Algorithmic Trading: Order-Flow
PIMS Summer School

Sebastian Jaimungal, U. Toronto

Álvaro Cartea,U. Oxford

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José Penalva,(U. Carlos III)
Luhui Gan (U. Toronto)
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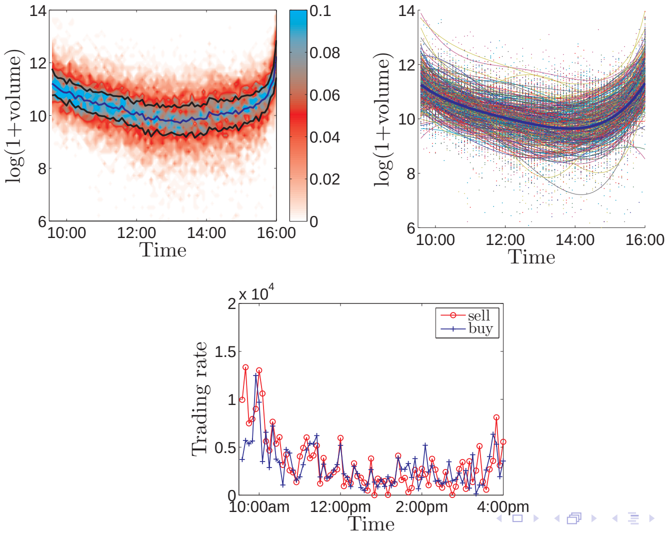
Order-Flow

- **Trade volume** and intensity **is stochastic** throughout the day
- **All order-flow affects prices**... not just "me"
- How can we **use order-flow information** and prediction **to improve execution performance**?

Order-Flow

Figure: ORCL (2013) traded volume using 5 minute buckets.



Order-Flow

We assume a **linear relationship** between **net order-flow** and **midprice changes**, thus for every trading day we perform the regression

$$\Delta S_n = b\,\mu_n + \varepsilon_n\,,$$

where

- $\Delta S_n = S_{n\tau} - S_{(n-1)\tau}$ is the **change in midprice**
- μ_n is **net order-flow** defined as the difference between the volume of buy and sell MOs during the time interval $[(n-1)\tau,\ n\tau]$, and
- ε_n is the **error term** (assumed normal)
- In the empirical analysis we choose $\tau = 5\,\text{min}$ intervals.

Order-Flow

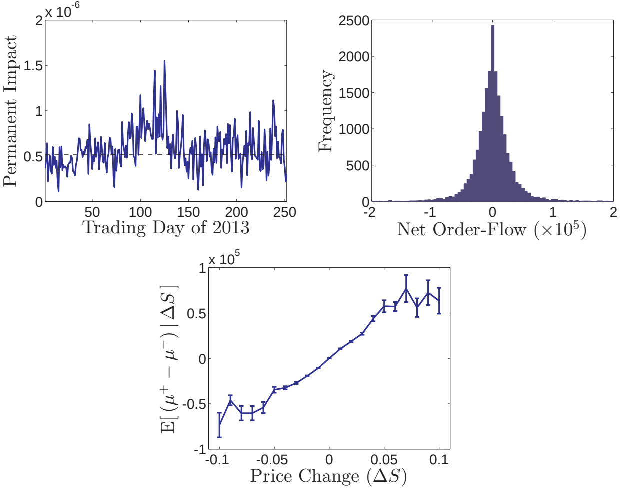


Figure: Order-Flow and effect on the drift of midprice of INTC.

Order-Flow

- Assume that **temporary price impact** is **linear** in the **trading rate** so that the **execution price** traders receive is

$$\hat{S}_t = S_t - \tfrac{1}{2}\Delta - k\,\nu\,,$$

- To estimate k we
  - Take snapshots of the LOB every second,
  - Determine the price per share for various volumes (by walking through the LOB),
  - Compute the difference between the price per share and the best quote at that time,
  - Perform a linear regression.

Order-Flow

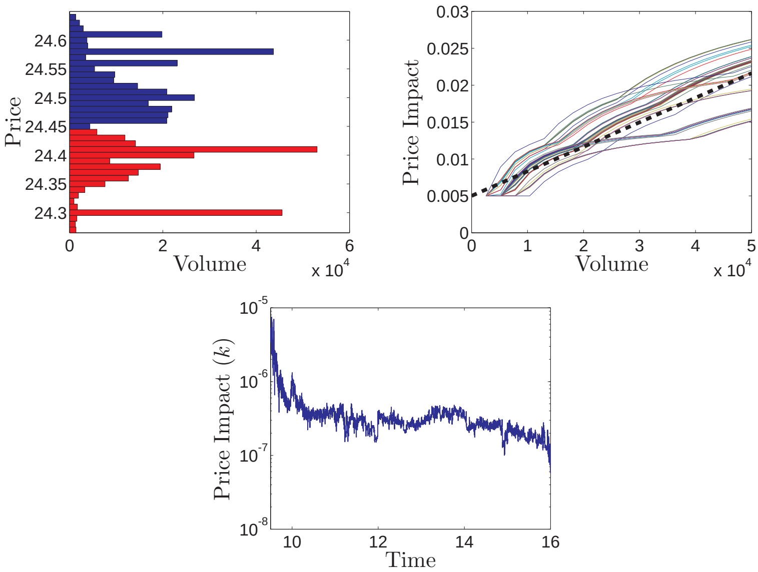


Figure: Temporary price impact: INTC on Nov 1, 2013. (a) at 11:00am,

Order-Flow

Table: Permanent and temporary price impact parameters for Nasdaq stocks, ADV of MOs, average midprice, σ volatility (hourly) of arithmetic price changes, mean arrival (hourly) of MOs $\lambda^{\pm}$, and average volume of MOs $\mathbb{E}[\eta^{\pm}]$. Data are from Nasdaq 2013.

	FARO		SMH		NTAP	
	mean	stdev	mean	stdev	mean	stdev
ADV	23,914	14,954	233,609	148,580	1,209,628	642,376
midprice	40.55	6.71	37.90	2.44	38.33	3.20
σ	0.151	0.077	0.067	0.039	0.078	0.045
b	1.41E-04	9.61E-05	5.45E-06	4.20E-06	5.93E-06	2.31E-06
k	1.86E-04	2.56E-04	8.49E-07	8.22E-07	3.09E-06	1.75E-06
b/k	1.02	0.83	7.43	6.24	2.04	0.77
λ^+	16.81	9.45	47.29	28.13	300.52	144.48
$\mathbb{E}[\eta^+]$	103.56	21.16	377.05	118.05	308.45	53.09
λ^-	17.62	10.69	46.37	27.62	293.83	136.13
$\mathbb{E}[\eta^-]$	104.00	21.79	381.70	126.74	312.81	49.86

Order-Flow

- **Temporary price impact**

$$\hat{S}_t^{\nu} = S_t^{\nu} - \tfrac{1}{2}\Delta - a\,\boldsymbol{\nu_t}$$

- **Price process** is affected by order-flow from all agents, i.e., **permanent price impact**

$$dS_t^{\nu} = b\,\underbrace{\left(\boldsymbol{\mu_t^+} - (\boldsymbol{\nu_t} + \boldsymbol{\mu_t^-})\right)}_{\textbf{net order-flow}}\,dt + dM_t$$

- **Cash process** is

$$X_t^{\nu} = \int_0^t \left(S_u^{\nu} - \tfrac{1}{2}\Delta - a\,\nu_u\right)\nu_u\,du$$

Order-Flow

- Agent aims to solve for

$$H(t,x,q,S,\boldsymbol{\mu}) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,q,S,\boldsymbol{\mu}}\Bigg[\overbrace{X_T^\nu + (S_T^\nu - \tfrac{1}{2}\Delta)\, q_T^\nu}^{\text{terminal book-value}} - \overbrace{\boldsymbol{\alpha\,(q_T^\nu)^2}}^{\text{terminal inventory penalty}} \\ - \underbrace{\boldsymbol{\phi\,\sigma^2 \int_t^T (q_u^\nu)^2\, du}}_{\text{running inventory penalty}} \Bigg]$$

- The DPP suggest that the value function should satisfy

$$0 = \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H + \mathscr{L}^{\boldsymbol{\mu}} H - \phi\, q^2 \\ + \sup_{\nu} \Big\{ (S - \tfrac{1}{2}\Delta - a\,\boldsymbol{\nu})\,\boldsymbol{\nu}\,\partial_x H \\ + (S + b\,(\mu^+ - \mu^- - \boldsymbol{\nu}))\,\partial_S H - \boldsymbol{\nu}\,\partial_q H \Big\}$$

subject to the terminal condition

$$H(T,x,q,S,\boldsymbol{\mu}) = x + q\,(S - \tfrac{1}{2}\Delta) - \alpha\, q^2$$

Order-Flow

Proposition

Solving the DPE. *The DPE admits the solution*

$$H(t, x, S, \boldsymbol{\mu}, q) = x + \boldsymbol{q}\ \left(S - \tfrac{1}{2}\Delta\right) + \boldsymbol{h_0(t, \mu)} + \boldsymbol{q}\ \boldsymbol{h_1(t, \mu)} + \boldsymbol{q}^2\ \boldsymbol{h_2(t)}\,, \quad (1)$$

where

$$h_2(t) = -\sqrt{k\,\phi}\,\frac{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}} - \tfrac{1}{2}\,b\,, \quad (2a)$$

$$h_1(t, \boldsymbol{\mu}) = b \int_t^T \left(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} \right) \mathbb{E}_{t,\boldsymbol{\mu}} \left[\boldsymbol{\mu}_u^+ - \boldsymbol{\mu}_u^- \right] du\,, \quad (2b)$$

$$h_0(t, \boldsymbol{\mu}) = \frac{1}{4k} \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}} \left[\boldsymbol{h_1^2(t, \mu_u)} \right] du\,, \quad (2c)$$

with the constants

$$\gamma = \sqrt{\frac{\phi}{k}}\,, \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}b + \sqrt{k\,\phi}}{\alpha - \frac{1}{2}b - \sqrt{k\,\phi}}\,. \quad (2d)$$

Order-Flow

Theorem

Verification. *The candidate value function is indeed the solution to the optimal control problem. Moreover, the optimal trading speed is given by*

$$\begin{aligned}\nu_t^* = & \gamma \frac{\zeta\, e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta\, e^{\gamma(T-t)} - e^{-\gamma(T-t)}}\, \boldsymbol{Q_t^{\nu^*}} \\ & - \frac{b}{2k} \int_t^T \left(\frac{\zeta e^{\gamma(T-u)} - e^{-\gamma(T-u)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} \right) \mathbb{E}\left[\boldsymbol{\mu_u^+ - \mu_u^-} \mid \boldsymbol{\mathcal{F}_t^{\mu}} \right] du\,,\end{aligned} \tag{3}$$

is the admissible optimal control we seek, and $\mathcal{F}_t^{\boldsymbol{\mu}}$ denotes the natural filtration generated by $\boldsymbol{\mu}$.

Order-Flow

- If the agent requires full liquidation

$$\lim_{\alpha\to\infty} \nu_t^* = \gamma \, \frac{\cosh(\gamma\,(T-t))}{\sinh(\gamma\,(T-t))} \, \boldsymbol{Q_t^{\nu^*}} - \frac{b}{2k} \int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))} \, \mathbb{E}\left[\boldsymbol{\mu_u^+ - \mu_u^-} \, | \boldsymbol{\mathcal{F}_t^\mu} \right] du \,.$$

- And if running penalty is absent...

$$\lim_{\phi\to 0} \lim_{\alpha\to\infty} \nu_t^* = \frac{1}{(T-t)} \, \boldsymbol{Q_t^{\nu^*}} - \frac{b}{2k} \int_t^T \frac{(T-u)}{(T-t)} \, \mathbb{E}\left[\boldsymbol{\mu_u^+ - \mu_u^-} \, | \boldsymbol{\mathcal{F}_t^\mu} \right] du \,.$$

Order-Flow

Simulation Study

- Assume that other market participants trades follow

$$d\mu_t^+ = -\kappa^+ \mu_t^+ \, dt + \eta^+_{1+N^+_{t-}} \, dN_t^+ ,$$
$$d\mu_t^- = -\kappa^- \mu_t^- \, dt + \eta^-_{1+N^-_{t-}} \, dN_t^- ,$$

  where

  - $\kappa^\pm \geq 0$ are the mean-reversion rates,
  - N_t^+ and N_t^- are independent homogeneous Poisson processes with intensities λ^+ and λ^-, respectively,
  - $\{\eta_1^\pm, \eta_2^\pm, \dots\}$ are non-negative i.i.d. random variables with distribution function F, with finite first moment, independent from all processes.
  - In addition, we require $\kappa^\pm > \lambda^\pm \, \mathbb{E}[\eta_1^\pm]$ is ergodic

Order-Flow

- The solution to order-flow is

$$\mu_s^\pm = e^{-\kappa^\pm (s-t)}\,\mu_t^\pm + \int_t^s e^{-\kappa^\pm(s-u)}\,\eta^\pm_{1+N^\pm_{u-}}\,dN_u^\pm\,,$$

  so that

$$\mathbb{E}[\,\mu_s^\pm\,|\,\mathcal{F}_t^\mu\,] = e^{-\kappa^\pm(s-t)}\,\left(\mu_t^\pm - \psi^\pm\right) + \psi^\pm\,,$$

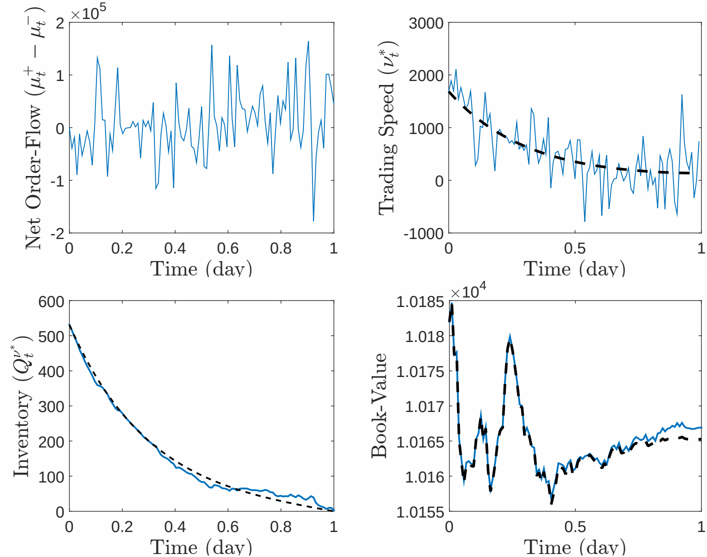
  where $\psi^\pm = \frac{1}{\kappa^\pm}\,\lambda^\pm\,\mathbb{E}[\eta^\pm]\,.$

- $\psi^\pm$ act as the expected long-run activity of buy and sell orders

- As a consequence, the optimal trading strategy is given by

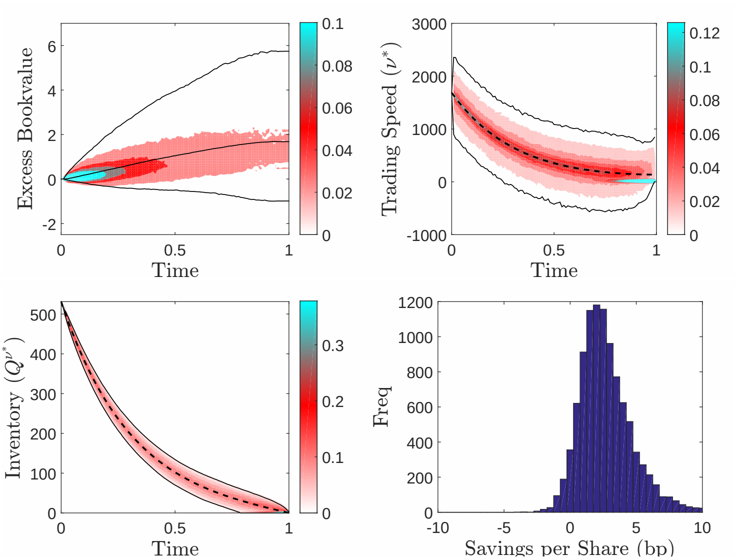
$$\lim_{\alpha\to\infty} \nu_t^* = \gamma\,\frac{\cosh(\gamma\,(T-t))}{\sinh(\gamma\,(T-t))}\,\boldsymbol{Q_t^{\nu^*}} - \frac{b}{2k}\left(\int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))}\,e^{-\kappa(u-t)}\,du\right)\boldsymbol{(\mu_u^+ - \mu_u^-)}\,.$$

Order-Flow



$\times 10^5$
Net Order-Flow $(\mu_t^+ - \mu_t^-)$
Time (day)
Trading Speed (ν_t^*)
Time (day)
Inventory $(Q_t^{\nu^*})$
Time (day)
$\times 10^4$
Book-Value
Time (day)

Order-flow



Excess Bookvalue
Time
Trading Speed (ν^*)
Time
Inventory (Q^{ν^*})
Time
Freq
Savings per Share (bp)