Funding, Repo and Credit Inclusion in Option Pricing via Dividends

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January 29, 2016

Abstract

This paper specializes a number of earlier contributions to the theory of valuation of financial products in presence of credit risk, repurchase agreements and funding costs. Earlier works, including our own, pointed to the need of tools such as Backward Stochastic Differential Equations (BSDEs) or semi-linear Partial Differential Equations (PDEs), which in practice translate to ad-hoc numerical methods that are time-consuming and which render the full valuation and risk analysis difficult. We specialize here the valuation framework to benchmark derivatives and we show that, under a number of simplifying assumptions, the valuation paradigm can be recast as a Black-Scholes model with dividends. In turn, this allows for a detailed valuation analysis, stress testing and risk analysis via sensitivities. We refer to the full paper [5] for a more complete mathematical treatment.

Keywords: Hedging, funding costs, counterparty risk, credit risk, re-purchase agreement, repo market, valuation adjustments, dividends

Mathematics Subjects Classification (2010): $91G40, 60J28$

*The research of Damiano Brigo and Marek Rutkowski was supported by the EPSRC Mathematics Platform Grant EP/I019111/1 *Mathematical Analysis of Funding Costs* at Imperial College London.

†The research of Marek Rutkowski was supported by the DVC Bridging Support Grant *BSDEs Approach to Models with Funding Costs* at the University of Sydney.

1 Introduction

The goal of this work is to show that alternative approaches to valuation in the presence of credit risk, re-purchase agreements and funding costs lead, under a few simplifying assumptions, to the same explicit formula, namely, the Black-Scholes pricing formula for vulnerable options on dividend-paying assets. This allows us to provide a closed-form solution for the benchmark product, the vulnerable call option. Section 2 investigates the replication of a defaultable bond using CDS contracts written on the same name as the bond, with special emphasis on the assumptions required for the default time. In Section 3, the replication approach is used to derive the PDE satisfied by the pre-default pricing function. We show that this PDE is equivalent to that obtained in [2] using the martingale measure approach. The solution to this PDE is then expressed as the Black and Scholes price when the underlying stock pays dividends, but with appropriately chosen parameters reflecting funding costs. In Section 4, the adjusted cash flow approach of [10] is used to obtain the price of the same option via an expected value of adjusted discounted cash flows, and show that it leads to exactly the same formula in terms of the Black-Scholes formula with dividends as in the replication/PDE approach of Section 3. The appeal of the Black-Scholes formula with dividends is its tractability, enabling the sensitivity analysis of the price in terms of funding and repo rates and credit spread, as outlined in Section 5.

2 Replication of a defaultable bond using CDS contracts

In this preliminary section, we discuss the issue of valuation of a defaultable bond in the simple model with funding account and traded CDSs. Special emphasis is put on the mathematical assumptions underpinning commonly used replication arguments, assumptions that are frequently neglected in the existing literature.

2.1 Dynamics of the defaultable bond price

Assume that we want to replicate a zero-recovery defaultable bond in a financial market with an unsecured funding account with rate f_t dubbed the treasury rate and a market CDS, which is traded at null price, on the company that issued the bond. The premium leg the CDS is assumed to pay a constant, continuous in time market spread r^{CDS} and the protection leg pays one at the default of the bond and nothing otherwise. Recall that the market spread is computed by equating the value of the protection leg with the value of the premium leg. As we shall show in Section 2.2, the present postulates regarding the market spread may only hold under specific assumptions on the probability distribution of the default time under the real-world probability.

The price process B of the zero-recovery defaultable bond maturing at T is given in terms of the point process J, which jumps to one when default occurs and stays zero otherwise. Specifically, we have

$$B_t = \mathbb{1}_{\{J_t=0\}}\widetilde{B}_t = \mathbb{1}_{\{\tau>t\}}\widetilde{B}_t$$

where the yet unspecified process $\widetilde{B}$ represents the pre-default price of the bond.

We will now provide intuitive replication arguments leading to the dynamics of the bond price; a more formal derivation is postponed to the next subsection. We assume here that there has been no default yet, but it may happen with a positive probability between the dates t and $t + dt$ for an arbitrarily small time increment dt. To show how to replicate a long position in the defaultable bond, let us consider the transactions an investor enters into at time $t < \tau \wedge T$:

1. borrow $\widetilde{B}_t$ from the treasury and use it to buy one defaultable bond;

2. buy a number $\widetilde{B}_t$ of CDS contracts on the same name.

We have established a long position in the defaultable bond, and everything else forms the reverse of the replicating portfolio. Hence, formally, the replicating portfolio consists of the short position in the CDS and the long position in the treasury.

We now look at investor's portfolio at time $t + dt$:

3. if there is a default ($J_{t+dt} = 1$) each of the $\widetilde{B}_t$ CDS contracts pays 1;

4. if there is no default ($J_{t+dt} = 0$), he sells the bond for $\widetilde{B}_{t+dt}$;

5. either way, he pays the premium leg $r^{CDS}dt$ for each of the $\widetilde{B}_t$ CDS contracts and pays back the loan to treasury, which amounts to $\widetilde{B}_t(1 + f_t\, dt)$.

The overall gain over the time interval $(t, t + dt)$ is

$$\widetilde{B}_t \mathbb{1}_{\{J_{t+dt}=1\}} + \widetilde{B}_{t+dt} \mathbb{1}_{\{J_{t+dt}=0\}} - \widetilde{B}_t r^{CDS} dt - \widetilde{B}_t(1 + f_t\, dt).$$

Equating this to zero to ensure replication and using the fact that the first indicator above is just dJ_t and that we assumed $J_t = 0$ (no default at time t), we obtain the dynamics for B

$$dB_t - B_t(r^{CDS} + f_t)\, dt + B_t\, dJ_t = 0, \tag{2.1}$$

and thus, since $B_T = \mathbb{1}_{\{\tau>T\}}$, we have for all $t \in [0, T]$

$$B_t = \mathbb{1}_{\{\tau>t\}} e^{-\int_t^T (r^{CDS}+f_u)\, du} \tag{2.2}$$

These dynamics were derived already in [9, Eq.(2.4)] using different arguments. They are also similar to those postulated a priori in [8]

$$\frac{dP_B}{P_B} = r_B(t)\, dt - dJ_t.$$

Note, however, that our rate is the CDS spread over the **treasury rate**, while in [8] it is the risk-neutral default intensity over the risk-free rate $r_B = \lambda_B + r$. This minor discrepancy is due to the fact that we derived the bond dynamics via replication using the market CDS, whereas in [8] it is assumed that the defaultable bond can be borrowed close to the "risk-free" rate using a repo. Moreover, the spread λ_B equals the risk-neutral default intensity under zero recovery convention.

2.2 Assumptions underpinning replication arguments

The above computations did not require the exact knowledge of a specific distribution of a random time modelling the default event. Nevertheless, certain conditions need in fact to be imposed on the default time for the replication argument to be valid. To explain why additional assumptions are needed, let us denote the CDS price process by $S(\kappa)$ with $\kappa = r^{CDS}$ and let the treasury rate be a constant $f > 0$.

Proposition 2.1 *The above replication of the defaultable bond holds whenever the probability distribution of τ is continuous and its support includes $[0, T]$.*

Proof. According to our assumptions, the process B^f satisfies $B^f_t = e^{ft}$ and the CDS price process jumps from zero to one at default and afterwards grows at the treasury rate

$$S(\kappa) = \mathbb{1}_{\{t\geq\tau\}} e^{f(t-\tau)}. \tag{2.3}$$

An essential assumption in this step is that the fair spread is constant. Since we should ensure that the model with two assets, B^f and $S(\kappa)$, is arbitrage-free, we postulate that there exists a probability measure $\mathbb{Q}$, equivalent to the real-world probability and such that $S_t(\kappa)$ is computed using the risk-neutral valuation under $\mathbb{Q}$. Standard computations show that, for a fixed spread κ, the equality

$S_t(\kappa) = 0$ may hold before default if and only if the intensity of default under $\mathbb{Q}$ is constant on $[0, T]$ (see, for instance, Section 2.4.2 in [3] or equation (2.7)). Consequently, the probability distribution of default time under the real-world probability is continuous with a positive density on $[0, T]$, so that this interval is included in the support of distribution of τ.

In the second step, we will show that the postulate that the interval $[0, T]$ is included in the support of the real-world probability distribution of τ is also required for the replication argument to be strict. To this end, let us consider a self-financing trading strategy $\varphi = (\varphi^1, \varphi^2)$ in assets $S(\kappa)$ and B^f, which is stopped at time $\tau \wedge T$ and replicates the defaultable bond, meaning that $V_{\tau\wedge T}(\varphi) = B_{\tau\wedge T} = \mathbb{1}_{\{\tau>T\}}$. A strategy $\varphi = (\varphi^1, \varphi^2)$ is self-financing if its value

$$V_t(\varphi) := \varphi^1_t S_t(\kappa) + \varphi^2_t B^f_t \tag{2.4}$$

satisfies

$$dV_t(\varphi) = \varphi^1_t \big(dS_t(\kappa) - \kappa\, d(t \wedge \tau)\big) + \varphi^2_t\, dB^f_t$$

since the CDS pays negative dividends at the constant rate $r^{CDS} = \kappa$ up to time τ. For the pre-default value of φ, denoted as $\widetilde{V}(\varphi)$, from (2.3) and (2.4), we obtain $\widetilde{V}_t(\varphi) = \varphi^2_t B^f_t$. Therefore,

$$dV_t(\varphi) = -\kappa\varphi^1_t\, dt + \varphi^2_t\, dB^f_t = -\kappa\varphi^1_t\, dt + f\widetilde{V}_t(\varphi)\, dt. \tag{2.5}$$

It is worth noting that our computations so far do not depend on the contingent claim we aim to replicate.

In the last step, we specialize our trading strategy to zero-recovery bond by postulating that $V_\tau(\varphi) = 0, \mathbb{P}$-a.s. This entails the jump of $V(\varphi)$ at τ satisfies

$$\Delta_\tau V(\varphi) = V_\tau(\varphi) - V_{\tau-}(\varphi) = -V_{\tau-}(\varphi) = -\widetilde{V}_{\tau-}(\varphi) = \varphi^1_\tau \Delta_\tau S(\kappa) = \varphi^1_\tau,$$

which in turn leads to the following condition (which formally holds a.e. with respect to the Lebesgue measure)

$$\varphi^1_t = -\widetilde{V}_{t-}(\varphi), \quad \forall t \in [0, T]. \tag{2.6}$$

Then the property that condition (2.6) holds a.e. is equivalent to the equality $\varphi^1_\tau = -\widetilde{V}_\tau(\varphi)$, provided that the distribution of τ under $\mathbb{P}$ is continuous and $[0, T]$ is included in its support. Then the replicating strategy is independent of a particular distribution of τ satisfying these conditions. In other words, the exact knowledge of this distribution is immaterial for the problem at hand.

Let us stress that equality (2.6) should not be postulated a priori when allowing for more general distributions of default time. For instance, when $\mathbb{P}(\tau \in (t_1, t_2)) = 0$, then we should set $\varphi^1_t = 0$ for all $t \in (t_1, t_2)$ (see also the analysis in [11] for the discontinuous case).

We are now in a position to derive explicitly the replicating strategy. By combining (2.5) with (2.6), we obtain

$$d\widetilde{V}_t(\varphi) = (\kappa + f)\widetilde{V}_t(\varphi)\, dt$$

with the terminal condition $\widetilde{V}_T(\varphi) = 1$. We recover (2.1)

$$\widetilde{V}_t(\varphi) = e^{-(\kappa+f)(T-t)} = \widetilde{B}_t$$

and so

$$B_t = \mathbb{1}_{\{\tau>t\}}\, e^{-(\kappa+f)(T-t)} = \mathbb{1}_{\{\tau>t\}}\, e^{-(r^{CDS}+f)(T-t)}.$$

We have thus shown that the strategy

$$\varphi^1_t = -e^{-(\kappa+f)(T-t)}, \quad \varphi^2_t = (B^f_t)^{-1} e^{-(\kappa+f)(T-t)}$$

is self-financing and replicates the defaultable bond B on $[0, \tau \wedge T]$. □

2.3 No-arbitrage and the martingale method

An arbitrage-free pricing model for the CDS and the defaultable bond can also be constructed using directly the so-called martingale mathod. In this modeling approach, one may take **any** probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ as a **postulated** martingale measure. To identify a martingale measure in our set-up, we require that $\mathbb{Q}$ should be consistent with the postulated properties of the CDS: the spread equals $\kappa(t,T) > 0$ and the CDS pays one unit of cash at the moment of default, provided that the default event occurs prior to or at T. As usual, the market CDS should have the value equal to zero at any time before the default event.

Simple computations show that, in general, the **market** (or **fair**) spread of the CDS can be computed from the following formula when the interest rate f is constant

$$\kappa(t,T) = -\frac{\int_{(t,T]} e^{-fu}\, dG(u)}{\int_{(t,T]} e^{-fu} G(u)\, du} \tag{2.7}$$

where $G(t) := \mathbb{Q}(\tau > t)$. As already mentioned, it is now possible to prove that the necessary and sufficient condition for the possibility of having a constant positive fair CDS rate $\kappa(t,T) = \kappa > 0$ is that the distribution of τ under $\mathbb{P}$ is continuous and has the support $[0,T]$, exactly what was assumed in Proposition 2.1. In essence, this is due to the fact that for any positive density function on $[0,T]$ there exists a unique probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the distribution of τ under $\mathbb{Q}$ is exponential with parameter $\lambda = \kappa$ where a constant $\kappa > 0$ is given in advance, provided that the interest rate f is constant (notwithstanding the level of f). Then $\kappa(t,T) = \kappa$ and λ has a natural interpretation as the default intensity under the CDS pricing measure $\mathbb{Q}$. The converse implication is valid as well.

Since we may show that the martingale measure $\mathbb{Q}$ is unique on $\mathcal{H}_T$, where $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau \le u\}}, u \le t)$ is the filtration generated by the default process, the model with two assets, the funding account and the CDS, is complete. Hence, from the Fundamental Theorem of Asset Pricing, any contingent claim X maturing at T can be replicated and its price, which is defined as the value of a replicating strategy, can be also computed using the following version of the classical risk-neutral valuation formula

$$V_t = B^f_t\, \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{B^f_T} \,\Big|\, \mathcal{H}_t \right). \tag{2.8}$$

For a claim with zero recovery, we obtain

$$V_t = B^f_t\, \mathbb{E}_{\mathbb{Q}}\left(\frac{X\mathbb{1}_{\{\tau > T\}}}{B^f_T} \,\Big|\, \mathcal{H}_t \right),$$

which reduces to

$$V_t = \mathbb{1}_{\{\tau > t\}}\, B^f_t (G_t)^{-1}\, \mathbb{E}_{\mathbb{Q}}\left(\frac{XG_T}{B^f_T} \right)$$

where

$$G_t = \mathbb{Q}(\tau > t) = e^{-\lambda t} = e^{-\kappa t}.$$

The defaultable bond corresponds to $X = 1$ and thus

$$B_t = \mathbb{1}_{\{\tau > t\}}\, B^f_t (G_t)^{-1} \frac{G_T}{B^f_T} = \mathbb{1}_{\{\tau > t\}}\, e^{-(\kappa + f)(T-t)}.$$

One may observe that the only claims with zero recovery in this model are zero-recovery bonds with differing, but constant, face values. Obviously, the valuation problem for claims with non-zero recovery can also be solved using (2.8).

3 Vulnerable call option pricing by replication via Black-Scholes formula with dividends

After a detailed analysis of valuation of the zero-recovery defaultable bond, we will now address a more advanced problem of valuation of vulnerable options on some risky asset. Once again, our goal is to compare various approaches and to identify the underlying assumptions, which are frequently neglected in the existing literature.

Denote by $\mathbb{F} = (\mathcal{F}_t)$ where $\mathcal{F}_t := \sigma(S_u, u \leq t)$ the natural filtration generated by the price process of a traded asset (stock). Let the maturity date T be fixed and let X be an $\mathcal{F}_T$-measurable integrable random variable. Assume that the default time τ is a positive random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The default time generates a filtration $\mathbb{H} = (\mathcal{H}_t)$ where $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau \leq u\}}, u \leq t)$, which is used to progressively enlarge $\mathbb{F}$ in order to obtain the full filtration $\mathbb{G} = (\mathcal{G}_t)$ where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$. We work under the assumption that $F_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t)$ is a continuous, increasing function and $F_t < 1$ for any t. Note that this assumption on the default time has already appeared in [9] in conjunction with the hypothesis (H) and is in line with what was assumed in Proposition 2.1.

Let A be a **contract** (vulnerable call option) that costs P_0 at time 0 and has the payoff X at maturity time T where

$$X = \mathbb{1}_{\{\tau > T\}}(S_T - K)^+.$$

Here τ is interpreted as the default time of the counterparty to the contract, that is, the issuer of the option. We wish to find the **price** P_t, $t \in [0, T]$, of this contract for an investor who replicates a long position using financial instruments available in the market.

We now consider a market with the following **primary assets** (A^1, A^2, A^3, A^4):

i) an unsecured funding account with the interest rate f;

ii) a stock (the underlying asset of the contract);

iii) a repo agreement on the stock with the repo rate h;

iv) a zero-recovery defaultable bond with the rate of return r^C issued by the counterparty.

At time t, the price P_t^i of the asset A^i is given by

$$P_t^1 = B_t^f, \;\; P_t^2 = S_t, \;\; P_t^3 = 0, \;\; P_t^4 = B_t$$

and the gains process since inception of A^i is denoted by G_t^i with $G_0^i = 0$ for all i.

As a preliminary step, we specify the model inputs: the treasury rate f, the repo rate h and the bond rate of return r^C. Note that the rates f, h and r^C are postulated to be constant (or, at least, deterministic) and they are known. We assume also that the process S is continuous (obviously, B^f is continuous as well). We will later assume, in addition, that the stock price volatility σ is known as well. Hence we seek for the pricing formula in terms of the model parameters f, h, r^C and σ and the option data: T and K.

Note that, in principle, all these quantities are observed in the market, provided that the volatility is understood as the implied volatility. By contrast, we do not need to assume that the CDS on the counterparty is traded, although this postulate would not change our derivation of the option pricing formula and the knowledge of the CDS spread r^{CDS} is immaterial. In fact, we know from the preceding section that, for a fixed level of the treasury rate f, there is one-to-one correspondence between r^C and r^{CDS}.

Let us now determine the gains processes. Buying one repo contract amounts to selling the shares of stock against cash, under the agreement of repurchasing them back at the higher price that includes the interest payments corresponding to the repo rate. (Selling the repo results in the opposite cash flows.) Any appreciation (or depreciation) in the stock price is part of the positive (or negative) gains, while the outgoing repo interest payments are negative gains: $dG_t^3 = dS_t - hS_t\,dt$.

Under the standing assumption that the pre-default rate of return r^C on the counterparty's bond is deterministic, we obtain

$$B_t = \mathbb{1}_{\{\tau>t\}}\, e^{-\int_t^T r_u^C\, du} = (1-J_t)e^{-\int_t^T r_u^C\, du}$$

where $J_t := \mathbb{1}_{\{\tau\le t\}}$ is the point process that models the jump to default of the counterparty. The gains have negative terms for outgoing cash flows corresponding to the drop in the bond value at the time of default. To summarize, the gains of primary assets are given by

$$dG_t^1 = fB_t^f\, dt,\;\; dG_t^2 = dS_t,\;\; dG_t^3 = dS_t - hS_t\, dt,\;\; dG_t^4 = r_t^C B_t\, dt - B_{t-}\, dJ_t. \tag{3.1}$$

A **trading strategy** $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$ gives the number of units of each primary asset purchased to build a portfolio. Let $\beta \in [0,1]$ be a constant. A trading strategy φ is **admissible** if at any date t the investor can only use the repo market for a fraction β of the stock amount required and the rest has to be obtained in the stock market with funding from the treasury. The **wealth** at time $t \in [0,T]$ of the portfolio resulted from an admissible strategy φ is denoted by V_t^φ and equals

$$V_t^\varphi = \sum_{i=1}^4 \varphi_t^i P_t^i$$

and the gains process associated with this strategy satisfies $G_0^\varphi = 0$ and

$$dG_t^\varphi := \sum_{i=1}^4 \varphi_t^i\, dG_t^i. \tag{3.2}$$

We then say that a strategy φ is **self-financing** if for all $t \in [0,T]$

$$V_t^\varphi = V_0^\varphi + G_t^\varphi. \tag{3.3}$$

An admissible trading strategy φ **replicates** the payoff of a contract A if $V_T^\varphi = X$. We define the time t price of a contract A as the wealth V_t^φ of the portfolio corresponding to the replicating strategy

$$P_t := V_t^\varphi. \tag{3.4}$$

The existence of the specific primary assets in our market ensures that any claim is **attainable**. In fact, the market under study is complete and no-arbitrage arguments show that the price of any contract is unique.

Recall that the replicating portfolio is the negative of the hedging portfolio that an investor holding the contract A would build to protect against market and counterparty risks. In other words, the replicating strategy replicates not only the payoff of the option, but also the market risk and the credit risk profiles of a long position in the option. At date t before default, the investor builds a replicating portfolio for a long position in the option knowing that the assumptions on τ imply that default may occur between t and $t+dt$ for an arbitrarily small dt. To replicate the contract the investor:

1. buys $\beta\Delta_t$ repos, borrows $\beta\Delta_t S_t$ from treasury to buy and deliver $\beta\Delta_t$ shares, and receives $\beta\Delta_t S_t$ cash which is paid back to treasury;
2. borrows $(1-\beta)\Delta_t S_t$ from treasury and buys $(1-\beta)\Delta_t$ shares;
3. buys P_t/B_t units of the counterparty bond in order to match the value of this portfolio and the option payoff.

Of course, at this moment the option price P_t is yet unknown, but it will be found from the matching condition (3.4) combined with the terminal payoff X. This replicating portfolio produces the following admissible strategy

$$\theta := \left(-\frac{(1-\beta)\Delta_t S_t}{B_t^f}, (1-\beta)\Delta_t, \beta\Delta_t, \frac{P_t}{B_t} \right). \tag{3.5}$$

At time $t+dt$ the investor:

4. receives $\beta\Delta_t$ shares from repo and sells them for $\beta\Delta_t S_{t+dt}$;
5. borrows from treasury $\beta\Delta_t S_t(1+hdt)$ to close the repo;
6. sells $(1-\beta)\Delta_t$ shares for $(1-\beta)\Delta_t S_{t+dt}$;
7. sells the counterparty's bond for $P_t B_{t+dt}/B_t$;
8. pays back to the treasury $(1-\beta)\Delta_t S_t(1+fdt)$.

From these transactions the change in the wealth of the replicating position is

$$\begin{aligned} V^\theta_{t+dt} - V^\theta_t &= \beta\Delta_t S_{t+dt} - \beta\Delta_t S_t(1+hdt) + (1-\beta)\Delta_t S_{t+dt} + \frac{P_t}{B_t}\,dB_t - (1-\beta)\Delta_t S_t(1+fdt) \\ &= \beta\Delta_t\,dS_t - \beta h\Delta_t S_t\,dt + (1-\beta)\Delta_t\,dS_t + \frac{P_t}{B_t}\,dB_t - (1-\beta)f\Delta_t S_t\,dt \\ &= \Delta_t\,dS_t - \big((1-\beta)f + \beta h\big)\Delta_t S_t\,dt + P_t(r^C dt - dJ_t). \end{aligned}$$

This can be derived formally by using (3.1) and computing the gains process (3.2) associated with the portfolio θ given by (3.5)

$$\begin{aligned} dG^\theta_t &= -\frac{(1-\beta)\Delta_t S_t}{B^f_t} fB^f_t\,dt + (1-\beta)\Delta_t\,dS_t + \beta\Delta_t(dS_t - hS_t\,dt) + \frac{P_t}{B_t}(r^C_t B_t\,dt - B_{t-}\,dJ_t) \\ &= \Delta_t\,dS_t - \big((1-\beta)f + \beta h\big)\Delta_t S_t\,dt + P_t(r^C_t dt - dJ_t) \end{aligned}$$

where we used the equality $B_{t-} = B_t$, which holds before default. Note also that the wealth of θ at default equals zero, which is consistent with the option payoff at default, and thus we may and do set $\theta_t = (0,0,0,0)$ for $t > \tau$.

Let us now focus on pricing before default. Since $dV^\theta_t = dG^\theta_t$ (from (3.3)) and $dP_t = dV^\theta_t$ (from (3.4)), we have

$$dP_t = \Delta_t\,dS_t - \big((1-\beta)f + \beta h\big)\Delta_t S_t\,dt + P_t(r^C_t dt - dJ_t). \tag{3.6}$$

To derive the pricing PDE, we assume that under the statistical probability $\mathbb{P}$ the stock price is governed by

$$dS_t = \mu_t S_t\,dt + \sigma S_t\,dW_t$$

and the price P_t can be expressed as

$$P_t = \mathbb{1}_{\{\tau>t\}}\widetilde{P}_t = \mathbb{1}_{\{\tau>t\}}v(t,S_t) = (1-J_t)v(t,S_t)$$

for some function $v(t,s)$ of class $\mathcal{C}^{1,2}$. Then the Ito formula yields

$$dP_t = (1-J_t)\,dv(t,S_t) + v(t,S_t)\,d(1-J_t) = (1-J_t)\,dv(t,S_t) - v(t,S_t)\,dJ_t$$

and

$$dP_t = (1-J_t)\Big(v_t(t,S_t) + \frac{\sigma^2 S_t^2}{2}v_{ss}(t,S_t)\Big)dt + (1-J_t)v_s(t,S_t)\,dS_t - v(t,S_t)\,dJ_t. \tag{3.7}$$

By equating the dS_t, dt and the jump terms in (3.6) and (3.7), we obtain the following equalities where the variables (t,S_t) were suppressed

$$\begin{aligned} \Delta_t &= (1-J_t)v_s, \\ (1-J_t)\Big(v_t + \frac{\sigma^2 S_t^2}{2}\,v_{ss} + \big((1-\beta)f + \beta h\big)S_t v_s\Big) - (1-J_t)r^C_t v &= 0, \\ -P_t\,dJ_t &= -v\,dJ_t. \end{aligned} \tag{3.8}$$

The pre-default pricing PDE for the function $v(t,s)$ is now obtained from (3.8) as

$$v_t + \big((1-\beta)f + \beta h\big)s\,\frac{\partial v}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial v^2}{\partial s^2} - r_t^C v = 0 \tag{3.9}$$

with terminal condition $v(T,s) = (s-K)^+$. One recognizes (3.9) as the Black-Scholes PDE when the underlying stock pays dividends. To see this, it suffices to take the discount rate to be the return on the defaultable bond $r := r^C$ and the instantaneous dividend yield to be the bond spread over the effective funding rate: $q := r^C - f^\beta$ where by the **effective funding rate** we mean the weighted average $f^\beta := (1-\beta)f + \beta h$. We conclude that the following result is valid.

Proposition 3.1 *The time t price of the vulnerable call option obtained by replication equals*

$$P_t = \mathbb{1}_{\{\tau>t\}}\Big(S_t e^{-q(T-t)}N(d_1^q) - Ke^{-r^C(T-t)}N(d_2^q)\Big) \tag{3.10}$$

with $q = r^C - f^\beta$ and

$$d_1^q = \frac{\log\frac{S_t}{K} + (r^C - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\quad d_2^q = d_1^q - \sigma\sqrt{T-t}.$$

It is worth noting that (3.10) may also be derived from (3.6) without resorting to the pricing PDE. From (3.6), we obtain the following equation for the pre-default price $\widetilde{P}$

$$d\widetilde{P}_t = -f^\beta \Delta_t S_t\,dt + \Delta_t\,dS_t + r_t^C \widetilde{P}_t\,dt. \tag{3.11}$$

Let $\mathbb{Q}^\beta$ be the probability measure equivalent to $\mathbb{P}$ and such that the drift of the risky asset S under $\mathbb{Q}^\beta$ is equal to the effective funding rate f^β. Then $\widetilde{P}$ is governed under $\mathbb{Q}^\beta$ by

$$d\widetilde{P}_t - r_t^C \widetilde{P}_t\,dt = \Delta_t \sigma S_t\,dW_t^\beta$$

with terminal condition $\widetilde{P}_T = (S_T-K)^+$ where W^β is the Brownian motion under $\mathbb{Q}^\beta$. This leads to the following probabilistic representation for $\widetilde{P}_t$

$$\widetilde{P}_t = e^{-r^C(T-t)}\,\mathbb{E}^\beta[(S_T-K)^+\,|\,\mathcal{F}_t] = e^{-(r^C-f^\beta)(T-t)}\,\mathbb{E}^\beta[e^{-f^\beta(T-t)}(S_T-K)^+\,|\,\mathcal{F}_t],$$

which in turn yields (3.10) through either standard computations of conditional expectation or by simply noting that it is given by the Black-Scholes formula with the interest rate f^β and no dividends.

Remarks 3.1 i) If we model the defaultable bond as in (2.2) with $r = r_t^C = r^{CDS} + f$ where the CDS spread r^{CDS} (rather than the bond return r^C) is taken as a model's input, then the pricing equation (3.10) holds with $q := r^{CDS} - \beta(h-f)$. In other words, the option pricing formula (3.10) is still valid when the defaultable bond is replaced by the counterparty's CDS in our trading model.

ii) PDE (3.9) is in fact equivalent to PDE (32) obtained in [2, Eg. 4.4] using the martingale approach. To see this, it suffices to rewrite (3.9) with the dynamics of the primary assets

$$\begin{aligned}
dB_t^f &= fB_t^f\,dt,\\
dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t,\\
dB_t &= B_{t-}(\mu_3\,dt - dM_t) = B_{t-}\big((\mu_3+\xi_t)dt - dJ_t\big),
\end{aligned}$$

where $\mu_3 = f$ and $M_t := J_t + \log G_{t\wedge\tau} = J_t - \xi_t$ where $G_t := \mathbb{P}(\tau > t|\mathcal{F}_t)$. The process M is commonly known as the compensated $\mathbb{G}$-martingale of the default process J.

iii) Though the stock S was assumed to pay no dividends, the present framework can be easily extended to the dividend-paying case. As a result, the effective funding rate f^β should be replaced by $f^\beta - \delta$. Then our PDE (3.8) would be similar to PDE (3.15) in [8] with zero-recovery, except that when replicating a long position in a vulnerable call option there is no exposure at the investor's default, so there is no need to consider a defaultable bond issued by the investor. To see the equivalence of the two PDEs, take their repo rate minus dividend rate $q_s - \gamma_s$ to be $\beta h + (1-\beta)f$ (they are the same for $\beta = 1$ and no dividends, i.e., $q_s = 0$) and their λ_B to be our λ. At time of default their market value results in a drop in derivative value of $\Delta\hat{V}_B = -\hat{V}$ (see (3.8) in [8]).

4 Vulnerable call option pricing by adjusted cash flows method via Black-Scholes formula with dividends

Let us consider again the problem of pricing the same vulnerable option, but this time using the **adjusted cash flows approach** originated in [10], derived rigorously in [6] and presented in a wider context in [7]. We do not make here an attempt to justify their approach, but we start instead with the pricing equation (11) of [7] and adapt it to the present context of a vulnerable call option, which is an uncollateralized contract. Note that the variation margin is M, while N^C and N^I are the initial margin accounts for the two counterparties, resulting in the total collateral account $C = M + N^C + N^I$. In our case, this means that the last two lines of equation (11) in [10] are simply zero. The cash flow at default equals zero (due to zero recovery convention for the vulnerable option) and the contract cash flow over time period $(t, t + dt)$ is

$$\Pi(t, t + dt) = (S_T - K)^+ \mathbb{1}_{\{t=T\}},$$

so the pricing equation (11) in [10] reduces to

$$V_t = E^h \big[\mathbb{1}_{\{\tau>T\}} D(t, T; f)(S_T - K)^+ \,|\, \mathcal{G}_t\big] \tag{4.1}$$

where $\mathbb{E}^h$ is the expectation under the probability measure $\mathbb{Q}^h$ that makes the drift of the risky asset equal to h, so that

$$dS_t = hS_t\, dt + \sigma S_t\, dW_t^h$$

where W^h is a Brownian motion under $\mathbb{Q}^h$. Moreover, $\mathbb{G} = (\mathcal{G}_t)$ is the full filtration that includes information on default times and the discount factor $D(s, t; f)$ equals

$$D(s, t; f) := \exp\left(-\int_s^t f_u\, du\right).$$

We assume a constant treasury rate f and we use the pre-default intensity λ under $\mathbb{Q}^h$ of the counterparty, which is defined in [7, (40)] by

$$\mathbb{1}_{\{\tau>t\}}\lambda\, dt := \mathbb{Q}^h(\tau \in dt \,|\, \tau > t, \mathcal{F}_t),$$

to obtain the survival probability $G_t^h = e^{-\lambda t}$ where $G_t^h := \mathbb{Q}^h(\tau > t \,|\, \mathcal{F}_t)$. Note that this is consistent with the assumptions on τ in the replication approach of Section 3. Using (4.1) and Cor. 3.1.1 of [1], we obtain

$$V_t = \mathbb{1}_{\{\tau>t\}}(G_t^h)^{-1}\, \mathbb{E}^h[D(t, T; f)(S_T - K)^+ G_T^h \,|\, \mathcal{F}_t].$$

If $\widetilde{V}$ denotes the $\mathbb{F}$-adapted pre-default price process such that for all $t \in [0, T]$

$$\mathbb{1}_{\{\tau>t\}} V_t = \mathbb{1}_{\{\tau>t\}} \widetilde{V}_t,$$

then from the above equation we immediately obtain

$$\widetilde{V}_t = (G_t^h)^{-1}\, \mathbb{E}^h[D(t, T; f)(S_T - K)^+ G_T^h \,|\, \mathcal{F}_t].$$

Since G^h is deterministic, for a constant treasure rate f the pre-default price can be written as

$$\widetilde{V}_t = e^{-(\lambda+f)(T-t)}\, \mathbb{E}^h[(S_T - K)^+ \,|\, \mathcal{F}_t]$$

or, equivalently,

$$\widetilde{V}_t = e^{-(\lambda+f-h)(T-t)}\, \mathbb{E}^h[e^{-h(T-t)}(S_T - K)^+ \,|\, \mathcal{F}_t].$$

The last expectation can be computed yielding the usual Black-Scholes formula when the drift of the stock equals h

$$\mathbb{E}^h[e^{-h(T-t)}(S_T - K)^+) \,|\, \mathcal{F}_t] = S_t N(d_1) - Ke^{-h(T-t)} N(d_2)$$

where
$$d_1 = \frac{\log \frac{S_t}{K} + (h + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$
We conclude that the pre-default price process satisfies
$$\widetilde{V}_t = e^{-(\lambda+f-h)(T-t)}\big(S_t N(d_1) - K e^{-h(T-t)} N(d_2)\big)$$
and thus
$$V_t = \mathbb{1}_{\{\tau>t\}}\Big(S_t e^{-(\lambda+f-h)(T-t)} N(d_1) - K e^{-(\lambda+f)(T-t)} N(d_2)\Big). \tag{4.2}$$
Upon setting $\lambda + f - h = q$ and $\lambda + f = r$, we deduce that (4.2) coincides with the pricing formula (3.10) obtained by replication for $\beta = 1$. It is also not difficult to show that $\lambda = r^{CDS}$ (in essence, this is due to the fact that the density of $\mathbb{Q}^h$ with respect to the martingale measure $\mathbb{Q}$ introduced in Section 2.2 is $\mathbb{F}$-adapted). This shows that the adjusted cash flow method and the replication approach lead to the same price for the vulnerable call option.

5 Sensitivity analysis

In the final step, we perform the sensitivity analysis for the vulnerable call option by focussing on the impact of the rates f and h. We leave aside the parameter r^C, since in our model the investor has the freedom to choose a particular combination of funding sources when purchasing shares, as formally represented by the parameter $\beta \in [0,1]$, but the spread r^C is given by the market and is assumed to be fixed.

Example 5.1 Figure 5.1 shows the dependence of the pre-default price (4.2) of a vulnerable call on the treasury rate f and repo rate h for $\beta = 1$ when $S_t = 80$, $K = 100$, $\sigma = 0.3$, $T - t = 0.1$, $r^{CDS} = 0.05$. The pre-default price of the vulnerable call is decreasing in the treasury rate f and increasing in the repo rate h.

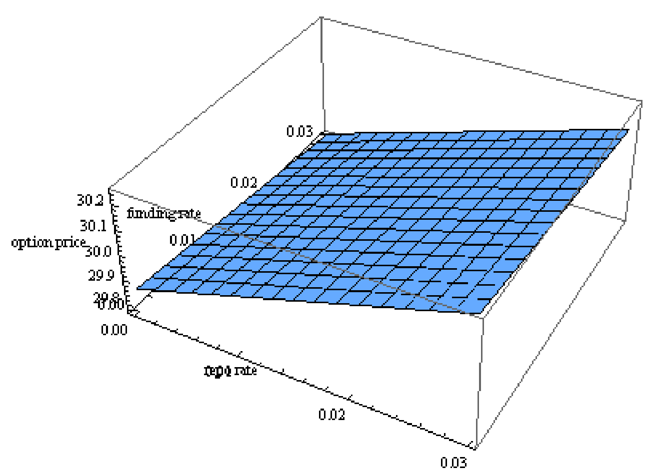

Figure 1: Option price is increasing in repo rate h and decreasing in funding rate f

To explain the dependence observed in Example 5.1 and perform a general sensitivity analysis, we first compute "funding Greeks" when $\beta = 1$, that is, all shares are purchased at repo. We obtain the following expressions

$$\partial_f \widetilde{V}_t = \partial_{r^C} \widetilde{V}_t = -(T-t)\widetilde{V}_t < 0, \tag{5.1}$$
$$\partial_h \widetilde{V}_t = e^{-(h-f-r^{CDS})(T-t)}(T-t)S_t N(d_1^q) > 0, \tag{5.2}$$

which means that the pre-default call price decreases in both the treasury rate f and the bond return r^C, but increases in the repo rate associated with the risky asset. Furthermore, the relative sensitivity to funding $\frac{\partial_f \widetilde{V}_t}{\widetilde{V}_t} = -(T-t)$ appears to be smaller in absolute value than the relative sensitivity to the repo rate $\frac{\partial_h \widetilde{V}_t}{\widetilde{V}_t} > T-t$, the more so the more the option is in the money. This simple benchmark case highlights that the repo rate may have an important impact on the contract value, often more significant than the treasury rate or the credit spread.

Let us now consider the price obtained in Section 3 where the additional parameter $\beta \in [0,1]$ dictates the structure of the funding arrangements for the investor. In view of (3.10) and Remark 3.1 i), we obtain the following funding Greeks

$$\begin{aligned} \partial_f \widetilde{V}_t &= -\beta(T-t)\widetilde{V}_t + (1-\beta)(T-t)e^{(\beta(h-f)-r^{CDS})(T-t)}KN(d_2^q), \\ \partial_h \widetilde{V}_t &= \beta e^{(\beta(h-f)-r^{CDS})(T-t)}(T-t)S_t N(d_1^q) \geq 0, \end{aligned} \tag{5.3}$$

where the last inequality is strict when $\beta > 0$. In particular, for $\beta = 1$ we recover (5.1)-(5.2) and for $\beta = 0$ (pure treasury funding), we get

$$\begin{aligned} \partial_f \widetilde{V}_t &= (T-t)e^{(f-r^C)(T-t)}KN(d_2^q) > 0, \\ \partial_h \widetilde{V}_t &= 0, \end{aligned}$$

where $f - r^C = r^{CDS} > 0$. In general, it is hard to determine the sign of the sensitivity $\partial_f \widetilde{V}_t$ given by (5.3), though it is clear that it changes from a positive value for $\beta = 0$ to a negative one for $\beta = 1$.

To give an interpretation of funding Greeks, we observe that the contract's payoff can be written as $X = B_T(S_T - K)^+$, so it can be seen as a hybrid contract which combines the call option on the stock with the long position in the counterparty bond. For any $0 < \beta \leq 1$ the price $\widetilde{V}_t$ increases in h, since the cost of hedging the option component $(S_T - K)^+$ is manifestly increasing with h.

The price dependence on f is harder to analyze. Indeed, from representation (3.5) of the hedging portfolio, we see that for $0 < \beta < 1$ the dependence on f is rather complex: the investor needs to borrow cash from B^f (which grows at the rate f) and thus the cost of hedging increases in f, but he simultaneously invests in the bond B (with the rate of return $r^C = f + r^{CDS}$ where r^{CDS} is constant) so that the cost of hedging decreases in f. The net impact of both legs may be negative, in the sense that the price of the option decreases when f increases. This is rather clear for $\beta = 1$, since in that case the investor does not use B^f for his hedging purposes (take $\beta = 1$ in (3.5)) and we see that the cost of hedging the component B_T in the payoff X falls when f increases. By contrast, when $\beta = 0$ the value of h is immaterial, and the increase of f makes the option more expensive. Finally, when only the CDS spread r^{CDS} increases and f, h are kept fixed, then the cost of hedging decreases as well, since the bond B becomes cheaper.

6 Concluding remarks

The mapping of the price computation to the Black-Scholes formula with dividends can be generalized to any local volatility model (e.g., the displaced diffusion model), which would thus cover both an increasing and a decreasing volatility smile. Stochastic volatility models would also be attractive to the industry, but any additional source of randomness would need to be hedged, thus requiring the inclusion of additional hedging instruments to our market model and solving a suitable modification of the pricing PDE. The local stochastic volatility models currently dominant in the industry would obviously pose the same problem.

In summary, we have shown that two alternative pricing approaches lead to the same result in the benchmark case of a vulnerable call option that includes funding, repo and credit risk. This confirms that even in the presence of funding costs and repo contracts the martingale method and the adjusted cash flow approach should be seen as alternative tools facilitating the computation of the replication price, rather than as alternative pricing paradigms. The reason for this is that all these approaches are in fact either explicitly or implicitly based on the concept of replication, as explained more generally in [5]. Furthermore, we show that the option price can be expressed as a Black-Scholes formula with dividends, thus facilitating the use of the funding Greeks in the valuation and sensitivity analyses. In this context, we highlight the potentially important pricing impact of the repo rate over the treasury rate and credit spread.

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