \zeta$.

Two nondecreasing processes A and A' are introduced in order to push the solution Y above ξ and below ζ in a minimal way. This *minimality property* of A and A' is ensured by the *Skorohod* conditions.

Definition (Doubly reflected BSDEs)

Let $T > 0$ be a fixed terminal time and g be a Lipschitz driver.
Let ξ and ζ be two adapted continuous on [t,T) processes with $\zeta_T = \xi_T$ a.s., $\xi \in \mathcal{S}^2$, $\zeta \in \mathcal{S}^2$, $\xi_t \leq \zeta_t$, $\forall t \in [0, T]$ a.s.
A process (Y, Z, A, A') in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathcal{A}^2 \times \mathcal{A}^2$ is a solution of the doubly reflected BSDE associated with driver g and barriers ξ, ζ if

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)dt + dA_t - dA'_t - Z_t dW_t;\ 0 \leq t \leq T \\ Y_T = \xi_T, \\ \xi_t \leq Y_t \leq \zeta_t, \quad 0 \leq t \leq T \text{ a.s.}, \\ dA_t \perp dA'_t \\ \int_0^T (Y_t - \xi_t)dA_t = 0 \text{ and } \displaystyle\int_0^T (\zeta_t - Y_t)dA'_t = 0 \text{ a.s. (Skorohod conditio} \end{cases}$$

$\mathcal{A}^2 = \{$nondecreasing continuous adapted processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty\}$

We start by the case of the driver $g = 0$ and $\xi_T = 0$ a.s.

Then the associated doubly reflected BSDE is related to a classical Dynkin game problem with payoff:

$$I(\tau, \sigma) = \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}.$$

Definition (Classical Dynkin game)

For any $S \in \mathcal{T}_0$, the upper and lower value functions at time S are defined respectively by

$$\overline{V}(S) := \mathit{ess}\inf_{\sigma\in\mathcal{T}_S} \mathit{ess}\sup_{\tau\in\mathcal{T}_S} \mathbb{E}[I(\tau,\sigma)|\mathcal{F}_S]$$

$$\underline{V}(S) := \mathit{ess}\sup_{\tau\in\mathcal{T}_S} \mathit{ess}\inf_{\sigma\in\mathcal{T}_S} \mathbb{E}[I(\tau,\sigma)|\mathcal{F}_S].$$

We clearly have the inequality $\underline{V}(S) \le \overline{V}(S)$ a.s.

We say that the Dynkin game is *fair* (or *there exists a value function*) at time S if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition (S-saddle point)

Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an *S-saddle point* if for all $(\tau, \sigma) \in \mathcal{T}_S^2$, we have

$$E[I(\tau, \sigma^*) \mid \mathcal{F}_S] \leq E[I(\tau^*, \sigma^*) \mid \mathcal{F}_S] \leq E[I(\tau^*, \sigma) \mid \mathcal{F}_S] \text{ a.s.}$$

When it exists, the value function of the Dynkin game can be written as the difference of 2 supermartingales J and J' solutions of a coupled optimal stopping problems: $\overline{V}(S) = \underline{V}(S) = J_S - J'_S$, where for all $\theta \in \mathcal{T}_0$,

$$J_\theta = ess \sup_{\tau \in \mathcal{T}_\theta} E\left[J'_\tau + \xi_\tau | \mathcal{F}_\theta\right] \quad \text{and} \quad J'_\theta = ess \sup_{\sigma \in \mathcal{T}_\theta} E\left[J_\sigma - \zeta_\sigma | \mathcal{F}_\theta\right] \text{ a.s.}$$

J and J' are finite (and in $\mathcal{S}^2$) iff Mokobodzki's condition holds, that is if there exist two nonnegative continuous supermartingales H and H' in $\mathcal{S}^2$ such that:

$$\xi_t \leq H_t - H'_t \leq \zeta_t \quad 0 \leq t \leq T \quad \text{a.s.} \tag{15}$$

(which is for example satisfied when ξ and ζ are semimartingales)

Theorem

Under this condition, the doubly reflected BSDE

$$\begin{cases} -dY_t = dA_t - dA'_t - Z_t dW_t;\, Y_T = 0 \\ \xi_t \leq Y_t \leq \zeta_t, \quad 0 \leq t \leq T \text{ a.s.}, \\ dA_t \perp dA'_t \\ \int_0^T (Y_t - \xi_t) dA_t = 0 \text{ and } \displaystyle\int_0^T (\zeta_t - Y_t) dA'_t = 0 \end{cases}$$

admits a unique solution (Y, Z, A, A') *in* $\mathcal{S}^2 \times \mathbb{H}^2 \times (\mathcal{A}^2)^2$.
$\forall S \in \mathcal{T}_0$, Y_S *is the common value function of the Dynkin game associated with the gain* $I(\tau,\sigma) = \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}$., *i.e.*

$$Y_S = \overline{V}(S) = \underline{V}(S) \quad a.s. \tag{16}$$

Moreover, $\forall S \in \mathcal{T}_0$, *the pair of stopping times* (τ_s^*, σ_s^*) *defined by*

$$\sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}; \quad \tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\} \tag{17}$$

is an S-saddle point.

Sketch of proof : Suppose that $J, J' \in \mathcal{S}^2$. Define

$$\overline{Y}_t := J_t - J'_t;\ 0 \leq t \leq T. \tag{18}$$

Let us show that there exist $(Z, A, A') \in \mathbb{H}^2 \times \mathcal{A}^2 \times \mathcal{A}^2$ such that $(\overline{Y}, Z, A, A')$ is a solution of DRBSDE (15) associated with driver process $g = 0$.

By assumption, J and $J^{'}$ are square integrable supermartingales. The process $\overline{Y}$ is thus well defined. We have $J_T = J'_T$ a.s. Hence, $\overline{Y}_T = \xi_T$ a.s. By the Doob-Meyer decomposition, there exist two square integrable martingales M and M' and two processes A and $A^{'} \in \mathcal{A}^2$ such that:

$$dJ_t = dM_t - dA_t \quad ; \quad dJ_t^{'} = dM_t^{'} - dA_t^{'}. \tag{19}$$

Set

$$\overline{M}_t := M_t - M_t^{'}.$$

By (19), (18), we derive $d\overline{Y}_t = d\overline{M}_t - d\alpha_t$, with $\alpha := A - A^{'}$. By the martingale representation theorem, there exist $Z \in \mathbb{H}^2$ such that $d\overline{M}_t = Z_t dW_t$. Hence,

$$-d\overline{Y}_t = d\alpha_t - Z_t dW_t.$$

By the optimal stopping theory, the process A increases only when the value function J is equal to the corresponding reward $J^{'} + \xi$. Now, $\{J_t = J_t^{'} + \xi\} = \{\overline{Y_t} = \xi_t\}$. Hence, $\int_0^T (\overline{Y_t} - \xi_t) dA_t = 0$ a.s. Similarly the process $A^{'}$ satisfies $\int_0^T (\overline{Y_t} - \zeta_t) dA_t^{'} = 0$ a.s.

We have $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s.
On $[S, \tau_S^*[$, we have $Y_t > \xi_t$ a.s., so A is constant on $[S, \tau_S^*]$.
Similarly, A' is constant on $[S, \sigma_S^*]$ a.s.
The process $(Y_t + \int_0^t g(s)ds, S \leq t \leq \tau_S^* \wedge \sigma_S^*)$ is thus a martingale. Hence, $Y_S = E[I_S(\tau_S^*, \sigma_S^*) \mid \mathcal{F}_S]$ a.s.
Similarly we can show that $\forall \tau, \sigma \in \mathcal{T}_S$, $E[I_S(\tau, \sigma_S^*) \mid \mathcal{F}_S] \leq Y_S$ and $Y_S \leq E[I_S(\tau_S^*, \sigma) \mid \mathcal{F}_S]$ a.s. , So (τ_S^*, σ_S^*) is an S-saddle point. $\square$

This result can easily be extended to the case when $g = (g_t)$ in $\mathbb{H}^2$ is a driver process and $\xi_T \neq 0$, by doing the change of variables:

$$\tilde{\xi}_t^g := \xi_t - E[\xi_T + \int_t^T g(s)ds \mid \mathcal{F}_t], \quad \tilde{\zeta}_t^g := \zeta_t - E[\zeta_T + \int_t^T g(s)ds \mid \mathcal{F}_t],$$

The associated payoff of the Dynkin game is then

$$I_S(\tau, \sigma) = \int_S^{\sigma \wedge \tau} g(u)du + \xi_\tau \mathbf{1}_{\{\tau \leq \sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma < \tau\}}.$$

Generalized Dynkin games and links with nonlinear DRBSDEs

From now on, we are given a standard Lipschitz driver g and we suppose that Mokobodzki's condition is satisfied.

Theorem (Existence and uniqueness for DRBSDEs)

Then, DRBSDE (15) admits a unique solution $(Y, Z, A, A') \in \mathcal{S}^2 \times \mathbb{H}^2 \times (\mathcal{A}^2)^2$.

Proof. based on a contraction argument (similar to the case of BSDEs and Reflected BSDEs) $\square$

Generalized Dynkin games

We introduce a *generalized Dynkin game* expressed in terms of $\mathcal{E}^g$-expectations. Let $g(t, y, z)$ be a given a Lipschitz driver.

For each pair (τ, σ) of stopping times valued in $[0, T]$, the criterium is given by

$$\mathcal{E}^g_{0,\tau\wedge\sigma}\left(\xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right)$$

with ξ, ζ two continuous adapted processes satisfying $\xi \leq \zeta$.

When the driver g does not depend on the solution, that is, when it is given by a process $(g(t))$, the criterium coincides with

$$\mathbb{E}\left(\int_0^{\tau\wedge\sigma} g_s ds + \xi_\tau \mathbf{1}_{\{\tau\leq\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}}\right).$$

For each stopping time $S \in \mathcal{T}_0$, the *upper* and *lower value functions* at time S are defined respectively by

$$\overline{V}(S) := ess \inf_{\sigma \in \mathcal{T}_S} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)); \tag{20}$$

$$\underline{V}(S) := ess \sup_{\tau \in \mathcal{T}_S} ess \inf_{\sigma \in \mathcal{T}_S} \mathcal{E}_{S,\tau\wedge\sigma}(I(\tau,\sigma)). \tag{21}$$

with

$$I(\tau,\sigma) := \xi_\tau \mathbf{1}_{\tau\leq\sigma} + \zeta_\sigma \mathbf{1}_{\sigma<\tau}.$$

We clearly have the inequality $\underline{V}(S) \leq \overline{V}(S)$ a.s.

By definition, we say that the game is *fair* (or *there exists a value function*) at time S if $\overline{V}(S) = \underline{V}(S)$ a.s.

Definition (S-saddle)

Let $S \in \mathcal{T}_0$. A pair $(\tau^*, \sigma^*) \in \mathcal{T}_S^2$ is called an *S-saddle point* for the generalized Dynkin game if for each $(\tau, \sigma) \in \mathcal{T}_S^2$ we have

$$\mathcal{E}^g_{S,\tau\wedge\sigma^*}(I(\tau,\sigma^*)) \le \mathcal{E}^g_{S,\tau^*\wedge\sigma^*}(I(\tau^*,\sigma^*)) \le \mathcal{E}^g_{S,\tau^*\wedge\sigma}(I(\tau^*,\sigma)) \quad \text{a.s.}$$

Definition

Let Y be an RCLL process in $\mathcal{S}^2$. The process Y is said to be a $\mathcal{E}^g$-supermartingale (resp $\mathcal{E}^g$-submartingale), if $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \le Y_\sigma$ (resp. $\mathcal{E}^g_{\sigma,\tau}(Y_\tau) \ge Y_\sigma$) a.s. on $\sigma \le \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.

We first provide a sufficient condition for the existence of an S-saddle point and for the characterization of the common value function as the solution of the DRBSDE.

Lemma

Let (Y, Z, A, A') be the solution of the DRBSDE. Let $S \in \mathcal{T}_0$. Let $(\hat{\tau}, \hat{\sigma}) \in \mathcal{T}_S$. Suppose that $(Y_t,\ S \leq t \leq \hat{\tau})$ is a $\mathcal{E}$-submartingale and that $(Y_t,\ S \leq t \leq \hat{\sigma})$ is a $\mathcal{E}$-supermartingale with $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s.

The pair $(\hat{\tau}, \hat{\sigma})$ is then an S-saddle point for the generalized Dynkin game (20)-(21) *and*

$$Y_S = \overline{V}(S) = \underline{V}(S) \text{ a.s.}$$

Proof. Since the process $(Y_t,\, S \leq t \leq \hat{\tau} \wedge \hat{\sigma})$ is a $\mathcal{E}$-martingale and since $Y_{\hat{\tau}} = \xi_{\hat{\tau}}$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s. , we have

$$Y_S = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(Y_{\hat{\tau}\wedge\hat{\sigma}}) = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(\xi_{\hat{\tau}}\mathbf{1}_{\hat{\tau}\leq\hat{\sigma}} + \zeta_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\hat{\tau}}) = \mathcal{E}^g_{S,\hat{\tau}\wedge\hat{\sigma}}(I(\hat{\tau},\hat{\sigma})) \quad \text{a.s.}$$

Let $\tau \in \mathcal{T}_S$. We want to show that for each $\tau \in \mathcal{T}_S$

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\hat{\sigma}}(I(\tau,\hat{\sigma})) \quad \text{a.s.} \tag{22}$$

Since the process $(Y_t,\, S \leq t \leq \tau \wedge \hat{\sigma})$ is a $\mathcal{E}$-supermartingale, we get

$$Y_S \geq \mathcal{E}_{S,\tau\wedge\hat{\sigma}}(Y_{\tau\wedge\hat{\sigma}}) \quad \text{a.s.} \tag{23}$$

Since $Y \geq \xi$ and $Y_{\hat{\sigma}} = \zeta_{\hat{\sigma}}$ a.s. , we also have

$$Y_{\tau\wedge\hat{\sigma}} = Y_\tau\mathbf{1}_{\tau\leq\hat{\sigma}} + Y_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\tau} \geq \xi_\tau\mathbf{1}_{\tau\leq\hat{\sigma}} + \zeta_{\hat{\sigma}}\mathbf{1}_{\hat{\sigma}<\tau} = I(\tau,\hat{\sigma}) \quad \text{a.s.}$$

By inequality (23) and the monotonicity property of $\mathcal{E}$, we derive inequality (22).
Similarly, one can show that for each $\sigma \in \mathcal{T}_S$, we have:

$$Y_S \leq \mathcal{E}_{S,\hat{\tau}\wedge\sigma}(I(\hat{\tau},\sigma)) \quad \text{a.s.}$$

The pair $(\hat{\tau},\hat{\sigma})$ is thus an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s. $\square$

Theorem (Existence of S-saddle points)

Let (Y, Z, A, A') be the solution of the DRBSDE (15). For each $S \in \mathcal{T}_0$, let

$$\tau_S^* := \inf\{t \geq S,\ Y_t = \xi_t\}; \quad \sigma_S^* := \inf\{t \geq S,\ Y_t = \zeta_t\}.$$

Then, for each $S \in \mathcal{T}_0$, the pair of stopping times (τ_S^*, σ_S^*) is a S-saddle point for the generalized Dynkin game and
$Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.
Moreover, $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$, $Y_{\tau_S^*} = \xi_{\tau_S^*}$, $A_{\tau_S^*} = A_S$ and $A'_{\sigma_S^*} = A'_S$ a.s.

Proof. Let $S \in \mathcal{T}_0$. Since Y and ξ are continuous processes, we have $Y_{\sigma_S^*} = \zeta_{\sigma_S^*}$ and $Y_{\tau_S^*} = \xi_{\tau_S^*}$ a.s. By definition of τ_S^*, $Y_t > \xi_t(\omega)$ for each $t \in [S, \tau_S^*[$.

Hence, since Y is solution of the DRBSDE, the process A is constant on $[S, \tau_S^*]$ a.s. because A is continuous. Hence, $A_{\tau_S^*} = A_S$ a.s. Similarly, $A'_{\sigma_S^*} = A'_S$ a.s.

By previous Lemma , (τ_S^*, σ_S^*) is an S-saddle point and $Y_S = \overline{V}(S) = \underline{V}(S)$ a.s.

$\square$

Application to nonlinear pricing of game options

We consider now a game option associated with continuous payoffs ξ and ζ in $\mathcal{S}^2$ with $\zeta_T = \xi_T$; $\xi_t \leq \zeta_t, 0 \leq t \leq T$ a.s. satisfying **Mokobodski's condition** (which is for example satisfied when ξ and ζ are semimartingales)
The game option consists for the seller to select a cancellation time $\sigma \in \mathcal{T}$ and for the buyer an exercise time $\tau \in \mathcal{T}$, so that the seller pays to the buyer at time $\tau \wedge \sigma$ the payoff

$$I(\tau, \sigma) := \xi_\tau \mathbf{1}_{\tau \leq \sigma} + \zeta_\sigma \mathbf{1}_{\sigma < \tau}.$$

A natural price for the seller of the game option, is the *g-value* given by

$$Y(0) := \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau, \sigma)] \tag{24}$$

Under the assumptions above, this *generalized Dynkin game* is *fair*, i.e

$$\inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^{g}_{0,\tau\wedge\sigma}[I(\tau,\sigma)]$$

and its *value* $Y(0)$ equals Y_0, where (Y, Z, A, A') is the unique solution in $\mathcal{S}^2 \times L^2(W) \times \mathcal{A}^2 \times \mathcal{A}^2$ of the Doubly Reflected BSDE associated to driver g and obstacles ξ, ζ.

Definition: *Superhedging price* u_0 of the game option := minimal initial wealth which enables the seller to choose a cancellation time σ and a portfolio which will cover his liability to pay to the buyer up to σ no matter what exercise time the buyer chooses (Kifer).

Definition: For each initial wealth x, a *super-hedge* against the game option is a pair (σ, φ) of a stopping time σ and a portfolio φ s.t. $V^{x,\varphi}_{t\wedge\sigma} \geq I(t,\sigma)$, $0 \leq t \leq T$ a.s. (or equivalently $V^{x,\varphi}_t \geq \xi_t$, $0 \leq t \leq \sigma$ a.s. and $V^{x,\varphi}_\sigma \geq \zeta_\sigma$ a.s.).

Let $\mathcal{A}(x)$ be the *set of all super-hedges associated with initial wealth* x.

The *superhedging price* is then defined as the infimum of initial wealths such that there exists a hedge against the game option, that is

$$u_0 = \inf\{x \in \mathbf{R},\ \exists(\sigma,\varphi) \in \mathcal{A}(x)\}.$$

Theorem

- **superhedging price** $u_0 = g$-value of the game option, that is
$$u_0 = \inf_{\sigma \in \mathcal{T}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^g_{0,\tau\wedge\sigma}[I(\tau,\sigma)].$$
- Let (Y, Z, A, A') be the solution of the DRBSDE associated with driver g and barriers ξ, ζ. We have
$$u_0 = Y_0.$$
Let
$$\sigma^* := \inf\{t \geq 0,\ \ Y_t = \zeta_t\} \ \text{ and } \varphi^*_t = (\sigma^*_t)^{-1} Z_t.$$
The pair (σ^*, φ^*) belongs to $\mathcal{A}(u_0)$, i.e. the cancellation time σ^* and the strategy φ^* allow the seller of the game option to be super-hedged.

The proof is based on links between generalized Dynkin games and DRBSDEs

Definition: (Kifer) A stopping time τ is a *rational* exercise time for the game option if $V_t^{Y_0,\varphi^*} \le \zeta_t$, $0 \le t \le \tau$ a.s. and $V_\tau^{Y_0,\varphi^*} = \xi_\tau$ a.s.

Lemma: The stopping time

$$\tau^* := \inf\{t \geq 0 : \quad Y_t = \xi_t\} \tag{25}$$

is a *rational* exercise time for the buyer of the game option.

The pair (τ^*, σ^*) is a saddle point for the generalized Dynkin game, that is for each $(\tau, \sigma) \in \mathcal{T}^2$ we have

$$\mathcal{E}^g_{0,\tau\wedge\sigma^*}[I(\tau,\sigma^*)] \leq Y_0 = \mathcal{E}^g_{0,\tau^*\wedge\sigma^*}[I(\tau^*,\sigma^*)] \leq \mathcal{E}^g_{0,\tau^*\wedge\sigma}[I(\tau^*,\sigma)].$$

In this case, τ^* is optimal for the optimal stopping problem $\sup_{\tau\in\mathcal{T}} \mathcal{E}^g[I(\tau,\sigma^*)]$ (as in the case of an American option) and σ^* is optimal for $\inf_{\sigma\in\mathcal{T}} \mathcal{E}^g[I(\tau^*,\sigma)]$.

Relations with variational inequalities (VI)

We consider now the Markovian case and study the links between generalized Dynkin games (or equivalently DRBSDEs) and obstacle problems.
Let $b : \mathbf{R} \to \mathbf{R}$, $\sigma : \mathbf{R} \to \mathbf{R}$ be continuous, globally Lipschitz.
$\forall (t,x) \in [0, T] \times \mathbf{R}$, let $(X_s^{t,x}, t \leq s \leq T)$ solution of

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r$$

and set $X_s^{t,x} = x$ for $s \leq t$.

We consider the DRBSDE associated with obstacles $\xi^{t,x}$, $\zeta^{t,x}$ of the following form:

$$\xi_s^{t,x} := h_1(s, X_s^{t,x}),\ \zeta_s^{t,x} := h_2(s, X_s^{t,x}),\ s < T, \xi_T^{t,x} = \zeta_T^{t,x} := g(X_T^{t,x})$$

with $g \in \mathcal{C}(\mathbf{R})$, $h_1, h_2 : [0, T]$ continuous with respect to t and Lipschitz continuous with respect to x, uniformly in t and g, h_1, h_2 have at most polynomial growth with respect to x.
Moreover, we assume that the obstacles $\xi_s^{t,x}$ and $\zeta_s^{t,x}$ satisfy Mokobodski's condition, which holds e.g; if h_1 and h_2 are $\mathcal{C}^{1,2}$.
Let $f : [0, T] \times \mathbf{R}^3 \to \mathbf{R}$ be continuous in t uniformly with respect to x, y, z, uniformly Lipschitz with respect to x, y, z uniformly in t, such that $f(t, x, 0, 0,)$ at most polynomial growth with respect to x.
The driver is defined by $f(s, X_s^{t,x}(\omega), y, z)$.

For each $(t,x) \in [0,T] \times \mathbf{R}$, there exists an unique solution $(Y^{t,x}, Z^{t,x}, A^{t,x}, A^{'\,t,x})$ of the associated DRBSDE.
We define:

$$u(t,x) := Y_t^{t,x}, \quad t \in [0,T],\ x \in \mathbf{R}. \tag{26}$$

which is a deterministic quantity.
u is continuous in (t,x) and has at most polynomial growth at infinity.

A solution of the obstacle problem is a function $u : [0, T] \times \mathbf{R} \to \mathbf{R}$ which satisfies the equality $u(T, x) = g(x)$ and

$$
(27) \qquad \begin{cases} h_1(t,x) \leq u(t,x) \leq h_2(t,x) \\ \text{ if } u(t,x) < h_2(t,x) \text{ then } \mathcal{H}u \geq 0 \\ \text{ if } h_1(t,x) < u(t,x) \text{ then } \mathcal{H}u \leq 0 \end{cases}
$$

where

- $L\phi(x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2}(x) + b(x)\frac{\partial \phi}{\partial x}(x),$
- $\mathcal{H}\phi(t,x) :=$ $-\frac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - f(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x)).$

Definition

• A continuous function u is said to be a *viscosity subsolution* (resp. *supersolution*) of (27) if

- $u(T, x) \leq g(x), \quad (\text{resp } \geq) \quad \forall x \in \mathbf{R}$,
- for all (t, x), $h_1(t, x) \leq u(t, x) \leq h_2(t, x)$
- for all $\phi \in C^{1,2}([0, T] \times \mathbf{R})$ such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi - u$ attains its minimum (resp. maximum) at (t_0, x_0), if $u(t_0, x_0) > h_1(t_0, x_0)$ (resp. $<$), then $(\mathcal{H}\phi)(t_0, x_0) \leq 0$ (resp $\geq$.)

Theorem

The function u defined by (26) *is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (27)*.*

This result provides a probabilistic interpretation of semi linear PDEs with two barriers in terms of game problems.

References

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- Links with Doubly RBSDEs when the driver g **does not depend on** y, z: Cvitanic and Karatzas (1996), Hamadène (2002) and Lepeltier (2000) (Hyp: Brownian+ regularity).
- *Generalized Dynkin games and DRBSDEs*, Dumitrescu-Quenez-A.S EJP 2016
  + *Game options in an imperfect market with default*, Dumitrescu-Quenez-A.S

Market model with default

Setup

Let $(\Omega, \mathcal{G}, P)$ be a probability space, equipped with two stochastic processes: a one-dimensional standard Brownian motion W and a jump process N defined by $N_t := \mathbf{1}_{\vartheta \leq t}$ for any $t \geq 0$, where ϑ is a random variable which represents a default time. We assume that this default can appear at any time that is $P(\vartheta \geq t) > 0$ for any $t \geq 0$.

We denote by $\mathbb{G} = (\mathcal{G}_t, t \geq 0))$ the complete natural filtration generated by W and N. We suppose that W is a $\mathbb{G}$-Brownian motion.

Let Λ be the predictable compensator of the nondecreasing process N (i.e. the unique RCLL increasing predictable process null at 0 such that $N_t - \Lambda$ is a martingale).
Note that $\Lambda_{t\wedge\vartheta}$ is then the predictable compensator of $N_{t\wedge\vartheta} = N_t$.
By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$.

We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$.
Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, λ vanishes after ϑ.
We denote by M the compensated martingale which satisfies

$$M_t = N_t - \int_0^t \lambda_s ds \, .$$

Let $T > 0$. We introduce the following sets :

- $\mathcal{S}^2$ is the set of $\mathbb{G}$-adapted RCLL processes φ such that $E[\sup_{0\le t\le T}|\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes such that $\|Z\|^2 := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty\,.$
- $\mathbb{H}^2_\lambda$ is the set of $\mathbb{G}$-predictable processes such that $\|U\|^2_\lambda := \mathbb{E}\Big[\int_0^T |U_t|^2 \lambda_t dt\Big] < \infty\,.$

We recall the martingale representation theorem (see e.g. Jeanblanc-Yor-Chesnay'09):

Lemma

Any $\mathbb{G}$-local martingale m has the representation

$$m_t = m_0 + \int_0^t z_s dW_s + \int_0^t l_s dM_s \,, \quad \forall\, t \in [0, T] \quad a.s.\,, \tag{28}$$

where z and l are predictable such that the two above stochastic integrals are well defined. If m is a square integrable martingale, then $z \in \mathbb{H}^2$ and $l \in \mathbb{H}^2_\lambda$.

Market model with default

Financial market

We consider a financial market with three assets with price process $S = (S^0, S^1, S^2)'$ governed by the equation:

$$\begin{cases} dS_t^0 = S_t^0 r_t dt \\ dS_t^1 = S_t^1 [\mu_t^1 dt + \sigma_t^1 dW_t] \\ dS_t^2 = S_{t^-}^2 [\mu_t^2 dt + \sigma_t^2 dW_t - dM_t] \end{cases}$$

S^2 : price of a defaultable asset with total default. It vanishes after ϑ.

All processes σ^1, σ^2, r, μ^1, μ^2 are predictable, $\sigma^1, \sigma^2 > 0$, and σ^1, σ^2, $r, \mu^1, \mu^2, (\sigma^1)^{-1}$, $(\sigma^2)^{-1}$ are bounded. We set $\sigma = (\sigma^1, \sigma^2)'$.

Market model with default

Risky asset strategy

We consider an investor, with initial wealth x, who can invest his wealth in the three assets of the market.
At each time $t < \vartheta$, he chooses the amount φ_t^1 (resp. φ_t^2) of wealth invested in the first (resp. second) risky asset. However, after time ϑ, the investor cannot invest his wealth in the defaultable asset since its price is equal to 0, and he only chooses the amount φ_t^1 of wealth invested in the first risky asset. Note that the process φ^2 can be defined on the whole interval $[0, T]$ by setting $\varphi_t^2 = 0$ for each $t \geq \vartheta$.
A process $\varphi_. = (\varphi_t^1, \varphi_t^2)'_{0 \leq t \leq T}$ is called a *risky assets stategy* if it belongs to $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$.
We denote by $V_t^{x,\varphi}$ (or simply V_t) the *wealth*, or equivalently the value of the portfolio, at time t. The amount invested in the non risky asset at time t is then given by $V_t - (\varphi_t^1 + \varphi_t^2)$.

Market model with default

The perfect market case

In the classical case of a perfect market model, the wealth process and the strategy satisfy the self financing condition:

$$
\begin{aligned}
dV_t &= (r_t V_t + \varphi^1_t(\mu^1_t - r_t) + \varphi^2_t(\mu^2_t - r_t))dt + (\varphi^1_t \sigma^1_t + \varphi^2_t \sigma^2_t) dW_t - \varphi^2_t d \\
&= (r_t V_t + \varphi^1_t \theta^1_t \sigma^1_t - \varphi^2_t \theta^2_t \lambda_t)dt + \varphi'_t \sigma_t dW_t - \varphi^2_t dM_t,
\end{aligned}
$$

where $\theta^1_t := \dfrac{\mu^1_t - r_t}{\sigma^1_t}$, $\theta^2_t := -\dfrac{\mu^2_t - \sigma^2_t \theta^1_t - r_t}{\lambda_t} \mathbf{1}_{\{t \leq \vartheta\}}$.

Consider a European option with maturity T and $\mathcal{G}_T$- measurable payoff ξ in L^2. The problem is to price and hedge this claim by constructing a replicating portfolio.

$\exists!$ process $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the following linear BSDE:

(29)
$$\begin{cases} -dX_t = -(r_t X_t + (Z_t - \sigma_t K_t)\theta^1_t + K_t \lambda_t \theta^2_t) dt - Z_t dW_t - K_t dM_t \\ X_T = \xi. \end{cases}$$

The solution (X, Z, K) provides the replicating portfolio. More precisely, the process X corresponds to its value, and the hedging risky assets strategy φ is such that

$$\varphi_t{}' \sigma_t = Z_t \ \ ; \ \ -\varphi^2_t = K_t, \tag{30}$$

This defines a change of variables $(Z, K) \mapsto \Phi(Z, K) := (\varphi^1, \varphi^2)$, where (φ^1, φ^2) is given by

$$\varphi^2_t = -K_t \ \ ; \ \ \varphi^1_t = \frac{Z_t - \varphi^2_t \sigma^2_t}{\sigma^1_t} = \frac{Z_t + \sigma^2_t K_t}{\sigma^1_t}. \tag{31}$$

X coincides with $V^{X_0,\varphi}$, the value of the (hedging) portfolio associated with initial wealth $x = X_0$ and portfolio strategy φ.

This process defines a price process called *hedging price* of ξ and denoted by $X(\xi)$. Since the driver of BSDE (29) is *linear* the representation property of the solution yields

$$X_t(\xi) = \mathbb{E}[e^{-\int_t^T r_s ds}\zeta_{t,T}\xi \,|\, \mathcal{G}_t],$$

where ζ satisfies

$$d\zeta_{t,s} = \zeta_{t,s^-}[-\theta_s^1 dW_s - \theta_s^2 dM_s]; \;\; \zeta_{t,t} = 1.$$

This defines a *linear* price system X: $\xi \mapsto X(\xi)$.
When $\theta_t^2 < 1$, $0 \le t \le \vartheta \;\; dt \otimes dP$-a.s. the price system X is increasing and corresponds to the classical free-arbitrage price system.

The imperfect market model $\mathcal{M}^g$

We assume now that there are imperfections in the market which are taken into account via the *nonlinearity* of the dynamics of the wealth.

We suppose that the *wealth* process $V_t^{x,\varphi}$ (or simply V_t) associated with an initial wealth x and a strategy $\varphi = (\varphi^1, \varphi^2)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ satisfies the following dynamics:

$$-dV_t = g(t, V_t, {\varphi_t}'\sigma_t, -\varphi_t^2)dt - {\varphi_t}'\sigma_t dW_t + \varphi_t^2 dM_t,\ \ V_0 = x$$

or equivalently, setting $Z_t = {\varphi_t}'\sigma_t$ and $K_t = -\varphi_t^2$,

$$(32) \qquad -dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t.$$

In the case of a perfect market , we have:

$$g(t, y, z, k) = -(r_t y + (z + \sigma_t^2 k)\theta_t^1 + \theta_t^2 \lambda_t k).$$

The imperfect market model $\mathcal{M}^g$.

Example

Different borrowing and lending interest rates, denoted respectively by R_t and r_t with $R_t \geq r_t$. Then the driver g is of the form

$$g(t, V_t, \varphi_t' \sigma_t, -\varphi_t^2) = -(r_t V_t + \varphi_t^1 \theta_t^1 \sigma_t^1 - \varphi_t^2 \lambda_t \theta_t^2) + (R_t - r_t)(V_t - \varphi_t^1 - \varphi_t^2)^-$$

where φ_t^2 vanishes after ϑ.

Definition (λ-*admissible* driver)

A function $g(\omega, t, y, z, k)$ is called a λ-*admissible driver* if

- g is $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^3)-$ measurable,
- $g(., 0, 0, 0) \in \mathbb{H}^2$,
- $\exists C \geq 0$ s.t. $dP \otimes dt$-a.s. , $\forall$ (y, z, k), (y_1, z_1, k_1), (y_2, z_2, k_2),

$|g(\omega, t, y, z_1, k_1) - g(\omega, t, y, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda}_t |k_1 - k_2|)$.

The above condition implies that for each $t > \vartheta$, g does depend on k, since $\lambda_t = 0$. In other terms, for each (y, z, k), we have:

$$g(t, y, z, k) = g(t, y, z, 0), \quad t > \vartheta \quad dP \otimes dt - \text{a.s.}$$

The pricing and hedging of European options in this imperfect market leads to BSDEs with nonlinear driver g. We give below a uniqueness and existence result in a model with a default time.

Theorem

Let g be a λ-admissible driver and let $\xi \in L^2(\mathcal{G}_T)$. There exists an unique solution $(X(T,\xi), Z(T,\xi), K(T,\xi))$ (denoted simply by (X,Z,K)) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE:

$$(33) \quad -dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \xi.$$

See proof in Dumitrescu-Quenez-Sulem 2016.

Consider a European option with maturity T and terminal payoff $\xi \in L^2(\mathcal{G}_T)$ in this market model.

Let (X, Z, K) be the solution of BSDE (33). The process X is equal to the wealth process associated with initial value $x = X_0$ and strategy $\varphi = \Phi(Z, K)$ (see (31)), that is $X = V^{X_0,\varphi}$.

Its initial value X_0 is thus a sensible price (at time 0) of the claim ξ for the seller since this amount allows him to construct a trading strategy φ, called hedging strategy, such that the value of the associated portfolio is equal to ξ at time T.

Similarly, X_t is a sensible price for the seller at time t.

This leads to a *nonlinear pricing* system denoted by $\mathcal{E}^g$:

$\forall S \in [0, T]$, $\forall \xi \in L^2(\mathcal{G}_S)$, $\mathcal{E}^g_{t,S}(\xi) := X_t(S, \xi)$ for each $t \in [0, S]$.

In order to ensure the monotonicity of the nonlinear pricing system $\mathcal{E}^g$, we make the following assumption:
Assumption: There exists a bounded map

$$\gamma : [0, T] \times \Omega \times \mathbf{R}^4 \to \mathbf{R}\,;\ (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^4)$-measurable and satisfying $dP \otimes dt$-a.s. , for each $(y, z, k_1, k_2) \in \mathbf{R}^4$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \tag{34}$$

and P-a.s. , for each $(y, z, k_1, k_2) \in \mathbf{R}^4$, $\gamma_t^{y,z,k_1,k_2} > -1$.

Recall that λ vanishes after ϑ and $g(t, \cdot)$ does not depend on k on $\{t > \vartheta\}$. Hence, (34) is always satisfied on $\{t > \vartheta\}$.
This assumption is satisfied e.g. if $g(t, \cdot)$ is non decreasing with respect to k, or if g is $\mathcal{C}^1$ in k with $\partial_k g(t, \cdot) > -\lambda_t$ on $\{t \leq \vartheta\}$.
By the comparison theorems for BSDEs, this assumption ensures that the nonlinear pricing system $\mathcal{E}^g$ is strictly monotone.