Algorithmic Trading: Execution Targets
PIMS Summer School

Sebastian Jaimungal, U. Toronto

Álvaro Cartea,U. Oxford

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José Penalva,(U. Carlos III)
Luhui Gan (U. Toronto)
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Damir Kinzebulatov (U. Laval)
Jason Ricci (Morgan Stanley)

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Targeting VWAP

- Many traders view Volume Weighted Average Price (**VWAP**) as a **fair price** over the duration of the tradeL

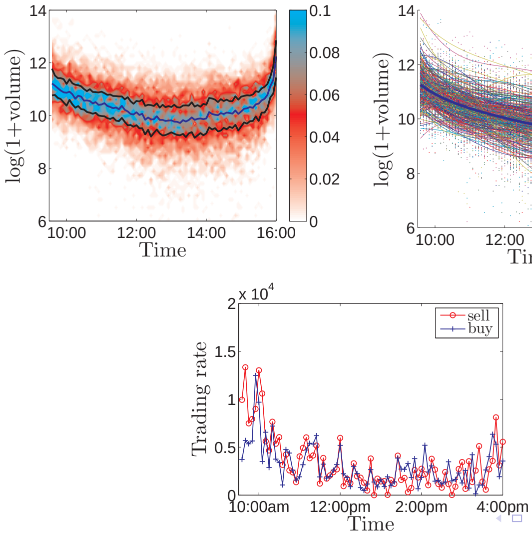
$$\textbf{VWAP} = \frac{\int_0^T S_u\,(\mu_u^+ + \mu_u^-)\,du}{\int_0^T (\mu_u^+ + \mu_u^-)\,du}\,,$$

  $\mu^\pm = (\mu_t^\pm)_{t\geq 0}$ is the rate of buy/sell order executions.

- How can we target this? The **volume** of future trades is **not known** a priori!

Targeting VWAP

Figure: ORCL (2013) traded volume using 5 minute buckets.



Targeting VWAP: POV

- **Instead target** percentage of volume (**POV**)... which would be exactly VWAP if the requested volume equals the given percentage of total volume traded
- Simultaneously **ensure** that total **volume traded** equals the requested amount
- Trader's **performance criteria** is

$$H^{\nu}(t,x,S,\boldsymbol{\mu},q) = \mathbb{E}_{t,x,S,\boldsymbol{\mu},q}\Bigg[\underbrace{X_T^{\nu}}_{\text{Cash}} + \overbrace{Q_T^{\nu}\,(S_T^{\nu} - \boldsymbol{\alpha}\,Q_T^{\nu})}^{\text{terminal liquidation}} - \tilde{\varphi}\underbrace{\int_t^T \left(\boldsymbol{\nu_u} - \boldsymbol{\chi_u^{\nu}}\right)^2 du}_{\text{POV penalty}}\Bigg],$$

where $\boldsymbol{\mu} = \{\mu^+, \mu^-\}$,

$$\chi_t^{\nu} := \tilde{\rho} \times \left(\mu_t^+ + \mu_t^- + \nu_t\right), \qquad \textbf{target rate}$$

Targeting VWAP: POV

- **Temporary price impact**

$$\hat{S}_t^{\nu} = S_t^{\nu} - a\,\nu_t$$

- **Price process** is affected by order-flow from all agents, i.e., **permanent price impact**

$$dS_t^{\nu} = b\,\underbrace{\left(\mu_t^{+} - (\nu_t + \mu_t^{-})\right)}_{\textbf{net order-flow}}\,dt + dM_t$$

- **Cash process** is

$$X_t^{\nu} = \int_0^t \left(S_u^{\nu} - a\,\nu_u\right)\nu_u\,du$$

Targeting VWAP: POV

- The **associated DPE** is

$$0 = \left(\partial_t + \mathcal{L}^M + \mathcal{L}^{\boldsymbol{\mu}}\right) H + \sup_{\nu} \Big\{ (S - k\,\nu)\,\nu\,\partial_x H - \nu\,\partial_q H$$
$$+ b\,((\mu^+ - \mu^-) - \nu)\,\partial_S H - \varphi\,(\nu - \rho\,\mu)^2 \Big\},$$

  with terminal condition

$$H(T, x, S, \boldsymbol{\mu}, q) = x + q\,(S - \alpha\,q)\,,$$

  - $\mu = \mu^+ + \mu^-$
  - $\mathcal{L}^{\boldsymbol{\mu}}$ generator of $\boldsymbol{\mu}$, $\mathcal{L}^M$ generator of M

- Optimal trading speed in feedback form is

$$\nu^*(t, x, S, \mu, q) = \frac{-(\partial_q H - S + b\,q) + 2\,\varphi\,\rho\,(\mu^+ + \mu^-)}{2(k + \varphi)}\,,$$

Targeting VWAP: POV

Theorem

Solving the DPE for POV target. *The DPE admits the solution*

$$H(t,x,S,\boldsymbol{\mu},q) = \underbrace{x + q\,S}_{\textbf{Book-value}} + \underbrace{\boldsymbol{h_0(t,\mu)} + \boldsymbol{h_1(t,\mu)}\,q + \boldsymbol{h_2(t)}\,q^2}_{\textbf{excess value}},$$

where

$$0 = (\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\,\boldsymbol{h_2} + \frac{\left(\boldsymbol{h_2} + \frac{1}{2}b\right)^2}{k+\varphi},$$

$$0 = (\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\,\boldsymbol{h_1} + \frac{\boldsymbol{h_1} - 2\,\varphi\,\rho\,(\mu^+ + \mu^-)}{k+\varphi}\left(h_2 + \tfrac{1}{2}\,b\right) + b\,(\mu^+ - \mu^-),$$

$$0 = (\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\,\boldsymbol{h_0} + \frac{1}{4(k+\varphi)}\left(h_1 - 2\varphi\,\rho\,(\mu^+ + \mu^-)\right)^2 - \varphi\,\rho^2\,(\mu^+ + \mu^-)^2.$$

subject to $h_0(t,\boldsymbol{\mu}) = h_1(t,\boldsymbol{\mu}) = 0$ *and* $h_2(t) = -\alpha$.

Targeting VWAP: POV

Proposition

The functions h_0, h_1 and h_2 admit the representation:

$$h_2(t) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}\,b\,,$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu}) &= \frac{\varphi\,\rho}{(T-t)+\zeta}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\boldsymbol{\mu}_s^+ + \boldsymbol{\mu}_s^-\right]ds \\ &\quad + \frac{b}{(T-t)+\zeta}\int_t^T ((T-s)+\zeta)\,\mathbb{E}_{t,\boldsymbol{\mu}}\left[\boldsymbol{\mu}_s^+ - \boldsymbol{\mu}_s^-\right]ds\,, \end{aligned}$$

$$\begin{aligned} h_0(t,\boldsymbol{\mu}) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\Bigg[&\frac{1}{4(k+\varphi)}\left(h_1(t,\boldsymbol{\mu}_s) - 2\varphi\,\rho\,(\mu_s^+ + \mu_s^-)\right)^2 \\ &- \varphi\,\rho^2\,(\mu_s^+ + \mu_s^-)^2\Bigg]ds\,, \end{aligned}$$

Targeting VWAP: POV

Theorem

Verification for POV target. *The candidate value function provided is indeed the solution to the original optimal control problem. Moreover, the trading speed is given by*

$$\nu_t^* = \frac{1}{(T-t)+\zeta}\, \boldsymbol{Q}_t^{\nu^*}$$

$$+ \frac{\varphi}{k+\varphi}\,\rho\left\{\left(\boldsymbol{\mu}_t^+ + \boldsymbol{\mu}_t^-\right) - \frac{1}{(T-t)+\zeta}\int_t^T \mathbb{E}\left[\left(\boldsymbol{\mu}_s^+ + \boldsymbol{\mu}_s^-\right) \,\middle|\, \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mu}}\right] ds\right\}$$

$$- \frac{b}{k+\varphi}\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\,\mathbb{E}\left[\left(\boldsymbol{\mu}_s^+ - \boldsymbol{\mu}_s^-\right) \,\middle|\, \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mu}}\right] ds\,,$$

where $\mathcal{F}_t^{\boldsymbol{\mu}}$ denotes the natural filtration generated by $\boldsymbol{\mu}$, is the admissible optimal control we seek, and $\zeta = (k+\varphi)/(\alpha - \frac{1}{2}b)$.

Targeting VWAP: POV

$$\lim_{\alpha\to\infty} \nu_t^* = \frac{1}{T-t} \boldsymbol{Q_t^{\nu^*}} \tag{4a}$$

$$+ \frac{\varphi}{k+\varphi}\,\rho \left\{ \boldsymbol{\mu_t} - \frac{1}{T-t}\int_t^T \mathbb{E}[\, \boldsymbol{\mu_s} \,|\, \boldsymbol{\mathcal{F}_t^{\mu}} \,]\, ds \right\} \tag{4b}$$

$$- \frac{b}{k+\varphi} \int_t^T \frac{T-s}{T-t}\, \mathbb{E}\left[\boldsymbol{\mu_s^+} - \boldsymbol{\mu_s^-} \,|\, \boldsymbol{\mathcal{F}_t^{\mu}} \right] ds\,, \tag{4c}$$

The strategy consists of three components:

1. The first component is **TWAP-like**
2. The second consists of correcting for future **expected total order-flow**
3. The third consists of corrections due to **permanent price impact of expected net order-flow**

Targeting VWAP: POV

- Simulation Study
- Assume that other market participants trades follow

$$d\mu_t^+ = -\kappa^+ \, \mu_t^+ \, dt + \eta^+_{1+N^+_{t-}} \, dN_t^+ \,,$$
$$d\mu_t^- = -\kappa^- \, \mu_t^- \, dt + \eta^-_{1+N^-_{t-}} \, dN_t^- \,,$$

where

- $\kappa^{\pm} \geq 0$ are the mean-reversion rates,
- N_t^+ and N_t^- are independent homogeneous Poisson processes with intensities λ^+ and λ^-, respectively,
- $\{\eta_1^{\pm}, \eta_2^{\pm}, \dots\}$ are non-negative i.i.d. random variables with distribution function F, with finite first moment, independent from all processes.
- In addition, we require $\kappa^{\pm} > \lambda^{\pm} \, \mathbb{E}[\eta_1^{\pm}]$ is ergodic

Targeting VWAP: POV

- The solution to order-flow is

$$\mu_s^{\pm} = e^{-\kappa^{\pm}(s-t)}\,\mu_t^{\pm} + \int_t^s e^{-\kappa^{\pm}(s-u)}\,\eta^{\pm}_{1+N^{\pm}_{u^-}}\,dN_u^{\pm}\,,$$

so that

$$\mathbb{E}[\,\mu_s^{\pm}\,|\,\mathcal{F}_t^{\mu}\,] = e^{-\kappa^{\pm}(s-t)}\,\left(\mu_t^{\pm} - \psi^{\pm}\right) + \psi^{\pm}\,,$$

where $\psi^{\pm} = \frac{1}{\kappa^{\pm}}\,\lambda^{\pm}\,\mathbb{E}[\eta^{\pm}]$.

- $\psi^{\pm}$ act as the expected long-run activity of buy and sell orders
- As a consequence, the optimal trading strategy is given by

$$\lim_{\varphi\to\infty}\lim_{\alpha\to\infty}\nu_t^* = \frac{1}{T-t}\,\boldsymbol{Q_t^{\nu^*}} + \rho\left(1-\frac{1-e^{-\kappa(T-t)}}{\kappa(T-t)}\right)\left[\left(\boldsymbol{\mu_t^+}+\boldsymbol{\mu_t^-}\right)-\left(\boldsymbol{\psi^+}+\boldsymbol{\psi^-}\right)\right],$$

Targeting VWAP: POV

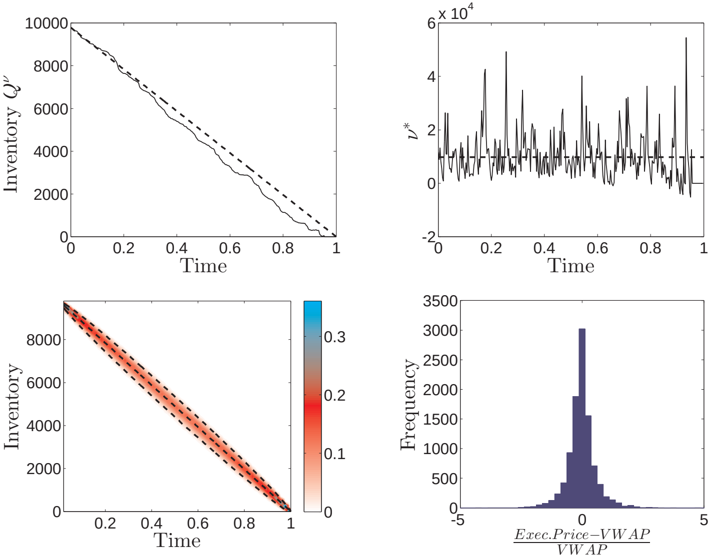
- Compute VWAP as

$$\text{VWAP} = \frac{\int_0^T S_u^\nu \, (\mu_u^+ + \mu_u^- + \nu_u) \; du}{\int_0^T \left(\mu_u^+ + \mu_u^- + \nu_u\right) \; du} \,,$$

- The strategy's execution price per share is

$$\text{Exec. Price} = \frac{X_T^\nu}{\mathfrak{N}} = \frac{\int_0^T \hat{S}_u^\nu \, \nu_u \, du}{\mathfrak{N}} \,.$$

Targeting VWAP: POV



Targeting VWAP: POCV

- **Cumulated volume** V of orders, excluding the agent's, is

$$V_t = \int_0^t \left(\mu_u^+ + \mu_u^-\right)\, du\,.$$

- The investor's performance criteria is now modified to

$$\begin{aligned} &H^\nu(t,x,S,\boldsymbol{\mu},q) \\ &= \mathbb{E}_{t,x,S,\boldsymbol{\mu},q}\Bigg[\overbrace{X_T^\nu}^{\text{Cash}} + \overbrace{Q_T^\nu\,(S_T^\nu - \boldsymbol{\alpha}\, Q_T^\nu)}^{\text{terminal liquidation}} \\ &\qquad - \underbrace{\tilde{\varphi}\int_t^T \Big((\mathfrak{N} - Q_u^\nu) - \tilde{\boldsymbol{\rho}}\,(\boldsymbol{V_u} + (\mathfrak{N} - \boldsymbol{Q_u^\nu}))\Big)^2 du}_{\text{POCV penalty}}\Bigg], \end{aligned}$$

Targeting VWAP: POCV

Proposition

The DPE admits the solution 1

$$H(t,x,S,\boldsymbol{\mu},V,q) = \underbrace{x + q\,S}_{\textbf{book value}} + \underbrace{\boldsymbol{h_0(t,\mu,V)} + \boldsymbol{h_1(t,\mu,V)}\,q + \boldsymbol{h_2(t)}\,q^2}_{\textbf{excess value: } h(t,\boldsymbol{\mu},V)},$$

where

$$h_2(t) = -\sqrt{k\,\varphi}\,\frac{\gamma\,e^{\xi\,(T-t)} + e^{-\xi\,(T-t)}}{\gamma\,e^{\xi(T-t)} - e^{-\xi\,(T-t)}} - \tfrac{1}{2}\,b\,,$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu},V) = 2\,\varphi \int_t^T &\ell(u,t)\,\left(\mathfrak{N} - \rho\,\mathbb{E}_{t,\boldsymbol{\mu},V}\,[V_u]\right)\,du \\ &+ b\int_t^T \ell(u,t)\,\mathbb{E}_{t,\boldsymbol{\mu},V}\left[(\boldsymbol{\mu}_u^+ - \boldsymbol{\mu}_u^-)\right]\,du\,, \end{aligned}$$

$$h_0(t,\boldsymbol{\mu},V) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\tfrac{1}{4k}\left(h_1(u,\boldsymbol{\mu}_u,V_u)\right)^2 - \varphi\left(\mathfrak{N} - \rho\,V_u\right)^2\right]\,du\,,$$

Targeting VWAP: POCV

- The DPE with the above ansatz reduces to

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) h - \varphi\left((\mathfrak{N} - q) - \rho\, V\right)^2 + b\,(\mu^+ - \mu^-)\, q + \sup_{\nu} \left\{ -\, k\, \nu^2 - (\partial_q h + b\, q)\, \nu \right\},$$

and the optimal control in feedback form is

$$\nu^*(t, \boldsymbol{\mu}, V) = -\tfrac{1}{2k}\,(\partial_q h + b\, q)$$

- Substituting further for h_0, h_1, and h_2...

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) \boldsymbol{h_2} + \tfrac{1}{k}\left(\boldsymbol{h_2} + \tfrac{1}{2} b\right)^2 - \varphi\,,$$

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) \boldsymbol{h_1} + \tfrac{1}{k}\left(h_2 + \tfrac{1}{2} b\right) \boldsymbol{h_1} + 2\,\varphi\,(\mathfrak{N} - \rho\, V) + b\,(\mu^+ - \mu^-)\,,$$

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) \boldsymbol{h_0} + \tfrac{1}{4k}\,(h_1)^2 - \varphi\,(\mathfrak{N} - \rho\, V)^2\,.$$

Targeting VWAP: POCV

- Optimal trading speed to ensure full liquidation is

$$
\begin{aligned}
\lim_{\alpha\to\infty} \nu_t^* = & \frac{\xi\, Q_t^{\nu^*}}{\sinh(\xi\,(T-t))} \\
& - \xi^2 \int_t^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-t))} \left\{ \left(\mathfrak{N} - Q_t^{\nu^*} \right) - \rho\, \mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\mu,V} \right] \right\} du \\
& - \frac{b}{2\,k} \int_t^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-t))}\, \mathbb{E}\left[(\mu_u^+ - \mu_u^-) \,\middle|\, \mathcal{F}_t^{\mu,V} \right] du\,,
\end{aligned}
$$

- First term is **Implementation Shortfall** like
- Second term corrects for deviations of current inventory from **future expected volume**
- Third term corrects for permanent impact from **expected future net order-flow**

Targeting VWAP: POCV

Using percentage of cumulative volume...