# FAIR BILATERAL PRICING UNDER FUNDING COSTS AND EXOGENOUS COLLATERALIZATION*

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**Abstract**

Bielecki and Rutkowski (2015) introduced and studied a generic non-linear market model, which includes several risky assets, multiple funding accounts and margin accounts. In this paper, we examine the pricing and hedging of contract both from the perspective of the hedger and the counterparty with arbitrary initial endowments. We derive inequalities for unilateral prices and we study the range of fair bilateral prices. We also examine the positive homogeneity and monotonicity of unilateral prices with respect to the initial endowments. Our study hinges on results from Nie and Rutkowski (2014a) for BSDEs driven by continuous martingales, but we also derive the pricing PDEs for path-independent contingent claims of a European style in a Markovian framework.

**Keywords**: hedging, fair prices, funding costs, margin agreement, BSDE, PDE

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# 1 Introduction

Bielecki and Rutkowski (2015) introduce and study a generic non-linear trading model for collateralized contracts, which includes several risky assets, multiple funding accounts and the margin account (for a related research, see also Brigo et al. (2011), Brigo and Pallavicini (2014), Burgard and Kjaer (2009, 2011), Crépey et al. (2014), Pallavicini et al. (2012), and Piterbarg (2010)). Using a suitable version of the concept of no-arbitrage, they first discuss the hedger's fair price for a contract in the market model without collateralization. Subsequently, under the assumption that a collateralized contract can be replicated, they define the hedger's ex-dividend price. They also show that the theory of backward stochastic differential equations (BSDEs) is an important tool to compute the ex-dividend price (see Propositions 5.2 and 5.4 in Bielecki and Rutkowski (2015)). It is worth mentioning that the pricing and hedging arguments in Bielecki and Rutkowski (2015) are given from the perspective of the hedger and no attempt was made there to study the range of fair bilateral prices.

In this work, we examine the issue of pricing and hedging of an over-the-counter contract from the perspective of the hedger and his counterparty. Since we work within a non-linear trading set-up, where the non-linearity may arise due to the different cash interest rates, funding costs for risky assets and asymmetric remuneration of the margin account (that is, collateral), the prices computed by the two parties will typically be different. One of our goals is to compare the unilateral prices and to derive the range for no-arbitrage prices. In the case of a model with different lending and borrowing rates, as studied by Bergman (1995), the no-arbitrage price of any contingent claims must belong to an arbitrage-free range with the lower (resp., upper) bound given by the counterparty's (resp., the hedger's) price of the contract. In a recent work by Mercurio (2013), the author extended the results from Bergman (1995) by examining the pricing of collateralized European options.

Bielecki and Rutkowski (2015) argue that the initial endowments of the counterparties play an important role in derivatives pricing within a non-linear set-up. Indeed, one may show that the ex-dividend price may depend on an initial endowment, in general. Note in this regard that the results established in Bergman (1995) and Mercurio (2013) only cover the case of null initial endowments. One of our main goals is to examine how the initial endowment of each party affects his unilateral price. For the sake of concreteness, we consider the model with partial netting and rehypothecated cash collateral, which was introduced in Bielecki and Rutkowski (2015). For a similar analysis within the set-up of Bergman's model, we refer to Nie and Rutkowski (2015). Although we study collateralized contracts, we focus on general properties of prices under funding costs, rather than the impact of the counterparty credit risk. For this reason, the default times and close-out payoffs are not introduced in our set-up (as opposed, for instance, to recent papers by Crépey (2015a, 2015b) or Bichuch et al. (2015a, 2015b)).

This work is organized as follows. In Section 2, we describe the non-linear trading model considered in this work. In Section 3, we study the no-arbitrage property and we extend some preliminary results from Bielecki and Rutkowski (2015) to contracts with an exogenous margin account. We also introduce and discuss the concepts of *fair bilateral prices* and *bilaterally profitable prices*. In Section 4, we show that the pricing and hedging problems for both parties can be represented by solutions of certain BSDEs and we establish the existence and uniqueness results for these BSDEs. Although the BSDEs are well known to be a convenient tool to deal with prices and hedging strategies (see, for instance, El Karoui et al. (1997)), we stress that the BSDEs studied in this work are formally derived using no-arbitrage arguments under a judiciously chosen martingale measure, whereas in some recent papers on funding costs the existence of a 'risk-neutral probability' is postulated a priori. In Section 5, which is the main part of this work, we examine the properties of unilateral prices. We establish there several inequalities for unilateral prices, which in turn allow us to obtain the range for fair bilateral prices. We also examine the monotonicity and positive homogeneity of prices with respect to the initial endowment. In Section 6, we derive the pricing PDEs for both parties within a Markovian framework. We conclude the paper by presenting an example of a contract with a non-empty interval of bilaterally profitable prices.

# 2 Trading under Funding Costs and Collateralization

Let $T > 0$ be a fixed finite trading horizon date for our model of the financial market. We denote by $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ a filtered probability space where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness. For convenience, we assume that the initial $\sigma$-field $\mathcal{G}_0$ is trivial. All probability measures are assumed to be defined on $(\Omega, \mathcal{G}_T)$. Moreover, all processes introduced in what follows are implicitly assumed to be $\mathbb{G}$-adapted. We use the following notation for the market data where $i = 1, 2, \ldots, d$:
$B^l$ (resp., $B^b$) – the unsecured *lending* (resp., *borrowing*) *cash account*,
$S^i$ – the *ex-dividend price* of the $i$th risky asset with the *cumulative dividend stream* $A^i$,
$B^{i,b}$ – the *funding account* for long (cash) positions in the $i$th risky asset,
$B^{c,l}$ (resp., $B^{c,b}$) – the *collateral remuneration* processes specifying the interest received (resp., paid) on the cash collateral pledged (resp., received) by the hedger.

The processes $S^1, S^2, \ldots, S^d$ model prices of arbitrary traded securities, such as, stocks, stock options, interest rate swaps, currency options, cross-currency swaps, CDSs, etc., and thus they are not assumed to be strictly positive.

**Assumption (M.1)** We postulate that:
(i) for each $i = 1, 2, \ldots, d$, the price process $S^i$ is a semimartingale and the cumulative dividend stream $A^i$ is a process of finite variation with $A^i_0 = 0$,
(ii) $B^l$, $B^b, B^{c,l}, B^{c,b}$ and $B^{i,b}$ are strictly positive, continuous processes of finite variation with $B^l_0 = B^b_0 = B^{c,l}_0 = B^{c,b}_0 = B^{i,b}_0 = 1$ for $i = 1, 2 \ldots, d$.

**Definition 2.1** By a *bilateral financial contract*, or simply a *contract*, we mean an arbitrary càdlàg process $A$ of finite variation, which represents the *cumulative cash flows* of a given contract from time 0 till its maturity date $T$.

The process $A$ models all cash flows, which are either paid out from the wealth or added to the wealth, as seen from the perspective of the *hedger*. Hence the process $-A$ plays an analogous role for the *counterparty*. We assume for the moment that a contract is uncollateralized; additional cash flows due to the presence of the margin account will be studied in foregoing subsections. Note also that the process $A$ includes the initial cash flow $A_0$ of a contract at its inception date $t_0 = 0$. For instance, if a contract has the initial price $p_0$ and stipulates that the hedger will receive the cash flows $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_k$ at times $t_1, t_2, \ldots, t_k \in (0, T]$, then we set $A_0 = p_0$ so that

$$A_t = p_0 \mathbb{1}_{[0,T]}(t) + \sum_{l=1}^{k} \bar{A}_l \, \mathbb{1}_{[t_l,T]}(t).$$

We use the symbol $p_0$, rather than $\bar{A}_0$, in order to emphasize that all future cash flows $\bar{A}_l$ for $l = 1, 2, \ldots, k$ are explicitly specified by the contract's covenants, but the initial cash flow (i.e., the contract's price) is yet to be formally defined and computed.

Unilateral valuation of a derivative contract $A$ at time $t$ means searching for the range of *fair values* of all future cash flows of $A$ (in general, given as $A - A_t$) from the viewpoint of either the hedger or the counterparty. Although we adopt the same valuation paradigm for the two parties, due either to the asymmetry in their trading costs and opportunities or simply the asymmetry of cash flows and a non-linear dynamics of wealth processes, the two parties will typically obtain different values for their respective unilateral prices for $A$.

## 2.1 Collateral with Rehypothecation

Let us now examine the situation when the hedger and the counterparty enter a collateralized contract with the *collateral process* $C$ representing the margin account. It is convenient to decompose $C$ as follows

$$C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^- \tag{2.1}$$

where for any real number $x$ we denote $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$, so that $x = x^+ - x^-$. By convention, $C_t^+ := C_t \mathbb{1}_{\{C_t \geq 0\}}$ is the cash collateral received at time $t$ by the hedger, whereas $C_t^- := -C_t \mathbb{1}_{\{C_t < 0\}}$ represents the cash collateral provided at time $t$ by the hedger. We postulate that $C_T = 0$ to ensure that the collateral is returned in full to the pledging party when a contract matures, provided that the default event has not occurred prior to or at time $T$. The cash flows $A$ are supplemented by the collateral process $C$, so that a collateralized contract is hereafter represented as a pair $(A, C)$. The prevailing market practice is *rehypothecation*, which refers to the situation where a bank is allowed to reuse the collateral pledged by its counterparties as collateral for its own borrowing, as opposed to *segregation* where the use of collateral is not possible. We work under a stylized convention of full rehypothecation, meaning that the cash collateral received by the hedger is used for trading without any restrictions. If the hedger is a collateral provider, then a particular convention regarding segregation or rehypothecation is obviously immaterial for the dynamics of the value process of his portfolio of traded assets.

The collateral account $B^{c,l}$ (resp., $B^{c,b}$) plays the following role: if the hedger provides (resp., receives) cash collateral with the amount $C^-$ (resp., $C^+$), then he receives (resp., pays) interest on this amount, as specified by the process $B^{c,l}$ (resp., $B^{c,b}$). This means that if the counterparty provides (resp., receives) collateral, then he receives (resp., pays) to the hedger the interest determined by $B^{c,b}$ (resp., $B^{c,l}$). Hence the counterparties are exposed to different conditions, unless $B^{c,l}$ coincides with $B^{c,b}$. For the sake of simplicity, the hedger and counterparty are implicitly assumed to be default-free before the maturity date $T$ of a contract at hand. In the presence of default events, we would need to specify the close-out payoff and to deal with the pricing BSDE up to a random time horizon. Finally, it is worth stressing that none of the processes $C, B^{c,l}$ and $B^{c,b}$ is assumed to be a traded asset – they should rather be seen as market frictions.

## 2.2 Market Model with Partial Netting

We are now in a position to introduce trading strategies based on a finite family of traded assets. A trading strategy is discussed from the perspective of the hedger who enters into a contract $(A, C)$ at time 0. To examine the situation of his counterparty, it suffices to replace $(A, C)$ by $(-A, -C)$ in the foregoing considerations. Our first goal is to obtain the dynamics of the value process of the hedger's trading strategy. For concreteness, we focus on a stylized market model with partial netting, which was introduced in Bielecki and Rutkowski (2015). Specifically, we assume that:
(i) short cash positions in risky assets $S^1, S^2, \dots, S^d$ are aggregated and the proceeds from short-selling are used for trading,
(ii) long cash positions in risky assets $S^i$ are assumed to be funded from their respective funding accounts $B^{i,b}$, which can be interpreted as secured loans in the repo market,
(iii) all positive and negative cash flows from a contract $(A, C)$ and a trading strategy $\varphi$, inclusive of the proceeds from short-selling of risky assets, are reinvested in traded assets.

A hedger's *trading strategy* is formally composed of his initial endowment $x$, a process $\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^l, \eta^b\big)$ and a contract $(A, C)$. The components of $\varphi$ represent positions in the risky assets $S^i$, $i = 1, 2, \dots, d$, the funding accounts $B^{i,b}$, $i = 1, 2, \dots, d$ for risky assets, the unsecured lending cash account $B^l$, the unsecured borrowing cash account $B^b$, and the collateral remuneration accounts $B^{c,l}$ and $B^{c,b}$. The following standing assumption formalizes the postulates of the trading model studied in this work.

**Assumption (M.2)** We postulate that:
(i) $\psi_t^l \geq 0$, $\psi_t^b \leq 0$ and $\psi_t^l \psi_t^b = 0$ for all $t \in [0, T]$,
(ii) for every $i = 1, 2, \dots, d$ and all $t \in [0, T]$

$$\psi_t^{i,b} = -(B_t^{i,b})^{-1} (\xi_t^i S_t^i)^+, \tag{2.2}$$

(iii) $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$ and $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$ for all $t \in [0, T]$.

The hedger's *initial endowment* $x$ is interpreted as either a positive or negative amount of cash he owns before entering into a contract $(A, C)$. After he engages in a transaction at time 0, his *initial wealth* amounts to $V_0 := x + p_0$ where $p_0$ is the initial price of $(A, C)$. Finally, since the collateral amount $C_0$ is either pledged or received and, in the latter case, it is rehypothecated by the hedger at time 0, the *initial value* of the hedger's trading strategy is equal to $V^p_0 := x + p_0 + C_0$ where the superscript $p$ stands for *portfolio*. Note also that in the case of a *segregated* collateral, the initial value of the hedger's portfolio would be equal to $x + p_0 - C_0^-$ since when the hedger is collateral taker, he is not allowed to use for his trading purposes the cash collateral pledged by the counterparty.

The next definition introduces a suitable version of the self-financing property for a trading strategy. Observe that the integrals $\int_0^t \eta^l_u \, dB^{c,l}_u$ and $\int_0^t \eta^b_u \, dB^{c,b}_u$ represent the accrued interest generated by the margin account. Note also that $B^{c,l}$ and $B^{c,b}$ do not appear in (2.3), since they are not traded assets, although they impact the dynamics of the value process, as can be seen from (2.4). Formally, the hedger's *portfolio* of traded assets is given as the vector $\big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b\big)$, but the value process of a portfolio depends on all four components of a self-financing trading strategy $(x, \varphi, A, C)$ and thus it is denoted as $V^p(x, \varphi, A, C)$.

**Definition 2.2** A hedger's trading strategy $(x, \varphi, A, C)$ is *self-financing* whenever the *portfolio's value* $V^p(x, \varphi, A, C)$, which is given by

$$V^p_t(x, \varphi, A, C) = \sum_{i=1}^d \xi^i_t S^i_t + \sum_{i=1}^d \psi^{i,b}_t B^{i,b}_t + \psi^l_t B^l_t + \psi^b_t B^b_t, \tag{2.3}$$

satisfies, for every $t \in [0, T]$,

$$\begin{aligned} V^p_t(x, \varphi, A, C) = x &+ \sum_{i=1}^d \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{i=1}^d \int_0^t \psi^{i,b}_u \, dB^{i,b}_u + \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u \\ &+ \int_0^t \eta^l_u \, dB^{c,l}_u + \int_0^t \eta^b_u \, dB^{c,b}_u + C_t + A_t. \end{aligned} \tag{2.4}$$

From equations (2.2) and (2.3), we obtain

$$V^p_t(x, \varphi, A, C) = \psi^l_t B^l_t + \psi^b_t B^b_t - \sum_{i=1}^d (\xi^i_t S^i_t)^-.$$

Since we postulated that $\psi^l_t \geq 0$, $\psi^b_t \leq 0$ and $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$, we also have that

$$\psi^l_t = (B^l_t)^{-1} \Big( V^p_t(x, \varphi, A, C) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^+ \tag{2.5}$$

and

$$\psi^b_t = -(B^b_t)^{-1} \Big( V^p_t(x, \varphi, A, C) + \sum_{i=1}^d (\xi^i_t S^i_t)^- \Big)^- . \tag{2.6}$$

Consequently, we obtain the following result showing that, for a given triplet $(x, A, C)$, the choice of the process $\xi$ uniquely determines the trading strategy $(x, \varphi, A, C)$ and thus also the unique value process $V^p(x, \varphi, A, C)$.

**Lemma 2.3** *Under Assumptions (M.1)–(M.2), the dynamics of a self-financing trading strategy* $(x, \varphi, A, C)$ *are uniquely determined by the initial endowment* $x$ *and processes* $\xi, A$ *and* $C$ *through* *the following equation*

$$
\begin{aligned}
dV_t^p(x,\varphi,A,C) = {} & \sum_{i=1}^d \xi_t^i\,(dS_t^i + dA_t^i) - \sum_{i=1}^d (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1}\,dB_t^{i,b} + dA_t^C \\
& + \Big(V_t^p(x,\varphi,A,C) + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^+ (B_t^l)^{-1}\,dB_t^l \\
& - \Big(V_t^p(x,\varphi,A,C) + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^- (B_t^b)^{-1}\,dB_t^b
\end{aligned}
\tag{2.7}
$$

*where* $A^C := A + C + F^C$ *and*

$$
F_t^C := \int_0^t \eta_u^l\,dB_u^{c,l} + \int_0^t \eta_u^b\,dB_u^{c,b} = \int_0^t C_u^- (B_u^{c,l})^{-1}\,dB_u^{c,l} - \int_0^t C_u^+ (B_u^{c,b})^{-1}\,dB_u^{c,b}.
$$

We also set $(-A)^{-C} := -A - C + F^{-C}$, so that the dynamics for the self-financing trading strategy $(x,\varphi,-A,-C)$ can be obtained by simply replacing $A^C$ by $(-A)^{-C}$ in equation (2.7). Note that $F^{-C} \neq -F^C$ and thus also $(-A)^{-C} \neq -A^C$, in general.

**Definition 2.4** The process $\Theta^C$ is given by the equation

$$
\Theta_t^C := A_t^C + (-A)_t^{-C} = F_t^C + F_t^{-C} = \int_0^t |C_u| (B_u^{c,l})^{-1}\,dB_u^{c,l} - \int_0^t |C_u| (B_u^{c,b})^{-1}\,dB_u^{c,b}.
$$

To examine the no-arbitrage property of a model, we need to introduce the process $V(x,\varphi,A,C)$ representing the hedger's *wealth*. In the financial interpretation, the hedger's wealth $V_t(x,\varphi,A,C)$ represents the cash value of the hedger's holdings under the assumption that the portfolio was liquidated at time $t$, the collateral $C_t$ was returned to the pledging party and the contract was deemed to be void. At time 0, the hedger's wealth equals $x+p$, as was already mentioned.

**Definition 2.5** For any self-financing trading strategy $(x,\varphi,A,C)$, the hedger's *wealth* is given by the equality $V(x,\varphi,A,C) = V^p(x,\varphi,A,C) - C$.

Obviously, the equality $V(x,\varphi,A,C) = V^p(x,\varphi,A,C)$ holds when the process $C$ vanishes. Also, the equality $V_T(x,\varphi,A,C) = V_T^p(x,\varphi,A,C)$ is always satisfied since $C_T = 0$.

**Remark 2.6** Formally, the self-financing property of the hedger's strategy can be defined either in terms of the dynamics of the portfolio's value process $V^p(x,\varphi,A,C)$ or, equivalently, in term of the dynamics of the hedger's wealth $V(x,\varphi,A,C)$. We prefer to focus on the process $V^p(x,\varphi,A,C)$ to emphasize the fact that the self-financing property is primarily concerned with specifying the manner in which the hedger's portfolio of traded assets can be continuously rebalanced by the hedger.

The following assumption will allow us to derive more explicit expressions for the wealth dynamics and thus also to compute the so-called *generator* (also known as the *driver*) for the associated BSDEs.

**Assumption (M.3)** We postulate that:
(i) the processes $B^l, B^b$ and $B^{i,b}$ are absolutely continuous and

$$
dB_t^l = r_t^l B_t^l\,dt, \quad dB_t^b = r_t^b B_t^b\,dt, \quad dB_t^{i,b} = r_t^{i,b} B_t^{i,b}\,dt,
$$

for some $\mathbb{G}$-adapted processes $r^l$, $r^b$ and $r^{i,b}$ such that $0 \le r^l \le r^b$ and $r^l \le r^{i,b}$ for $i = 1,2,\dots,d$,
(ii) the processes $B^{c,b}$ and $B^{c,l}$ are absolutely continuous and

$$
dB_t^{c,b} = r_t^{c,b} B_t^{c,b}\,dt, \quad dB_t^{c,l} = r_t^{c,l} B_t^{c,l}\,dt,
$$

for some $\mathbb{G}$-adapted processes $r^{c,b}$ and $r^{c,l}$ satisfying $r^{c,l} \leq r^{c,b}$.

We henceforth work under Assumptions (M.1)–(M.3). From part (ii) in Assumption (M.3), we infer the following representation for the process $\Theta^C$

$$\Theta_t^C = \int_0^t (r_u^{c,l} - r_u^{c,b})|C_u|\, du. \tag{2.8}$$

The inequality $r^{c,l} \leq r^{c,b}$ means that the counterparty enjoys an advantage over the hedger with reference to the margin account. Indeed, when pledging (resp., receiving) the collateral, the counterparty obtains a higher (resp., lower) interest than the hedger in analogous circumstances. Since $r^{c,l} \leq r^{c,b}$, the process $\Theta^C$ is decreasing with $\Theta_0^C = 0$, so that $\Theta_t^C \leq 0$ for all $t \in [0,T]$. It is worth noting that if $r^{c,l} \geq r^{c,b}$, so that the hedger has an advantage over the counterparty in regard to the margin account, the process $\Theta^C$ is increasing and some comparison results established in what follows are no longer valid. We will sometimes work under the following additional assumption, which restores the symmetry between the two parties in regard to remuneration of the margin account.

**Assumption 2.7** *The collateral accounts $B^{c,l}$ and $B^{c,b}$ satisfy $B^{c,l} = B^{c,b} = B^c$ where $dB_t^c = r_t^c B_t^c\, dt$ so that, for all $t \in [0,T]$,*

$$F_t^C = -\int_0^t C_u (B_u^c)^{-1}\, dB_u^c = -\int_0^t r_u^c C_u\, du = -F_t^{-C}$$

*and thus the equality $(-A)^{-C} = -A^C$ holds.*

The discounted cumulative prices of risky assets are given by the following expressions

$$\widetilde{S}_t^{i,l,\mathrm{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1}\, dA_u^i$$

and

$$\widetilde{S}_t^{i,b,\mathrm{cld}} := (B_t^b)^{-1} S_t^i + \int_{(0,t]} (B_u^b)^{-1}\, dA_u^i.$$

We also denote

$$A_t^{C,l} := \int_{(0,t]} (B_u^l)^{-1}\, dA_u^C, \quad A_t^{C,b} := \int_{(0,t]} (B_u^b)^{-1}\, dA_u^C. \tag{2.9}$$

In view (2.7), the following lemma is straightforward (see Lemma 5.1 in Bielecki and Rutkowski (2015)).

**Lemma 2.8** *The discounted wealth $Y^l := \widetilde{V}^{p,l}(x,\varphi,A,C) = (B^l)^{-1} V^p(x,\varphi,A,C)$ satisfies*

$$dY_t^l = \sum_{i=1}^d \xi_t^i\, d\widetilde{S}_t^{i,l,cld} + \widetilde{f}_l(t, Y_t^l, \xi_t)\, dt + dA_t^{C,l}$$

*where the mapping $\widetilde{f}_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is given by*

$$\widetilde{f}_l(t,y,z) := (B_t^l)^{-1} f_l(t, B_t^l y, z) - r_t^l y \tag{2.10}$$

*and $f_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ equals*

$$f_l(t,y,z) := \sum_{i=1}^d r_t^l z^i S_t^i - \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+ + r_t^l \Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^+ - r_t^b \Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^-.$$

When studying a party with a negative initial endowment, the following lemma can be used instead of Lemma 2.8 (see Remark 5.3 in Bielecki and Rutkowski (2015)).

**Lemma 2.9** *The discounted wealth* $Y^b := \widetilde{V}^{p,b}(x,\varphi,A,C) = (B^b)^{-1}V^p(x,\varphi,A,C)$ *satisfies*

$$dY_t^b = \sum_{i=1}^d \xi_t^i \, d\widetilde{S}_t^{i,b,cld} + \widetilde{f}_b(t,Y_t^b,\xi_t)\, dt + dA_t^{C,b}$$

*where the mapping* $\widetilde{f}_b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is given by*

$$\widetilde{f}_b(t,y,z) := (B_t^b)^{-1} f_b(t, B_t^b y, z) - r_t^b y \tag{2.11}$$

*and* $f_b : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *equals*

$$f_b(t,y,z) := \sum_{i=1}^d r_t^b z^i S_t^i - \sum_{i=1}^d r_t^{i,b}(z^i S_t^i)^+ + r_t^l\Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^+ - r_t^b\Big(y + \sum_{i=1}^d (z^i S_t^i)^-\Big)^-.$$

# 3 Arbitrage Opportunities and Ex-Dividend Prices

We henceforth consider self-financing trading strategies $(x,\varphi,A,C)$, as specified by Definition 2.2. In all results presented in what follows, we work under Assumptions (M.1)–(M.3). By contrast, Assumption 2.7 is not postulated, unless explicitly stated otherwise.

## 3.1 Netted Wealth and Arbitrage Opportunities

We first extend some concepts and results from Section 3 of Bielecki and Rutkowski (2015) to the case of a collateralized contract. For a detailed financial interpretation of the *netted wealth*, the reader is referred to that paper. Recall that $p_0 \in \mathbb{R}$ stands for a generic price of a contract at time 0, as seen from the perspective of the hedger. We wish to address the issue whether he can consistently outperform the benchmark strategy of investing his initial endowment in the cash account by taking long and short positions in a contract and cleverly choosing hedging strategies for both legs. According to Definition 3.1, the answer to this question does not depend on an initial price $p_0$ of a contract $(A,C)$. Moreover, in the case of a linear trading model (even with idiosyncratic funding costs for risky assets and collateral), Definition 3.1 reduces to the classical one. For any initial endowment $x \in \mathbb{R}$, we define the benchmark wealth process $V^0(x)$ by setting $V_t^0(x) = xB_t^l \mathbb{1}_{\{x\ge 0\}} + xB_t^b \mathbb{1}_{\{x<0\}}$. It is clear that $V^0(x)$ represents the wealth of the hedger if he decides not to engage in any contract and simply invest his initial endowment $x$ in the cash account.

**Definition 3.1** An *extended arbitrage opportunity* with respect to a contract $(A,C)$ for the hedger with an initial endowment $x$ is a pair $(\widehat{x},\widehat{\varphi},A,C)$ and $(\widetilde{x},\widetilde{\varphi},-A,-C)$ of trading strategies such that $x = \widehat{x} + \widetilde{x}$ and

$$\mathbb{P}(V_T^{\rm net} \ge V_T^0(x)) = 1 \quad \text{and} \quad \mathbb{P}(V_T^{\rm net} > V_T^0(x)) > 0 \tag{3.1}$$

where the *netted wealth* $V^{\rm net} = V^{\rm net}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C)$ is given by

$$V^{net}(\widehat{x},\widetilde{x},\widehat{\varphi},\widetilde{\varphi},A,C) := V(\widehat{x},\widehat{\varphi},A,C) + V(\widetilde{x},\widetilde{\varphi},-A,-C).$$

Observe that

$$V_0^{\rm net}(x,\varphi,A,C) = V_0(x,\varphi,A,C) + V_0(0,\widetilde{\varphi},-A,-C) = x + p_0 + C_0 - p_0 - C_0 = x,$$

meaning that the initial netted wealth $V_0^{\rm net}(x,\varphi,A,C)$ is independent of the pair $(p_0,C_0)$ and it coincides with the hedger's initial endowment. The next technical definition is a natural extension of the standard concept of admissibility of a trading strategy.

**Definition 3.2** A pair $(\widehat{x}, \widehat{\varphi}, A, C)$ and $(\widetilde{x}, \widetilde{\varphi}, -A, -C)$ of self-financing trading strategies is *admissible for the hedger* if $x = \widehat{x} + \widetilde{x}$ and the discounted netted wealth

$$\widehat{V}^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\varphi}, \widetilde{\varphi}, A, C) := (B^l)^{-1} V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\varphi}, \widetilde{\varphi}, A, C) \mathbb{1}_{\{x \geq 0\}} + (B^b)^{-1} V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\varphi}, \widetilde{\varphi}, A, C) \mathbb{1}_{\{x < 0\}}$$

is bounded from below by a constant.

The following assumption will be used to study the no-arbitrage property of the model.

**Assumption 3.3** *There exists a probability measure $\widetilde{\mathbb{P}}^l$, which is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and such that the processes $\widetilde{S}^{i,l,cld}$, $i = 1, 2, \ldots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales.*

It appears that Assumption 3.3 is a sufficient condition for the non-existence of an extended arbitrage opportunity for the hedger with a non-negative initial endowment provided that $r^l \leq r^{i,b}$ for all $i$.

**Proposition 3.4** *If Assumption 3.3 holds, then no extended arbitrage opportunity in regard to any contract $(A, C)$ exists for the hedger with a non-negative initial endowment.*

*Proof.* Observe that the process $\widehat{V}^p := V^p(\widehat{x}, \widehat{\varphi}, A, C)$ is governed by

$$\begin{aligned} d\widehat{V}^p_t = & \sum_{i=1}^d \widehat{\xi}^i_t \big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\widehat{\xi}^i_t S^i_t)^+ \, dt + dA^C_t \\ & + r^l_t \Big(\widehat{V}^p_t + \sum_{i=1}^d (\widehat{\xi}^i_t S^i_t)^-\Big)^+ dt - r^b_t \Big(\widehat{V}^p_t + \sum_{i=1}^d (\widehat{\xi}^i_t S^i_t)^-\Big)^- dt, \end{aligned}$$

whereas the process $\widetilde{V}^p := V^p(\widetilde{x}, \widetilde{\varphi}, -A, -C)$ has the following dynamics

$$\begin{aligned} d\widetilde{V}^p_t = & \sum_{i=1}^d \widetilde{\xi}^i_t \big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\widetilde{\xi}^i_t S^i_t)^+ \, dt + d(-A)^{-C}_t \\ & + r^l_t \Big(\widetilde{V}^p_t + \sum_{i=1}^d (\widetilde{\xi}^i_t S^i_t)^-\Big)^+ dt - r^b_t \Big(\widetilde{V}^p_t + \sum_{i=1}^d (\widetilde{\xi}^i_t S^i_t)^-\Big)^- dt. \end{aligned}$$

The netted wealth $V^{\mathrm{net}} = V^{\mathrm{net}}(\widehat{x}, \widetilde{x}, \widehat{\varphi}, \widetilde{\varphi}, A, C)$ satisfies

$$V^{\mathrm{net}} := V(\widehat{x}, \widehat{\varphi}, A, C) + V(\widetilde{x}, \widetilde{\varphi}, -A, -C) = \widehat{V}^p - C + \widetilde{V}^p + C = \widehat{V}^p + \widetilde{V}^p$$

and thus

$$\begin{aligned} dV^{\mathrm{net}}_t = & \sum_{i=1}^d (\widehat{\xi}^i_t + \widetilde{\xi}^i_t)\big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\widehat{\xi}^i_t S^i_t)^+ \, dt - \sum_{i=1}^d r^{i,b}_t (\widetilde{\xi}^i_t S^i_t)^+ \, dt \\ & + r^l_t \Big(\widehat{V}^p_t + \sum_{i=1}^d (\widehat{\xi}^i_t S^i_t)^-\Big)^+ dt - r^b_t \Big(\widehat{V}^p_t + \sum_{i=1}^d (\widehat{\xi}^i_t S^i_t)^-\Big)^- dt \\ & + r^l_t \Big(\widetilde{V}^p_t + \sum_{i=1}^d (\widetilde{\xi}^i_t S^i_t)^-\Big)^+ dt - r^b_t \Big(\widetilde{V}^p_t + \sum_{i=1}^d (\widetilde{\xi}^i_t S^i_t)^-\Big)^- dt + d\Theta^C_t. \end{aligned}$$

Since $r^l \leq r^b$ and $r^{c,l} \leq r^{c,b}$, we also have that

$$\begin{aligned} dV^{\mathrm{net}}_t \leq & \sum_{i=1}^d (\widehat{\xi}^i_t + \widetilde{\xi}^i_t)\big(dS^i_t + dA^i_t\big) - \sum_{i=1}^d r^{i,b}_t (\widehat{\xi}^i_t S^i_t)^+ \, dt - \sum_{i=1}^d r^{i,b}_t (\widetilde{\xi}^i_t S^i_t)^+ \, dt \\ & + r^l_t \Big(\widehat{V}^p_t + \widetilde{V}^p_t + \sum_{i=1}^d (\widehat{\xi}^i_t S^i_t)^- + \sum_{i=1}^d (\widetilde{\xi}^i_t S^i_t)^-\Big) \, dt \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
dV_t^{\mathrm{net}} &\leq \sum_{i=1}^d (\widehat{\xi}_t^i + \widetilde{\xi}_t^i)\big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^{i,b}(\widehat{\xi}_t^i S_t^i)^+\,dt - \sum_{i=1}^d r_t^{i,b}(\widetilde{\xi}_t^i S_t^i)^+\,dt \\
&\quad + r_t^b\Big(\widehat{V}_t^p + \widetilde{V}_t^p + \sum_{i=1}^d (\widehat{\xi}_t^i S_t^i)^- + \sum_{i=1}^d (\widetilde{\xi}_t^i S_t^i)^-\Big)\,dt.
\end{aligned}
\tag{3.3}
$$

Using (3.2) and the equality $V^{\mathrm{net}} = \widehat{V}^p + \widetilde{V}^p$, for the process $\widetilde{V}^{l,\mathrm{net}} := (B^l)^{-1}V^{\mathrm{net}}$ we obtain

$$
\begin{aligned}
d\widetilde{V}_t^{l,\mathrm{net}} &= (B_t^l)^{-1}dV_t^{\mathrm{net}} - r_t^l(B_t^l)^{-1}V_t^{\mathrm{net}}\,dt \\
&\leq (B_t^l)^{-1}\Big(\sum_{i=1}^d \widehat{\xi}_t^i\big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^{i,b}(\widehat{\xi}_t^i S_t^i)^+\,dt + \sum_{i=1}^d r_t^l(\widehat{\xi}_t^i S_t^i)^-\,dt\Big) \\
&\quad + (B_t^l)^{-1}\Big(\sum_{i=1}^d \widetilde{\xi}_t^i\big(dS_t^i + dA_t^i\big) - \sum_{i=1}^d r_t^{i,b}(\widetilde{\xi}_t^i S_t^i)^+\,dt + \sum_{i=1}^d r_t^l(\widetilde{\xi}_t^i S_t^i)^-\,dt\Big) \\
&= (B_t^l)^{-1}\Big(\sum_{i=1}^d \widehat{\xi}_t^i\big(dS_t^i + dA_t^i - r_t^l S_t^i\,dt\big) - \sum_{i=1}^d r_t^{i,b}(\widehat{\xi}_t^i S_t^i)^+\,dt + \sum_{i=1}^d r_t^l(\widehat{\xi}_t^i S_t^i)^+\,dt\Big) \\
&\quad + (B_t^l)^{-1}\Big(\sum_{i=1}^d \widetilde{\xi}_t^i\big(dS_t^i + dA_t^i - r_t^l S_t^i\,dt\big) - \sum_{i=1}^d r_t^{i,b}(\widetilde{\xi}_t^i S_t^i)^+\,dt + \sum_{i=1}^d r_t^l(\widetilde{\xi}_t^i S_t^i)^+\,dt\Big).
\end{aligned}
$$

Since $r^l \leq r^{i,b}$, we conclude that

$$
\widetilde{V}_t^{l,\mathrm{net}} - \widetilde{V}_0^{l,\mathrm{net}} \leq \sum_{i=1}^d \int_{(0,t]} (\widehat{\xi}_u^i + \widetilde{\xi}_u^i)\,d\widetilde{S}_u^{i,l,\mathrm{cld}}.
\tag{3.4}
$$

The assumption that the process $\widetilde{V}^{l,\mathrm{net}}$ is bounded from below, implies that the right-hand side in (3.4) is a $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-supermartingale, which is null at $t=0$. Since $x \geq 0$, we have $V_T^0(x) = B_T^l x$ and thus, from (3.4), we obtain

$$
(B_T^l)^{-1}\big(V_T^{\mathrm{net}}(x,\varphi,A) - V_T^0(x)\big) \leq \sum_{i=1}^d \int_{(0,T]} \xi_t^i\,d\widetilde{S}_t^{i,l,\mathrm{cld}}.
$$

Since $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$, we conclude that either $V_T^{\mathrm{net}}(x,\varphi,A,C) = V_T^0(x)$ or $\mathbb{P}(V_T^{\mathrm{net}}(x,\varphi,A,C) < V_T^0(x)) > 0$. This means that an extended arbitrage opportunity may not arise and thus the market model with partial netting is arbitrage-free for the hedger in regard to any contract $(A,C)$. □

The following version of Assumption 3.3 appears to be more suitable when dealing with the hedger with a negative initial endowment.

**Assumption 3.5** *There exists a probability measure $\widetilde{\mathbb{P}}^b$, which is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G}_T)$, and such that the processes $\widetilde{S}^{i,b,cld}$, $i = 1,2,\dots,d$ are $(\widetilde{\mathbb{P}}^b,\mathbb{G})$-local martingales.*

An analogue of Proposition 3.4 for the hedger with a negative initial endowment can be established if we postulate that Assumption 3.5 is met and the inequality $r^b \leq r^{i,b}$ holds for every $i$. It then suffices to use inequality (3.3) and to consider the discounted process $\widetilde{V}^{b,\mathrm{net}} := (B^b)^{-1}V^{\mathrm{net}}$ in the last step of the proof of Proposition 3.4. Note, however, that the postulate that $r^b \leq r^{i,b}$ is too strong from the practical point of view and thus we will focus on the case of nonnegative endowments in what follows.

## 3.2 Fair and Profitable Bilateral Prices

Our next goal is to examine the concept of the range of arbitrage prices for a collateralized contract. We first first recall the definition of a fair price.

**Definition 3.6** We say that a real number $p^{A,C} = A_0$ is a *hedger's fair price* for $(A,C)$ at time 0 whenever for any self-financing trading strategy $(x,\varphi,A,C)$, such that the discounted wealth

$$\widehat{V}(x,\varphi,A,C) := (B^l)^{-1}V(x,\varphi,A,C)\mathbb{1}_{\{x\geq 0\}} + (B^b)^{-1}V(x,\varphi,A,C)\mathbb{1}_{\{x<0\}}$$

is bounded from below, we have that

$$\mathbb{P}\big(V_T(x,\varphi,A,C) = V_T^0(x)\big) = 1 \quad \text{or} \quad \mathbb{P}\big(V_T(x,\varphi,A,C) < V_T^0(x)\big) > 0. \tag{3.5}$$

It is rather clear that a hedger's fair price may depend on the hedger's initial endowment $x$ and it may fail to be unique, in general. One may observe that the two conditions appearing in (3.1) are analogous to conditions in (3.5), but in fact they have different financial interpretation (for a more detailed discussion, see Bielecki and Rutkowski (2015)). Let us recall a generic concept of replication of a contract on $[t,T]$ (see Definition 5.1 in Bielecki and Rutkowski (2015)).

**Definition 3.7** For a fixed $t\in[0,T]$, a self-financing trading strategy $(V_t^0(x)+p_t^{A,C},\varphi,A-A_t,C)$, where $p_t^{A,C}$ is a $\mathcal{G}_t$-measurable random variable, is said to *replicate a collateralized contract* $(A,C)$ on $[t,T]$ whenever $V_T(V_t^0(x)+p_t^{A,C},\varphi,A-A_t,C) = V_T^0(x)$.

We henceforth denote the initial endowment of the hedger (resp., the counterparty) by $x_1$ (resp., $x_2$) where $x_1,x_2\in\mathbb{R}$. In the next definition, we consider the situation when the hedger with the initial endowment $x_1$ at time 0 enters into a contract $(A,C)$ at time $t$ and he is able to replicate it.

**Definition 3.8** Any $\mathcal{G}_t$-measurable random variable for which a replicating strategy for $(A,C)$ over $[t,T]$ exists is called a *hedger's ex-dividend price* at time $t$ for the contract $(A,C)$ and it is denoted by $P_t^h(x_1,A,C)$, so that for some strategy $\varphi$ replicating $(A,C)$

$$V_T(V_t^0(x_1)+P_t^h(x_1,A,C),\varphi,A-A_t,C) = V_T^0(x_1).$$

Similarly, for an arbitrary level $x_2$ of the counterparty's initial endowment and any trading strategy $\varphi$ replicating $(-A,-C)$, the *counterparty's ex-dividend price* $P_t^c(x_2,-A,-C)$ at time $t$ for the contract $(-A,-C)$ is implicitly given by the equality

$$V_T(V_t^0(x_2)-P_t^c(x_2,-A,-C),\varphi,-A+A_t,-C) = V_T^0(x_2).$$

It is clear that we deal here with *unilateral* prices evaluated by the hedger and the counterparty, respectively. Note that if $x_1=x_2=x$, then $P_t^h(x,A,C)=p_t^{A,C}$ and $P_t^c(x,-A,-C)=-p_t^{-A,-C}$. Due to this convention, the equality $P_t^h(x_1,A,C)=P_t^c(x_1,-A,-C)$ holds when Definition 3.8 is applied to a standard market model with a single cash account where in fact the arbitrage prices are known to be independent of initial endowments $x_1$ and $x_2$. Definition 3.9 hinges on an implicit assumption that the unilateral prices of $(A,C)$ are unique; we will address this important issue in the next section.

**Definition 3.9** The hedger is willing to *sell* (resp., to *buy*) a contract $(A,C)$ if $P_t^h(x_1,A,C)\geq 0$ (resp., $P_t^h(x_1,A,C)\leq 0$). The counterparty is willing to *sell* (resp., to *buy*) a contract $(-A,-C)$ if $P_t^c(x_2,-A,-C)\leq 0$ (resp., $P_t^c(x_2,-A,-C)\geq 0$).

Since we place ourselves in a non-linear framework, a natural asymmetry arises between the hedger and his counterparty. No wonder that the price discrepancy may occur, that is, it may happen that $P^h_t(x_1,A,C) \neq P^c_t(x_2,-A,-C)$. However, it is expected that, typically, the two prices will yield a no-arbitrage range determined by the (higher) seller's price and the (lower) buyer's price, although it may also happen that both parties are willing to be sellers (or both are willing to be buyers) of a given contract. In addition, since a positive excess cash generated by one contract may be offset (either partially or completely) by a negative excess cash associated with another contract, it is natural to conjecture that the seller's (resp., buyer's) price for the combination of two contracts should be lower (resp., higher) than the sum of the seller's (resp., buyer's) prices of individual contracts.

**Remark 3.10** Consider a contract $(A,C)$ with the process $A$ given by

$$A_t = p_0 \mathbb{1}_{[0,T]}(t) + X \mathbb{1}_{[T]}(t).$$

If $X = -(S^i_T - K)^+$, then we deal with a European call option sold by the hedger. A natural guess is that the prices $P^h_0(x_1,A,C)$ and $P^c_0(x_2,-A,-C)$ should be positive. Similarly, if $X = (S^i_T - K)^+$, so that the counterparty is the option's seller, it is natural to expect that $P^h_0(x_1,A,C)$ and $P^c_0(x_2,-A,-C)$ should be negative. Furthermore, if $C=0$ and

$$A_t = p_0 \mathbb{1}_{[0,T]}(t) - (S^i_T - K)^+ \mathbb{1}_{[T]}(t)$$

then we conjecture that the price $P^h_0(x_1,A,0)$ should be independent of $x_1$, provided that $x_1 \geq 0$. Indeed, as a consequence of (2.2), the hedger cannot use his initial endowment to buy shares for the purpose of hedging. Note that, in view of that constraint, the model considered here does not cover the standard case of different borrowing and lending rates when $r^{i,b} = r^b > r^l$ and trading is assumed to be unrestricted, so that the hedger's initial endowment can be used to purchase shares. In the standard Bergman's model, we expect that the hedger's price of the call option will depend on the hedger's initial endowment $x_1$. To conclude, the properties of ex-dividend prices may be very different in each particular market model, but several features of prices can be analyzed using general results for BSDEs.

Recall that $x_1$ and $x_2$ stand for the initial endowments of the hedger and the counterparty, respectively. Due to a generic nature of a contract $(A,C)$, it is impossible to make any plausible a priori conjectures about relative sizes and/or signs of prices. The equality $P^h_t(x_1,A,C) = P^c_t(x_2,-A,-C)$ means that both parties agree on a common price for the contract. Otherwise, that is, if the equality $P^h_t(x_1,A,C) = P^c_t(x_2,-A,-C)$ fails to hold, then the following situations may arise:

**(H.1)** $0 \leq P^c_t(x_2,-A,-C) < P^h_t(x_1,A,C)$,

**(H.2)** $P^c_t(x_2,-A,-C) \leq 0 < P^h_t(x_1,A,C)$,

**(H.3)** $P^c_t(x_2,-A,-C) < P^h_t(x_1,A,C) \leq 0$,

and, symmetrically,

**(C.1)** $0 \leq P^h_t(x_1,A,C) < P^c_t(x_2,-A,-C)$,

**(C.2)** $P^h_t(x_1,A,C) \leq 0 < P^c_t(x_2,-A,-C)$,

**(C.3)** $P^h_t(x_1,A,C) < P^c_t(x_2,-A,-C) \leq 0$.

Before analyzing each situation, let us recall that the cash flows of a contract $(A,C)$ are invariably considered from the perspective of the hedger, so the counterparty faces the cash flows given by $(-A,-C)$. Consequently, in case (H.1), we may say that the hedger is the seller of $(A,C)$ and the counterparty is the buyer of $(-A,-C)$, but the counterparty is not willing to pay the amount demanded by the hedger. In case (H.2), both parties are willing to be sellers of the contract, meaning in practice that the hedger is ready to sell $(A,C)$ and the counterparty is willing to sell $(-A,-C)$. Finally, case (H.3) refers to the situation the counterparty is willing to be the seller of $(-A,-C)$, whereas the hedger can now be seen as a buyer of $(A,C)$, but he is not willing to pay the price that is needed by the counterparty to replicate the contract.

Assume that the market model is arbitrage-free for both parties in the sense of Definition 3.1. Then in all three cases, (H.1)–(H.3), any $\mathcal{G}_t$-measurable random variable $P^f_t$ satisfying

$$P^f_t \in \big[P^c_t(x_2, -A, -C), P^h_t(x_1, A, C)\big] \tag{3.6}$$

can be considered to be a *fair price* for both the hedger and his counterparty, in the sense that a bilateral transaction executed at $P^f_t$ will not generate an arbitrage opportunity for either of them. Hence the interval $[P^c_t(x_2, -A, -C), P^h_t(x_1, A, C)]$ represents the range of fair prices of the contract $(A, C)$ for both parties, as seen from the perspective of the hedger (a particular instance of this interval is called the *arbitrage-band* in Bergman (1995)).

**Definition 3.11** The $\mathcal{G}_t$-measurable interval $\mathcal{R}^f_t(x_1, x_2) := \big[P^c_t(x_2, -A, -C), P^h_t(x_1, A, C)\big]$ is called the *range of fair bilateral prices* at time $t$ of an OTC contract $(A, C)$ between the hedger and the counterparty.

Although the analysis for the cases (C.1)–(C.3) can be done analogously, the financial interpretation and conclusions are quite different. In case (C.1), the hedger is willing to be the seller of $(A, C)$ and the counterparty is willing to be the buyer and he is ready to pay even more than it is asked for by the hedger. In case (C.2), both parties are ready to be buyers at their respective prices, meaning that each party is ready to pay a positive premium to another. Finally, in case (C.3), the counterparty is willing to be the seller, whereas the hedger can now be seen as a buyer of $(A, C)$ and he is ready to pay more than the counterparty requests. Therefore, any $\mathcal{G}_t$-measurable random variable $P^p_t$ satisfying

$$P^p_t \in \big[P^h_t(x_1, A, C), P^c_t(x_2, -A, -C)\big] \tag{3.7}$$

can be interpreted as a bilaterally acceptable price. Note that, unless $P^h_t(x_1, A, C) = P^c_t(x_2, -A, -C)$, the price $P^p_t$ is not a fair bilateral price, in the sense explained above, since an arbitrage opportunity may arise for at least one party involved when an OTC contract $(A, C)$ is traded between them at the price $P^p_t$. This observation motivates the following definition.

**Definition 3.12** Assume that the inequality $P^h_t(x_1, A, C) \neq P^c_t(x_2, -A, -C)$ holds. Then the $\mathcal{G}_t$-measurable interval $\mathcal{R}^p_t(x_1, x_2) := \big[P^h_t(x_1, A, C), P^c_t(x_2, -A, -C)\big]$ is called the *range of bilaterally profitable prices* at time $t$ of an OTC contract $(A, C)$ between the hedger and the counterparty.

Note that in our discussion so far, we dealt with three different types of arbitrage:

**(A.1)** A *classical* arbitrage opportunity produced by trading in the primary assets.

**(A.2)** An *extended* arbitrage opportunity, which may arise when the long and short hedged positions in some contract are combined. The contract's price at time 0 is here considered to be exogenously given by the market and its value is immaterial (see Definition 3.1).

**(A.3)** A *bilateral* arbitrage opportunity originating from the fact that the hedger and the counterparty may require different premia to implement their respective replicating strategies. Note that here arbitrage opportunities are available simultaneously to both parties who execute a contract at a bilaterally negotiated price (see Definition 3.12).

Let us finally observe that if (C.2) occurs, then a *reselling* arbitrage opportunity arises for the third party. Specifically, if $P^h_t(x_1, A, C) \leq 0$ and $P^c_t(x_2, -A, -C) > 0$, then a third party can make a deal with the hedger to face $(-A, -C)$ and receive $-P^h_t(x_1, A, C) \geq 0$ and, at the same time, enter the contract with the counterparty to face $(A, C)$ and get $P^c_t(x_2, -A, -C) > 0$. This offsetting strategy generates an immediate profit of $P^c_t(x_2, -A, -C) - P^h_t(x_1, A, C) > 0$ to the third party.

# 4 Pricing BSDEs and Replicating Strategies

Our next aim is to show that the hedger's and counterparty's prices and their replicating strategies can be found by solving pricing BSDEs. We will use for this purpose auxiliary results on solutions to BSDEs driven by multi-dimensional continuous martingales. In Propositions 4.5 and 4.7, we will show that if $x_1 \geq 0$ and $x_2 \geq 0$, then the price processes $P^h(x_1, A, C)$ and $P^c(x_2, -A, -C)$ are given by the solutions of two BSDEs that are driven by the $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingale $\widetilde{S}^{l,\mathrm{cld}}$.

## 4.1 Modeling of Risky Assets

To establish the existence of a solution to pricing BSDEs for the hedger and the counterparty with positive endowments, we work under Assumption 3.3 complemented by additional technical conditions on the underlying market model. To be more specific, we will introduce Assumption 4.2, which will allow us to use results from Section 5 in Nie and Rutkowski (2014a). We first recall the following standard definition.

**Definition 4.1** We say that a process $\gamma$ satisfies the *ellipticity* condition if there exists a constant $\Lambda > 0$ such that for all $t \in [0, T]$ and every $a \in \mathbb{R}^d$

$$\sum_{i,j=1}^{d} (\gamma_t \gamma_t^*)_{ij}\, a_i a_j \geq \Lambda \|a\|^2 = \Lambda a^* a. \tag{4.1}$$

Let the matrix-valued process $\mathbb{S}$ be given by

$$\mathbb{S}_t := \begin{pmatrix} S_t^1 & 0 & \dots & 0 \\ 0 & S_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_t^d \end{pmatrix}.$$

We consider the following assumption, which corresponds to Assumption 5.1 in Nie and Rutkowski (2014a). Recall that the existence of a probability measure $\widetilde{\mathbb{P}}^l$ was postulated in Assumption 3.3.

**Assumption 4.2** *We postulate that:*
*(i) the process $\widetilde{S}^{l,cld}$ is a continuous, square-integrable, $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-martingale and has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$,*
*(ii) there exists an $\mathbb{R}^{d\times d}$-valued, $\mathbb{G}$-adapted process $m^l$ such that*

$$\langle \widetilde{S}^{l,cld} \rangle_t = \int_0^t m_u^l (m_u^l)^*\, du \tag{4.2}$$

*where the $\mathbb{G}$-adapted process $m^l$ is such that $m^l(m^l)^*$ is invertible and satisfies the equality $m^l(m^l)^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$ where a $d$-dimensional square matrix $\gamma$ of $\mathbb{G}$-adapted processes satisfies condition* (4.1)*.*

An alternative version of Assumption 4.2 can be obtained by replacing the superscript $l$ by $b$ everywhere.

**Remark 4.3** We will show that Assumption 4.2 can be easily met when the prices of risky assets are given by the diffusion-type model. For example, we may assume that each risky asset $S^i$, $i = 1, 2, \dots, d$ has the ex-dividend price dynamics under the real-world probability $\mathbb{P}$ given by

$$dS_t^i = S_t^i \Big( \mu_t^i\, dt + \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j \Big), \quad S_0^i > 0,$$

or, equivalently, the $d$-dimensional process $S = (S^1, \dots, S^d)^*$ satisfies

$$dS_t = \mathbb{S}_t(\mu_t\, dt + \sigma_t\, dW_t)$$

where $W=(W^1,\dots,W^d)^*$ is the $d$-dimensional Brownian motion, $\mu=(\mu^1,\dots,\mu^d)^*$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^W$-adapted process, $\sigma=[\sigma^{ij}]$ is a $d$-dimensional square matrix of $\mathbb{F}^W$-adapted processes satisfying the ellipticity condition. In addition, we assume that the processes $\mu$, $\sigma$ and $\kappa$ are bounded. We now set $\mathbb{G}=\mathbb{F}^W$ and we recall that the $d$-dimensional Brownian motion $W$ enjoys the predictable representation property with respect to its natural filtration $\mathbb{F}^W$; this property is shared by the process $\widetilde{W}$ defined by (4.4).

Assuming that the corresponding dividend processes are given by $A^i_t=\int_0^t \kappa^i_u S^i_u\,du$, we obtain

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t=(B^l_t)^{-1}\big(dS^i_t+dA^i_t-r^l_tS^i_t\,dt\big)=(B^l_t)^{-1}S^i_t\bigg(\big(\mu^i_t+\kappa^i_t-r^l_t\big)\,dt+\sum_{j=1}^d\sigma^{ij}_t\,dW^j_t\bigg).$$

Hence, if we denote $\widetilde{S}^{l,\mathrm{cld}}=(\widetilde{S}^{1,l,\mathrm{cld}},\dots,\widetilde{S}^{d,l,\mathrm{cld}})^*$ and $\mu+\kappa-r^l=(\mu^1+\kappa^1-r^l,\dots,\mu^d+\kappa^d-r^l)^*$, then

$$d\widetilde{S}^{l,\mathrm{cld}}_t=(B^l_t)^{-1}\mathbb{S}_t\Big(\big(\mu_t+\kappa_t-r^l_t\big)\,dt+\sigma_t\,dW_t\Big).$$

Let us define $l_t:=\sigma_t^{-1}(\mu_t+\kappa_t-r^l_t)$ for all $t\in[0,T]$. Since $\mu$, $\sigma$, $\kappa$ are bounded and $\sigma$ satisfies the ellipticity condition, we see that the process $l$ is bounded and thus we can define the probability measure $\widetilde{\mathbb{P}}^l$ on $(\Omega,\mathcal{F}^W_T)$ by setting

$$\frac{d\widetilde{\mathbb{P}}^l}{d\mathbb{P}}=\exp\bigg(-\int_0^T l_t\,dW_t-\frac{1}{2}\int_0^T|l_t|^2\,dt\bigg). \tag{4.3}$$

Then the probability measure $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}^W_T)$ and, from Girsanov's theorem, the process $\widetilde{W}^l:=(\widetilde{W}^{l,1},\widetilde{W}^{l,2},\dots,\widetilde{W}^{l,d})^*$ is a $d$-dimensional Brownian motion under $\widetilde{\mathbb{P}}^l$, where

$$d\widetilde{W}^l_t:=dW_t+l_t\,dt=dW_t+\sigma_t^{-1}(\mu_t+\kappa_t-r^l_t)\,dt. \tag{4.4}$$

It is clear that under $\widetilde{\mathbb{P}}^l$

$$d\widetilde{S}^{l,\mathrm{cld}}_t=(B^l_t)^{-1}\mathbb{S}_t\sigma_t\,d\widetilde{W}^l_t.$$

Hence the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i=1,2,\dots,d$ are continuous, square-integrable, $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-martingales. Furthermore, the quadratic variation of $\widetilde{S}^{l,\mathrm{cld}}$ equals

$$\langle\widetilde{S}^{l,\mathrm{cld}}\rangle_t=\int_0^t m^l_u(m^l_u)^*\,du$$

where $m^l(m^l)^*=\mathbb{S}\gamma\gamma^*\mathbb{S}$ with $\gamma:=(B^l)^{-1}\sigma$. Obviously, the matrix $m^l(m^l)^*$ is invertible and thus Assumption 4.2 is satisfied.

## 4.2 Hedger's Price and Replicating Strategy

We denote by $\mathcal{H}^{2,d}(\mathbb{P})$ the subspace of all $\mathbb{R}^d$-valued, $\mathbb{G}$-adapted processes $X$ with

$$\|X\|^2_{\mathcal{H}^{2,d}(\mathbb{P})}:=\mathbb{E}_{\mathbb{P}}\bigg[\int_0^T\|X_t\|^2\,dt\bigg]<\infty. \tag{4.5}$$

For simplicity, we denote $\mathcal{H}^2(\mathbb{P}):=\mathcal{H}^{2,1}(\mathbb{P})$. Also, let $L^2(\mathbb{P})$ stand for the space of all real-valued, $\mathcal{G}_T$-measurable random variables $\eta$ such that $\|\eta\|^2_{L^2(\mathbb{P})}=\mathbb{E}_{\mathbb{P}}(\eta^2)<\infty$. In the present set-up, we have $Q_t=t$ for every $t\in[0,T]$ in Assumption 3.1 in Nie and Rutkowski (2014a) and thus also in all results in Sections 3–5 therein. This is consistent with equation (4.2) in Assumption 4.2. We henceforth assume that the processes $r^l$, $r^b$ and $r^{i,b}$ for $i=1,2,\dots,d$ are non-negative and bounded.

**Definition 4.4** A contract $(A,C)$ is *admissible under* $\widetilde{\mathbb{P}}^l$ if the process $A^{C,l}$ given by (2.9) belongs to $\mathcal{H}^2(\widetilde{\mathbb{P}}^l)$ and the random variable $A^{C,l}_T$ belongs to $L^2(\widetilde{\mathbb{P}}^l)$.

The next result describes the price and the replicating strategy for the hedger.

**Proposition 4.5** *Let Assumptions 3.3 and 4.2 be satisfied. Then for any initial endowment $x_1 \geq 0$ and any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$, the hedger's ex-dividend price satisfies, for every $t \in [0, T)$,*

$$P_t^h(x_1, A, C) = B_t^l(Y_t^{h,l,x_1} - x_1) - C_t$$

*where the pair $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to the BSDE*

$$\begin{cases} dY_t^{h,l,x_1} = Z_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,cld} + \widetilde{f}_l\big(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}\big)\, dt + dA_t^{C,l}, \\ Y_T^{h,l,x_1} = x_1, \end{cases} \tag{4.6}$$

*with the generator $\widetilde{f}_l$ given by* (2.10). *The unique replicating strategy equals*

$$\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\big)$$

*where, for every $t \in [0, T]$ and $i = 1, 2, \dots, d$,*

$$\xi_t^i = Z_t^{h,l,x_1,i},\ \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+,\ \eta_t^l = (B_t^{c,l})^{-1} C_t^-,\ \eta_t^b = -(B_t^{c,b})^{-1} C_t^+,$$

*and*

$$\psi_t^l = (B_t^l)^{-1}\Big(B_t^l Y_t^{h,l,x_1} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^+,$$

$$\psi_t^b = -(B_t^b)^{-1}\Big(B_t^l Y_t^{h,l,x_1} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^-.$$

*Proof.* From Theorems 4.1 and 5.1 in Nie and Rutkowski (2014a), we know that if Assumptions 3.3 and 4.2 are satisfied, $A^{C,l} \in \mathcal{H}^2(\widetilde{\mathbb{P}}^l)$ and $A_T^{C,l} \in L^2(\widetilde{\mathbb{P}}^l)$, then BSDE (4.6) has a unique solution $(Y^{h,l,x_1}, Z^{h,l,x_1})$. Thus, from Proposition 5.2 in Bielecki and Rutkowski (2015), we obtain $P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$. Moreover, the unique replicating strategy $\varphi$ can be constructed using Lemma 2.3. Note that the uniqueness of $\varphi$ is in the sense explained in Remark 4.6. □

**Remark 4.6** Let us comment on the uniqueness of a replicating strategy in Proposition 4.5. The uniqueness of a solution of BSDE (4.6) means that if $(Y^1, Z^1)$ and $(Y^2, Z^2)$ are two solutions to BSDE (4.6), then

$$\mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T |Y_t^1 - Y_t^2|^2\, dt + \int_0^T \|(m_t^l)^* Z_t^1 - (m_t^l)^* Z_t^2\|^2\, dt\bigg] = 0. \tag{4.7}$$

Under Assumption 4.2, we have that $m^l(m^l)^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$ and thus

$$\mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T \|(m_t^l)^*(Z_t^1 - Z_t^2)\|^2\, dt\bigg] = \mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T (Z_t^1 - Z_t^2)^*\mathbb{S}\gamma\gamma^*\mathbb{S}(Z_t^1 - Z_t^2)\, dt\bigg].$$

Since $\gamma$ satisfies the ellipticity condition, there exists a constant $\Lambda > 0$ such that

$$\mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T (Z_t^1 - Z_t^2)^*\mathbb{S}_t\gamma\gamma^*\mathbb{S}_t(Z_t^1 - Z_t^2)\, dt\bigg] \geq \Lambda\, \mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T \|\mathbb{S}_t Z_t^1 - \mathbb{S}_t Z_t^2\|^2\, dt\bigg].$$

We conclude that under Assumption 4.2 for any two solutions of BSDE (4.6) we have

$$\mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T \|\mathbb{S}_t Z_t^1 - \mathbb{S}_t Z_t^2\|^2\, dt\bigg] = 0 \tag{4.8}$$

Using the structure of the replicating strategy in Proposition 4.5, we conclude that the uniqueness of $\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\big)$ holds up to an equivalence with respect to the product measure $\mathbb{P}^l \otimes \ell$. Moreover, for $\xi = (\xi^1, \dots, \xi^d)^*$, $\psi^l$, $\psi^b$ and $\psi = (\psi^{1,b}, \dots, \psi^{d,b})^*$ the uniqueness holds in the following norm

$$\|\varphi\|^2 := \mathbb{E}_{\widetilde{\mathbb{P}}^l}\bigg[\int_0^T \|\mathbb{S}_t\xi_t\|^2 dt + \int_0^T (|\psi_t^l|^2 + |\psi_t^b|^2) dt + \int_0^T \|\psi_t\|^2 dt\bigg].$$

### 4.3 Counterparty's Price and Replicating Strategy

In order to compare the prices computed by the hedger and his counterparty, we will postulate that Assumption 2.7 is satisfied as well, so that $(-A)^{-C} = -A^C$. Notice that no arbitrage opportunity may arise for either of the parties if Assumption 3.3 is postulated. Arguing as in the proof of Proposition 4.5, one can easily establish the following result for the counterparty's price of the contract.

**Proposition 4.7** *Let Assumptions 2.7, 3.3 and 4.2 be satisfied. Then for any initial endowment $x_2 \geq 0$ the counterparty's ex-dividend price satisfies, for every $t \in [0,T)$,*

$$P_t^c(x_2, -A, -C) = -B_t^l(Y_t^{c,l,x_2} - x_2) - C_t$$

*where the pair $(Y^{c,l,x_2}, Z^{c,l,x_2})$ is the unique solution to the BSDE*

$$\begin{cases} dY_t^{c,l,x_2} = Z_t^{c,l,x_2,*}\, d\widetilde{S}_t^{l,cld} + \widetilde{f}_l\big(t, Y_t^{c,l,x_2}, Z_t^{c,l,x_2}\big)\, dt + d(-A)_t^{l,-C}, \\ Y_T^{c,l,x_2} = x_2, \end{cases} \tag{4.9}$$

*where*

$$(-A)_t^{l,-C} := \int_{(0,t]} (B_u^l)^{-1}\, d(-A)_u^{-C} = -A_t^{C,l}$$

*and where the generator $\widetilde{f}_l$ is given by (2.10). The unique replicating strategy for the counterparty equals*

$$\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\big)$$

*where, for every $t \in [0,T]$ and $i = 1, 2, \dots, d$,*

$$\xi_t = Z_t^{c,l,x_2},\ \psi_t^{i,b} = -(B_t^{i,b})^{-1}(\xi_t^i S_t^i)^+,\ \eta_t^l = (B_t^{c,l})^{-1} C_t^+,\ \eta_t^b = -(B_t^{c,b})^{-1} C_t^-,$$

*and*

$$\psi_t^l = (B_t^l)^{-1}\Big(B_t^l Y_t^{c,l,x_2} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^+,$$

$$\psi_t^b = -(B_t^b)^{-1}\Big(B_t^l Y_t^{c,l,x_2} + \sum_{i=1}^d (\xi_t^i S_t^i)^-\Big)^-.$$

## 5 Properties of Unilateral and Bilateral Prices

Recall that we consider the special case of an exogenous margin account with rehypothecated cash collateral. This means, in particular, that the process $C$ does not depend on trading strategies chosen by the hedger and the counterparty. As before, we denote the initial endowments of the hedger and the counterparty by $x_1$ and $x_2$, respectively. Unless explicitly stated otherwise, we assume that $x_1 \geq 0$ and $x_2 \geq 0$. The first goal is to show that the range of fair bilateral prices is non-empty under mild assumptions. Next, we will study the monotonicity property of unilateral prices with respect to the initial endowments, the asymptotic properties and the independence of the initial endowments. We continue working under the standing Assumptions (M.1)–(M.3).

### 5.1 Range of Fair Bilateral Prices

In the next result, we deal with both unilateral prices and thus we work under Assumption 2.7.

**Proposition 5.1** *Let Assumptions 2.7, 3.3 and 4.2 be satisfied. If $x_1 \geq 0$ and $x_2 \geq 0$, then for any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$ we have, for all $t \in [0, T]$,*

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}, \tag{5.1}$$

*so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty almost surely.*

*Proof.* From Propositions 4.5 and 4.7, we already know that the hedger's price equals $P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$ where the pair $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to the BSDE

$$\begin{cases} dY_t^{h,l,x_1} = Z_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + \widetilde{f}_l\big(t, Y_t^{h,l,x_1}, Z_t^{h,l,x_1}\big)\, dt + dA_t^{C,l}, \\ Y_T^{h,l,x_1} = x_1, \end{cases} \tag{5.2}$$

where the mapping $\widetilde{f}_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is given by equation (2.10). Similarly, the counterparty's price equals $P^c(x_2, -A, -C) = -B^l(Y^{c,l,x_2} - x_2) - C$ where the pair $(Y^{c,l,x_2}, Z^{c,l,x_2})$ is the unique solution to the BSDE

$$\begin{cases} dY_t^{c,l,x_2} = Z_t^{c,l,x_2,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + \widetilde{f}_l\big(t, Y_t^{c,l,x_2}, Z_t^{c,l,x_2}\big)\, dt - dA_t^{C,l}, \\ Y_T^{c,l,x_2} = x_2. \end{cases} \tag{5.3}$$

Therefore, to establish inequality (5.1), it suffices to show that

$$-B_t^l(Y_t^{c,l,x_2} - x_2) - C_t \leq B_t^l(Y_t^{h,l,x_1} - x_1) - C_t,$$

which in turn is equivalent to $-Y_t^{c,l,x_2} + x_2 \leq Y_t^{h,l,x_1} - x_1$. If we denote $\bar{Y}^{h,l,x_1} := Y^{h,l,x_1} - x_1$ and $\bar{Z}^{h,l,x_1} = Z^{h,l,x_1}$, then the pair $(\bar{Y}^{h,l,x_1}, \bar{Z}^{h,l,x_1})$ is the unique solution of the following BSDE

$$\begin{cases} d\bar{Y}_t^{h,l,x_1} = \bar{Z}_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + \widetilde{f}_l\big(t, \bar{Y}_t^{h,l,x_1} + x_1, \bar{Z}_t^{h,l,x_1}\big)\, dt + dA_t^{C,l}, \\ \bar{Y}_T^{h,l,x_1} = 0. \end{cases} \tag{5.4}$$

Similarly, the pair $(\bar{Y}^{c,l,x_2}, \bar{Z}^{c,l,x_2}) := \big( - Y^{c,l,x_2} + x_2,\, \bar{Z}_t^{c,l,x_2} = -Z^{c,l,x_2}\big)$ is the unique solution of the BSDE

$$\begin{cases} d\bar{Y}_t^{c,l,x_2} = \bar{Z}_t^{c,l,x_2,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} - \widetilde{f}_l\big(t, -\bar{Y}_t^{c,l,x_2} + x_2, -\bar{Z}_t^{c,l,x_2}\big)\, dt + dA_t^{C,l}, \\ \bar{Y}_T^{c,l,x_2} = 0. \end{cases} \tag{5.5}$$

Note that (5.4) and (5.5) have the same term $dA_t^{C,l}$ and identical terminal conditions $\bar{Y}_T^{h,l,x_1} = \bar{Y}_T^{c,l,x_2} = 0$. It is also easy to check that the generator $\widetilde{f}_l$ satisfies the assumptions of Theorem 3.3 in Nie and Rutkowski (2014a). Therefore, if either (see Theorem 3.3 in Nie and Rutkowski (2014a))

$$-\widetilde{f}_l\big(t, \bar{Y}_t^{h,l,x_1} + x_1, \bar{Z}_t^{h,l,x_1}\big) \geq \widetilde{f}_l\big(t, -\bar{Y}_t^{h,l,x_1} + x_2, -\bar{Z}_t^{h,l,x_1}\big), \quad \widetilde{\mathbb{P}}^l \otimes \ell - \text{a.e.} \tag{5.6}$$

or (see Remark 3.2 therein)

$$-\widetilde{f}_l\big(t, \bar{Y}_t^{c,l,x_2} + x_1, \bar{Z}_t^{c,l,x_2}\big) \geq \widetilde{f}_l\big(t, -\bar{Y}_t^{c,l,x_2} + x_2, -\bar{Z}_t^{c,l,x_2}\big), \quad \widetilde{\mathbb{P}}^l \otimes \ell - \text{a.e.} \tag{5.7}$$

then the inequality $\bar{Y}^{h,l,x_1} \geq \bar{Y}^{c,l,x_2}$ holds $\widetilde{\mathbb{P}}^l \otimes \ell$−a.e. To establish both (5.6) and (5.7), it is enough to show that

$$-\widetilde{f}_l\big(t, y + x_1, z\big) \geq \widetilde{f}_l\big(t, -y + x_2, -z\big) \ \text{ for all } (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \widetilde{\mathbb{P}}^l \otimes \ell - \text{a.e.} \tag{5.8}$$

To complete the proof, it suffices to note that (5.8) holds, as is shown in Lemma 5.2. □

**Lemma 5.2** *Assume that $x_1 \geq 0$ and $x_2 \geq 0$. Then the mapping $\widetilde{f}_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ given by equation* (2.10) *satisfies inequality* (5.8).

*Proof.* Let us denote $\widetilde{z}^i_t = (B^l_t)^{-1} z^i S^i_t$. Then

$$\begin{aligned}\delta &:= \widetilde{f}_l\big(t, y+x_1, z\big) + \widetilde{f}_l\big(t, -y+x_2, -z\big)\\ &= -r^l_t(y+x_1) + f_l(t, B^l_t(y+x_1), z) - r^l_t(-y+x_z) + f_l(t, B^l_t(-y+x_2), -z)\\ &= -r^l_t(x_1+x_2) - \sum_{i=1}^d r^{i,b}_t |\widetilde{z}^i_t| + r^l_t(\delta_1^+ + \delta_2^+) - r^b_t(\delta_1^- + \delta_2^-)\end{aligned}$$

where we write

$$\delta_1 := y + x_1 + \sum_{i=1}^d (\widetilde{z}^i_t)^-, \quad \delta_2 := -y + x_2 + \sum_{i=1}^d (-\widetilde{z}^i_t)^-.$$

From $r^l \leq r^b$, we have

$$\begin{aligned}\delta &= -r^l_t(x_1+x_2) - \sum_{i=1}^d r^{i,b}_t |\widetilde{z}^i_t| + r^l_t(\delta_1^+ + \delta_2^+) - r^b_t(\delta_1^- + \delta_2^-)\\ &\leq -r^l_t(x_1+x_2) - \sum_{i=1}^d r^{i,b}_t |\widetilde{z}^i_t| + r^l_t(\delta_1 + \delta_2)\\ &= -r^l_t(x_1+x_2) - \sum_{i=1}^d r^{i,b}_t |\widetilde{z}^i_t| + r^l_t(x_1 + x_2) + \sum_{i=1}^d r^l_t |\widetilde{z}^i_t|\\ &= \sum_{i=1}^d (r^l_t - r^{i,b}_t)|\widetilde{z}^i_t| \leq 0.\end{aligned}$$

We thus conclude that $\delta \leq 0$ and thus inequality (5.8) is satisfied. □

## 5.2 Monotonicity of the Price with Respect to the Initial Endowment

We continue working under Assumption 2.7. We will now examine in more details the impact of initial endowments of the two parties on their respective ex-dividend prices. From Propositions 4.5 and 4.7, it follows that $P^h(x_1, A, C) = B^l(Y^{h,l,x_1} - x_1) - C$ where $(Y^{h,l,x_1}, Z^{h,l,x_1})$ is the unique solution to BSDE (5.2), whereas $P^c(x_2, -A, -C) = -B^l(Y^{c,l,x_2} - x_2) - C$ where $(Y^{c,l,x_2}, Z^{c,l,x_2})$ is the unique solution to BSDE (5.3). If we denote $\widetilde{Y}^{h,l,x_1} = B^l(Y^{h,l,x_1} - x_1)$, then $P^h(x_1, A, C) = \widetilde{Y}^{h,l,x_1} - C$. It is easy to check that

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t = (B^l_t)^{-1}\left(dS^i_t - r^l_t S^i_t\,dt + dA^i_t\right).$$

Therefore, using also (2.9) and (2.10), we obtain

$$\begin{aligned}d\widetilde{Y}^{h,l,x_1}_t &= -x_1\,dB^l_t + Y^{h,l,x_1}_t\,dB^l_t + B^l_t\,dY^{h,l,x_1}_t\\ &= -x_1 r^l_t B^l_t\,dt + r^l_t B^l_t Y^{h,l,x_1}_t\,dt + B^l_t Z^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + B^l_t \widetilde{f}_l\big(t, Y^{h,l,x_1}_t, Z^{h,l,x_1}_t\big)\,dt + dA^C_t\\ &= -x_1 r^l_t B^l_t\,dt + B^l_t Z^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + f_l\big(t, B^l_t Y^{h,l,x_1}_t, Z^{h,l,x_1}_t\big)\,dt + dA^C_t\\ &= B^l_t Z^{h,l,x_1,*}_t\,d\widetilde{S}^{l,\mathrm{cld}}_t + g(t, x_1, B^l_t Y^{h,l,x_1}_t, Z^{h,l,x_1}_t)\,dt + dA^C_t\end{aligned}$$

where

$$\begin{aligned}g(t,x,y,z) &:= -x r^l_t B^l_t + f_l\big(t,y,z\big) = -x r^l_t B^l_t + \sum_{i=1}^d r^l_t z^i S^i_t - \sum_{i=1}^d r^{i,b}_t (z^i S^i_t)^+\\ &\quad + r^l_t\Big(y + \sum_{i=1}^d (z^i S^i_t)^-\Big)^+ - r^b_t\Big(y + \sum_{i=1}^d (z^i S^i_t)^-\Big)^-. \qquad (5.9)\end{aligned}$$

Upon denoting $\widetilde{Z}^{h,l,x_1} = B^l Z^{h,l,x_1}$, we obtain

$$d\widetilde{Y}_t^{h,l,x_1} = \widetilde{Z}_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + g\big(t, x_1, \widetilde{Y}_t^{h,l,x_1} + x_1 B_t^l, (B_t^l)^{-1} \widetilde{Z}_t^{h,l,x_1}\big)\, dt + dA_t^C.$$

We conclude that if $x_1 \geq 0$, then for any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$ we have $P^h(x_1, A, C) = \widetilde{Y}^{h,l,x_1} - C$ where the pair $(\widetilde{Y}^{h,l,x_1}, \widetilde{Z}^{h,l,x_1})$ is the unique solution of the following BSDE

$$\begin{cases} d\widetilde{Y}_t^{h,l,x_1} = \widetilde{Z}_t^{h,l,x_1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + g^{h,l}(t, x_1, \widetilde{Y}_t^{h,l,x_1}, \widetilde{Z}_t^{h,l,x_1})\, dt + dA_t^C, \\ \widetilde{Y}_T^{h,l,x_1} = 0, \end{cases} \tag{5.10}$$

where $g^{h,l}(t, x, y, z) := g(t, x, y + xB_t^l, z(B_t^l)^{-1})$. Using analogous arguments, one can show that if $x_2 \geq 0$, then $P^c(x_2, -A, -C) = \widetilde{Y}^{c,l,x_2} - C$ where the pair $(\widetilde{Y}^{c,l,x_2}, \widetilde{Z}^{c,l,x_2})$ is the unique solution to the BSDE

$$\begin{cases} d\widetilde{Y}_t^{c,l,x_2} = \widetilde{Z}_t^{c,l,x_2,*} d\widetilde{S}_t^{l,\mathrm{cld}} + g^{c,l}(t, x_2, \widetilde{Y}_t^{c,l,x_2}, \widetilde{Z}_t^{c,l,x_2})\, dt + dA_t^C, \\ \widetilde{Y}_T^{c,l,x_2} = 0, \end{cases} \tag{5.11}$$

where $g^{c,l}(t, x, y, z) := -g(t, x, -y + xB_t^l, -z(B_t^l)^{-1})$. It is easy to check that the admissibility of $(A, C)$ under $\widetilde{\mathbb{P}}^l$ implies that $A^C \in \mathcal{H}^2(\widetilde{\mathbb{P}}^l)$ and $A_T^C \in L^2(\widetilde{\mathbb{P}}^l)$. Moreover, one can show that $g^{h,l}$ and $g^{c,l}$ satisfy the conditions of Theorems 4.1 in Nie and Rutkowski (2014a). Consequently, the well-posedness of BSDEs (5.10) and (5.11) holds.

In the next result, Assumption 2.7 is not required for the hedger's inequality (5.12), but it is needed when analyzing the counterparty's price.

**Proposition 5.3** *Let Assumptions 2.7, 3.3 and 4.2 be satisfied and let a contract $(A, C)$ be admissible under $\widetilde{\mathbb{P}}^l$. Then the hedger's price satisfies: if $\bar{x} \geq x \geq 0$, then for all $t \in [0, T]$*

$$P_t^h(\bar{x}, A, C) \leq P_t^h(x, A, C), \tag{5.12}$$

*and the counterparty's price satisfies: if $\bar{x} \geq x \geq 0$, then for all $t \in [0, T]$*

$$P_t^c(\bar{x}, -A, -C) \geq P_t^c(x, -A, -C). \tag{5.13}$$

*Proof.* We wish to show that the mapping $x \mapsto g^{h,l}(t, x, y, z)$ is increasing and the mapping $x \mapsto g^{c,l}(t, x, y, z)$ is decreasing, for any fixed $(t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d$, $\widetilde{\mathbb{P}}^l$-a.s. Let us denote

$$\begin{aligned} K_0(t, z) &:= (B_t^l)^{-1} \sum_{i=1}^d r_t^l z^i S_t^i - (B_t^l)^{-1} \sum_{i=1}^d r_t^{i,b} (z^i S_t^i)^+, \\ K(t, y, z) &:= y + (B_t^l)^{-1} \sum_{i=1}^d (z^i S_t^i)^-. \end{aligned}$$

For conciseness, we write $K := K(t, y, z)$ and $\widetilde{K} := K(t, -y, -z)$. Then

$$\begin{aligned} g^{h,l}(t, x, y, z) &= K_0(t, z) - x r_t^l B_t^l + r_t^l (x B_t^l + K)^+ - r_t^b (x B_t^l + K)^- \\ &= K_0(t, z) - r_t^l (x B_t^l + K) + r_t^l (x B_t^l + K)^+ - r_t^b (x B_t^l + K)^- + r_t^l K \\ &= K_0(t, z) + r_t^l (x B_t^l + K)^- - r_t^b (x B_t^l + K)^- + r_t^l K \\ &= K_0(t, z) + (r_t^l - r_t^b)(x B_t^l + K)^- + r_t^l K \end{aligned}$$

and

$$\begin{aligned} g^{c,l}(t, x, y, z) &= -K_0(t, -z) + x r_t^l B_t^l - r_t^l (x B_t^l + \widetilde{K})^+ + r_t^b (x B_t^l + \widetilde{K})^- \\ &= -K_0(t, -z) + r_t^l (x B_t^l + \widetilde{K}) - r_t^l (x B_t^l + \widetilde{K})^+ + r_t^b (x B_t^l + \widetilde{K})^- - r_t^l \widetilde{K} \\ &= -K_0(t, -z) + (r_t^b - r_t^l)(x B_t^l + \widetilde{K})^- - r_t^l \widetilde{K}. \end{aligned}$$

Therefore, the function $g^{h,l}$ is increasing with respect to $x$, whereas the function $g^{c,l}$ is decreasing with respect to $x$. Consequently, from the comparison theorem for BSDEs, if $\bar{x} \geq x \geq 0$, then $\widetilde{Y}^{h,l,x} \leq \widetilde{Y}^{h,l,\bar{x}}$ where $(\widetilde{Y}^{h,l,x}, \widetilde{Z}^{h,l,x})$ is the unique solution of BSDE (5.10). Moreover, $\widetilde{Y}^{c,l,x} \geq \widetilde{Y}^{c,l,\bar{x}}$ where $(\widetilde{Y}^{c,l,x}, \widetilde{Z}^{c,l,x})$ is the unique solution of BSDE (5.11). It is now clear that inequalities (5.12) and (5.13) hold. □

By combining Propositions 5.1 and 5.3, we obtain the following result, which summarizes the properties of unilateral prices.

**Corollary 5.4** *Let Assumptions 2.7, 3.3 and 4.2 be satisfied. Then for any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$ the following statement is valid: if $\bar{x} \geq x \geq 0$, then for all $t \in [0, T]$*

$$P^c_t(x, -A, -C) \leq P^c_t(\bar{x}, -A, -C) \leq P^h_t(\bar{x}, A, C) \leq P^h_t(x, A, C). \tag{5.14}$$

*In particular, for any $x \geq 0$ and any date $t \in [0, T]$*

$$P^c_t(0, -A, -C) \leq P^c_t(x, -A, -C) \leq P^h_t(x, A, C) \leq P^h_t(0, A, C), \tag{5.15}$$

*so that $\mathcal{R}^f_t(x, x) \subset \mathcal{R}^f_t(0, 0)$.*

Corollary 5.4 shows that an investor with a positive initial endowment has a relative advantage over an investor with null initial endowment when entering into an arbitrary contract $(A, C)$ at any time $t$. This conclusion is intuitively plausible, since the borrowing rate is higher than the lending rate and thus for the same strategy, when an investor with null initial endowment needs to borrow money in order to hedge a contract, an investor with a positive initial endowment may use cash from his initial endowment for the same purpose and this creates a comparative advantage for him.

## 5.3 Asymptotic Properties of Unilateral Prices

Using Proposition 5.3, we will examine the asymptotic properties of $P^h_t(x, A, C)$ and $P^c_t(x, -A, -C)$ when the initial endowment $x$ tends to $\infty$. In practice, this can be interpreted as pricing of a contract with a relatively small nominal value.

**Proposition 5.5** *Let Assumptions 2.7, 3.3 and 4.2 be satisfied. Then for any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$ and any date $t \in [0, T]$, there exist $\mathbb{G}$-adapted processes, denoted by $P^{h,A,C,+}_t$ and $P^{c,-A,-C,+}_t$, such that*

$$P^{h,A,C,+}_t,\, P^{c,-A,-C,+}_t \in [P^c_t(0, -A, -C), P^h_t(0, A, C)] = \mathcal{R}^f_t(0, 0)$$

*and*

$$\lim_{x\to+\infty} P^c_t(x, -A, -C) = P^{c,-A,-C,+}_t \leq P^{h,A,C,+}_t = \lim_{x\to+\infty} P^h_t(x, A, C).$$

*Proof.* The statement easily follows from Proposition 5.3 and Corollary 5.4. □

## 5.4 Price Independence of an Initial Endowment

We will now show that for a certain class of contracts a unilateral price is independent of an initial endowment. It is worth noting that an analogous result fails to hold in Bergman's model studied by Nie and Rutkowski (2015). Recall that in Bergman's model the lending and borrowing rates $r^l$ and $r^b$ are different, but the funding rates $r^{i,b}$ for risky assets are not introduced. Since we now deal with the hedger's price only, Assumption 2.7 is relaxed.

**Proposition 5.6** *Let Assumptions 3.3 and 4.2 be satisfied. Consider an arbitrary contract* $(A, C)$ *admissible under* $\widetilde{\mathbb{P}}^l$ *and such that the process* $A^C - A^C_0$ *is decreasing. The price* $P^h_t(x_1, A, C)$ *is independent of the hedger's initial endowment* $x_1 \geq 0$*, so that* $P^h_t(x_1, A, C) = P^h_t(0, A, C)$ *for all* $x_1 \geq 0$ *and* $t \in [0, T]$*.*

*Proof.* Since $x_1 \geq 0$, the hedger's price of any contract $(A, C)$ admissible under $\widetilde{\mathbb{P}}^l$ satisfies $P^h(x_1, A, C) = \widetilde{Y}^{h,l,x_1} - C$ where $(\widetilde{Y}^{h,l,x_1}, \widetilde{Z}^{h,l,x_1})$ is the unique solution to BSDE (5.10). Since $g^{h,l}(t, x_1, 0, 0) = 0$ and the process $A^C - A^C_0$ is decreasing, we deduce from the comparison theorem for BSDEs (see Theorem 3.3 in Nie and Rutkowski (2014a) with $U^1 = A^C - A^C_0$ and $U^2 = 0$) that $\widetilde{Y}^{h,l,x_1} \geq 0$. Since $x_1 \geq 0$, BSDE (5.10) can also be represented as follows

$$
(5.16) \qquad \begin{cases} d\widetilde{Y}^{h,l,x_1}_t = \widetilde{Z}^{h,l,x_1,*}_t \, d\widetilde{S}^{l,\mathrm{cld}}_t + \widetilde{g}^{h,l}(t, x_1, \widetilde{Y}^{h,l,x_1}_t, \widetilde{Z}^{h,l,x_1}_t) \, dt + dA^C_t, \\ \widetilde{Y}^{h,l,x_1}_T = 0, \end{cases}
$$

where the generator $\widetilde{g}^{h,l}(t, x_1, y, z)$ equals (recall that $\bar{z}^i_t = z^i S^i_t$)

$$
\widetilde{g}^{h,l}(t, x_1, y, z) := r^l_t y + (B^l_t)^{-1} \bigg( \sum_{i=1}^d r^l_t \bar{z}^i_t - \sum_{i=1}^d r^{i,b}_t (\bar{z}^i_t)^+ + r^l_t \sum_{i=1}^d (\bar{z}^i_t)^- \bigg).
$$

Since $\widetilde{g}^{h,l}$ does not depend on $x_1$, the unique solution to BSDE (5.16) is independent of $x_1$ as well, and thus the price $P^h(x_1, A, C) = \widetilde{Y}^{h,l,x_1} - C$ has the same property. $\square$

**Remark 5.7** Note that the conclusion of Proposition 5.6 hinges on the assumption that $x_1 \geq 0$. Indeed, when $x_1 \leq 0$ and the process $A^C - A^C_0$ is decreasing, then the price $P^h(x_1, A, C)$ does not enjoy the independence property. Moreover, if the process $A^C - A^C_0$ is increasing, then the counterparty's price $P^c(x_2, -A, -C)$ is independent of the initial wealth $x_2 \leq 0$.

Let us comment on the financial interpretation of Proposition 5.6. We conjecture that the independence of the hedger's price of his non-negative positive wealth is a consequence of the trading constraint implicit in equation (2.2) and the fact that the portfolio's value $V^p$ is always greater than or equal to $x_1 B^l_t$. On the one hand, equation (2.2) states that the hedger cannot use his initial endowment to buy shares for the purpose of hedging. When he sells shares to replicate a contract (for instance, in order to hedge a put option) then, obviously, the fact that his initial endowment is positive is also irrelevant. On the other hand, the decreasing property of $A^C - A^C_0$ and $x_1 \geq 0$ make the price $P^h_t(x_1, A, C)$ high enough, so that $V^p_t - x_1 B^l_t = P^h_t(x_1, A, C) + C_t = \widetilde{Y}^{h,l,x_1}_t \geq 0$. In view of equation (2.6), this means that no borrowing of cash is required for replication of $(A, C)$, even when the initial endowment is null. This in turn essentially means that the initial endowment $x_1$ can simply be invested in the account $B^l$ and thus its level has no bearing on the hedger's price $P^h_t(x_1, A, C)$.

## 5.5 Positive Homogeneity of the Hedger's Price

We conclude this section by showing that the hedger's price is positively homogeneous with respect to the contract's size and the non-negative initial endowment. Let us stress that this property is no longer true if only the contract's size, but not the initial endowment, is scaled by a non-negative number $\lambda$ (of course, unless the price is independent of the initial endowment as, for instance, under the assumptions of Proposition 5.6).

**Proposition 5.8** *Let Assumptions 3.3 and 4.2 be satisfied and let* $(A, C)$ *be an arbitrary contract admissible under* $\widetilde{\mathbb{P}}^l$*. If* $x_1 \geq 0$ *and* $C \in \widehat{\mathcal{H}}^2_0$*, then the hedger's price is positively homogeneous, in the sense that the equality* $P^h_t(\lambda x_1, \lambda A, \lambda C) = \lambda P^h_t(x_1, A, C)$ *is valid for all* $\lambda \in \mathbb{R}_+$*.*

*Proof.* It is obvious that the asserted equality holds for $\lambda = 0$. Suppose that $\lambda > 0$. We know that $P^h(x_1, A, C) = \widetilde{Y}^{h,l,x_1} - C$ where $(\widetilde{Y}^{h,l,x_1}, \widetilde{Z}^{h,l,x_1})$ is the unique solution to (5.10). Moreover, $P^h(\lambda x_1, \lambda A, \lambda C) = \widetilde{Y}^{h,l,\lambda x_1} - \lambda C$ where $(\widetilde{Y}^{h,l,\lambda x_1}, \widetilde{Z}^{h,l,\lambda x_1})$ is the unique solution of the following BSDE

$$\begin{cases} d\widetilde{Y}_t^{h,l,\lambda x_1} = \widetilde{Z}_t^{h,l,\lambda x_1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + g^{h,l}(t, \lambda x_1, \widetilde{Y}_t^{h,l,\lambda x_1}, \widetilde{Z}_t^{h,l,\lambda x_1})\, dt + \lambda\, dA_t^C, \\ \widetilde{Y}_T^{h,l,\lambda x_1} = 0. \end{cases}$$

Then $P^h(x_1, A, C) = Y^1$ where $(Y^1, Z^1)$ is the unique solution to the BSDE (since $(A, C)$ is admissible under $\widetilde{\mathbb{P}}^l$, the well-posedness of this BSDE is easy to check)

$$\begin{cases} dY_t^1 = Z_t^{1,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + g^{h,l}(t, x_1, Y_t^1 + C_t, Z_t^1)\, dt + d(A_t + F_t^C), \\ Y_T^1 = 0. \end{cases}$$

Similarly, $P^h(\lambda x_1, \lambda A, \lambda C) = Y^2$ where $(Y^2, Z^2)$ is the unique solution to the BSDE

$$\begin{cases} dY_t^2 = Z_t^{2,*}\, d\widetilde{S}_t^{l,\mathrm{cld}} + g^{h,l}(t, \lambda x_1, Y_t^2 + \lambda C_t, Z_t^2)\, dt + \lambda\, d(A_t + F_t^C), \\ Y_T^2 = 0. \end{cases} \tag{5.17}$$

For $Y := \lambda Y^1$ and $Z := \lambda Z^1$, we have

$$\begin{cases} dY_t = Z_t^*\, d\widetilde{S}_t^{l,\mathrm{cld}} + \lambda g^{h,l}(t, x_1, \lambda^{-1} Y_t + C_t, \lambda^{-1} Z_t)\, dt + \lambda\, d(A_t + F_t^C), \\ Y_T = 0. \end{cases} \tag{5.18}$$

Therefore, to complete the proof, it suffices to observe that the equality

$$\lambda g^{h,l}(t, x_1, \lambda^{-1} y + C_t, \lambda^{-1} z) = g^{h,l}(t, \lambda x_1, y + \lambda C_t, z)$$

is satisfied for all $\lambda > 0$. □

# 6 Quasi-Linear Pricing PDEs for European Claims

For simplicity of presentation, we henceforth assume that $d = 1$, so that there is only one risky asset $S = S^1$. It is clear, however, that the results obtained in this section can be easily extended to a model with several risky assets. We now postulate that the lending and borrowing rates $r^l$ and $r^b$, as well as the funding rate $r^{1,b}$, are bounded deterministic functions of time. We examine the valuation and hedging of an uncollateralized European contingent claim starting from a fixed time $t \in [0, T]$. A generic path-independent claim of European style pays a single cash flow $H(S_T)$ at its expiration date $T > 0$, so that

$$A_t - A_0 = -H(S_T)\mathbb{1}_{[T,T]}(t).$$

For any fixed $t < T$, the risky asset $S$ has the ex-dividend price dynamics under $\mathbb{P}$ given by the following expression, for $u \in [t, T]$,

$$dS_u = \mu(u, S_u)\, du + \sigma(u, S_u)\, dW_u, \quad S_t = s \in \mathcal{O}, \tag{6.1}$$

where $W$ is a one-dimensional Brownian motion and $\mathcal{O} \in \mathbb{R}$ is the domain of real values that are attainable by the diffusion process $S$ (typically, $\mathcal{O} = \mathbb{R}_+$ when $S$ models a stock price). Finally, the dividend process for the asset $S$ is given by the equality $A_u^1 = \int_t^u \kappa(v, S_v)\, dv$.

Throughout this section, we assume that the coefficients $\mu, \sigma, \kappa : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions: $\mu(t, s), \sigma(t, s), \kappa(t, s)$ and $H(s)$ are uniformly Lipschitz continuous in $s$, and $\sup_{t \in [0,T]} \{|\mu(t, 0)| + |\sigma(t, 0)| + |\kappa(t, 0)| + |H(0)|\}$ is bounded. Our first goal is to derive the hedger's and counterparty's pricing PDEs for a path-independent European claim. We observe that

$$\begin{aligned} d\widetilde{S}_u^{l,\mathrm{cld}} &= (B_u^l)^{-1}\big(dS_u + dA_u^1 - r_u^l\, du\big) \\ &= (B_u^l)^{-1}\Big(\big(\mu(u, S_u) + \kappa(u, S_u) - r_u^l\big)\, du + \sigma(u, S_u)\, dW_u\Big). \end{aligned}$$

Let us denote

$$l(u,S_u) := (\sigma(u,S_u))^{-1}\big(\mu(u,S_u)+\kappa(u,S_u)-r^l_u\big).$$

Under suitable assumptions on $\mu,\sigma,\kappa$, the process $l$ satisfies Novikov's condition

$$\mathbb{E}_{\mathbb{P}}\bigg\{\exp\bigg(\frac{1}{2}\int_0^T |l(t,S_t)|^2\,dt\bigg)\bigg\}<\infty$$

and thus we may define the probability measure $\widetilde{\mathbb{P}}^l$ as

$$\frac{d\widetilde{\mathbb{P}}^l}{d\mathbb{P}} = \exp\bigg(-\int_t^T l(u,S_u)\,dW_u - \frac{1}{2}\int_t^T |l(u,S_u)|^2\,du\bigg).$$

Then, from Girsanov's theorem, $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}^W_T)$ and the process $\widetilde{W}^l$ is a Brownian motion on $[t,T]$ under $\widetilde{\mathbb{P}}^l$, where $d\widetilde{W}^l_u := dW_u + l(u,S_u)\,du$. It is clear that

$$d\widetilde{S}^{l,\mathrm{cld}}_u = (B^l_u)^{-1}\sigma(u,S_u)\,d\widetilde{W}^l_u$$

and thus $\widetilde{S}^{l,\mathrm{cld}}$ is a $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-martingale and $\langle \widetilde{S}^{l,\mathrm{cld}}\rangle_u = \int_t^u (B^l_v)^{-2}\sigma^2(v,S_v)\,dv$. Hence Assumptions 3.3 and 4.2 are satisfied, provided that we assume that the Brownian motion $\widetilde{W}^l$ has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$. Of course, the latter assumption is not restrictive in the present set-up.

## 6.1 Hedger's Pricing PDE

Since $A$ has only a single cash flow at time $T$ and $C=0$, for any initial endowment $x_1\geq 0$, the hedger's ex-dividend price satisfies $P^h_t(x_1,A,0)=\widetilde{Y}^{h,l,x_1}_t$ for every $t\in[0,T)$ where $(\widetilde{Y}^{h,l,x_1},\widetilde{Z}^{h,l,x_1})$ is the unique solution to the BSDE (see (5.10))

$$\begin{cases} d\widetilde{Y}^{h,l,x_1}_u = \widetilde{Z}^{h,l,x_1}_u (B^l_u)^{-1}\sigma(u,S_u)\,d\widetilde{W}^l_u + g^{h,l}(u,x_1,\widetilde{Y}^{h,l,x_1}_u,\widetilde{Z}^{h,l,x_1}_u)\,du, \\ \widetilde{Y}^{h,l,x_1}_T = H(S_T), \end{cases} \tag{6.2}$$

where

$$\begin{aligned} g^{h,l}(u,x_1,s,y,z) &:= -r^l_u x_1 B^l_u + (B^l_u)^{-1}(r^l_u zs - r^{1,b}_u (zs)^+) \\ &\quad + r^l_u\Big(y + x_1B^l_u + (B^l_u)^{-1}(zs)^-\Big)^+ - r^b_u\Big(y + x_1B^l_u + (B^l_u)^{-1}(zs)^-\Big)^-. \end{aligned}$$

It is clear that the solution $(\widetilde{Y}^{h,l,x_1},\widetilde{Z}^{h,l,x_1})$ will now depend on the asset's initial price $s$ at time $t$; to emphasize this feature, we write $(\widetilde{Y}^{h,l,x_1,s},\widetilde{Z}^{h,l,x_1,s})$. Furthermore, if we set $(Y^{h,x_1,s}_u,Z^{h,x_1,s}_u):=(\widetilde{Y}^{h,l,x_1,s}_u,\widetilde{Z}^{h,l,x_1,s}_u(B^l_u)^{-1}\sigma(u,S^{s,t}_u))$, then BSDE (6.2) yields

$$\begin{cases} dY^{h,x_1,s}_u = Z^{h,x_1,s}_u\,d\widetilde{W}^l_u + \overline{g}^{h,l}(u,x_1,S^{s,t}_u,Y^{h,x_1,s}_u,Z^{h,x_1,s}_u)\,du, \\ Y^{h,x_1,s}_T = H(S^{s,t}_T). \end{cases} \tag{6.3}$$

where

$$\begin{aligned} \overline{g}^{h,l}(u,x_1,s,y,z) &:= g^{h,l}(u,x_1,s,y,zB^l_u\sigma^{-1}(u,s)) \\ &:= -r^l_u x_1 B^l_u + r^l_u zs\sigma^{-1}(u,s) - r^{1,b}_u(zs\sigma^{-1}(u,s))^+ \\ &\quad + r^l_u\Big(y + x_1B^l_u + (zs\sigma^{-1}(u,s))^-\Big)^+ - r^b_u\Big(y + x_1B^l_u + (zs\sigma^{-1}(u,s))^-\Big)^-. \end{aligned}$$

The well-posedness of BSDE (6.3) (or, equivalently, (6.2)) holds under mild assumptions, since we postulated that $\widetilde{W}^l$ has the predictable representation property under $(\mathbb{G},\widetilde{\mathbb{P}}^l)$. For instance, if $s\sigma^{-1}(u,s)$ is uniformly bounded, then the generator $\overline{g}^{h,l}$ is uniformly Lipschitz continuous with respect to $(y,z)$. Noticing that $H(S_T^{s,t})$ is also square-integrable under $\widetilde{\mathbb{P}}^l$, we deduce from Theorem 2.1 in El Karoui et al. (1997) that BSDE (6.3) (or, equivalently, BSDE (6.2)) has a unique solution in a suitable space of stochastic processes.

Proposition 4.5 and the equality $\widetilde{Z}^{h,l,x_1} = B^l Z^{h,l,x_1}$ show that the unique replicating strategy for the hedger equals $\varphi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ where $\xi_u = (B^l_u)^{-1}\widetilde{Z}^{h,l,x_1}_u$, $\psi^{1,b}_u = -(B^{1,b}_u)^{-1}(\xi_u S_u)^+$ and

$$\psi^l_u = (B^l_u)^{-1}\big(\widetilde{Y}^{h,l,x_1}_u + x_1 B^l_u + (\xi_u S_u)^-\big)^+, \quad \psi^b_u = -(B^b_u)^{-1}\big(\widetilde{Y}^{h,l,x_1}_u + x_1 B^l_u + (\xi_u S_u)^-\big)^-.$$

In the next step, we fix $t \in [0,T)$ and we assume that $S^{s,t}_t = s \in \mathcal{O}$. Note that under $\widetilde{\mathbb{P}}^l$ we have, for all $u \in [t,T]$,

$$dS^{s,t}_u = (r^l_u - \kappa(u, S^{s,t}_u))\,du + \sigma(u, S^{s,t}_u)\,d\widetilde{W}^l_u.$$

We now invoke the non-linear Feynman-Kac formula (see Peng (1991) or Pardoux and Peng (1992)) to argue that, under additional conditions imposed on $\sigma$, the *hedger's pricing function* $v(t,s) := Y^{h,x_1,s}_t$ is the unique viscosity solution of the following *pricing PDE*

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \mathcal{L}v(t,s) = \overline{g}^{h,l}\big(t, x_1, s, v(t,s), \sigma(t,s)\frac{\partial v}{\partial s}\big), & (t,s) \in [0,T] \times \mathcal{O}, \\ v(T,s) = H(s), \quad s \in \mathcal{O}, \end{cases} \tag{6.4}$$

where the differential operator $\mathcal{L}$ is given by the following expression

$$\mathcal{L} := \frac{1}{2}\sigma^2(t,s)\frac{\partial^2}{\partial s^2} + (r^l_t - \kappa(t,s))\frac{\partial}{\partial s}.$$

For example, if the map $s\sigma^{-1}(u,s)$ is uniformly continuous in $s$ and uniformly bounded, one can check that the generator $\overline{g}^{h,l}(u,x_1,s,y,z)$ satisfies the conditions of Theorem 4.2 in El Karoui et al. (1997) and thus the *hedger's pricing function* $v(t,s) := Y^{h,x_1,s}_t$ is the unique viscosity solution to PDE (6.4). For the existence of a classical solution to PDE (6.4), the reader is referred to monographs by Krylov (1987) and Ladyzenskaya et al. (1968) and the references therein.

In view of the definition of $\overline{g}^{h,l}$, it is clear that PDE (6.4) is in turn equivalent to, for $(t,s) \in [0,T] \times \mathcal{O}$,

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s) = \kappa(t,s)\frac{\partial v}{\partial s}(t,s) - x_1 r^l_t B^l_t - r^{1,b}_t\big(s\frac{\partial v}{\partial s}(t,s)\big)^+ \\ \quad + r^l_t\Big(v(t,s) + x_1 B^l_t + \big(s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^+ - r^b_t\Big(v(t,s) + x_1 B^l_t + \big(s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^-, \\ v(T,s) = H(s), \quad s \in \mathcal{O}. \end{cases} \tag{6.5}$$

Conversely, if $v \in C^{1,2}([0,T]\times\mathcal{O})$ solves PDE (6.5) and both $v(u,s)$ and $\frac{\partial v}{\partial s}(u,s)$ have polynomial growth in $s$, then the pair $(v(u,S_u), \sigma(u,S_u)\frac{\partial v}{\partial s}(u,S_u))$ is the unique solution to BSDE (6.3) on $u \in [t,T]$ where, for brevity, we write $S = S^{s,t}$. From the above considerations, $(v(u,S_u), B^l_u\frac{\partial v}{\partial s}(u,S_u))$ is also the unique solution to BSDE (6.2) on $u \in [t,T]$ for any initial asset's price $S_t = s$. Consequently, the unique replicating strategy for the hedger can be represented as follows: $\varphi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ where for all $u \in [t,T]$

$$\begin{aligned} \xi_u &= \frac{\partial v}{\partial s}(u,S_u), \quad \psi^{1,b}_t = -(B^{1,b}_u)^{-1}\Big(S_u\frac{\partial v}{\partial s}(u,S_u)\Big)^+, \\ \psi^l_u &= (B^l_u)^{-1}\Big(v(u,S_u) + x_1 B^l_u + \big(S_u\frac{\partial v}{\partial s}(u,S_u)\big)^-\Big)^+, \\ \psi^b_u &= -(B^b_u)^{-1}\Big(v(u,S_u) + x_1 B^l_u + \big(S_u\frac{\partial v}{\partial s}(u,S_u)\big)^-\Big)^-. \end{aligned} \tag{6.6}$$

## 6.2 Counterparty's Pricing PDE

Let us now focus on the pricing PDE for the counterparty with an initial endowment $x_2 \geq 0$. Recall that $P^c_t(x_2, -A, 0) = \widetilde{Y}^{c,l,x_2}_t$ for all $t \in [0,T)$ where $(\widetilde{Y}^{c,l,x_2}, \widetilde{Z}^{c,l,x_2})$ is the unique solution to the following BSDE under $\widetilde{\mathbb{P}}^l$ (see (5.11))

$$\begin{cases} d\widetilde{Y}^{c,l,x_2}_u = \widetilde{Z}^{c,l,x_2}_u (B^l_u)^{-1}\sigma(u,S_u)\, d\widetilde{W}^l_u + g^{c,l}(u, x_2, S_u, \widetilde{Y}^{c,l,x_2}_u, \widetilde{Z}^{c,l,x_2}_u)\, du, \\ \widetilde{Y}^{c,l,x_2}_T = H(S^{s,t}_T), \end{cases} \tag{6.7}$$

with the generator given by

$$\begin{aligned} g^{c,l}(u, x_1, s, y, z) &:= r^l_u x_2 B^l_u + (B^l_u)^{-1}(r^l_u zs - r^{1,b}_u(-zs)^+) \\ &\quad - r^l_u\Big(-y + x_2 B^l_u + (B^l_u)^{-1}(-zs)^-\Big)^+ + r^b_u\Big(-y + x_2 B^l_u + (B^l_u)^{-1}(-zs)^-\Big)^-. \end{aligned}$$

Hence the unique replicating strategy for the counterparty is given as $\varphi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ where $\xi_u = -(B^l_u)^{-1}\widetilde{Z}^{c,l,x_2}_u$, $\psi^{1,b}_u = -(B^{1,b}_u)^{-1}(\xi_u S_u)^+$ and

$$\psi^l_u = (B^l_u)^{-1}\Big(-\widetilde{Y}^{h,x_2}_u + x_2 B^l_u + (\xi_u S_u)^-\Big)^+, \quad \psi^b_u = -(B^b_u)^{-1}\Big(-\widetilde{Y}^{h,x_2}_u + x_2 B^l_u + (\xi_u S_u)^-\Big)^-.$$

For a fixed $(t,s) \in [0,T) \times \mathcal{O}$, we write $(Y^{c,x_2,s}_u, Z^{c,x_2,s}_u) := (\widetilde{Y}^{c,l,x_2}_u, \widetilde{Z}^{c,l,x_2}_u (B^l_u)^{-1}\sigma(u, S^{s,t}_u))$ and

$$\overline{g}^{c,l}(u, x_2, s, y, z) = g^{c,l}(u, x_2, s, y, z(B^l_u)^{-1}\sigma^{-1}(u,s)).$$

Then BSDE (6.7) becomes

$$\begin{cases} dY^{c,x_2,s}_u = Z^{c,x_2,s}_u\, d\widetilde{W}^l_u + \overline{g}^{c,l}(u, x_2, S^{s,t}_u, Y^{c,x_2,s}_u, Z^{c,x_2,s}_u)\, du, \\ Y^{c,x_2,s}_T = H(S^{s,t}_T). \end{cases} \tag{6.8}$$

Using the same arguments as for the hedger, we deduce that the pricing function $v(t,s) := Y^{c,x_2,s}_t$ is the unique viscosity solution to the following PDE

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \mathcal{L}v(t,s) = \overline{g}^{c,l}\big(t, x_2, s, v(t,s), \sigma(t,s)\frac{\partial v}{\partial s}\big), \quad (t,s) \in [0,T] \times \mathcal{O}, \\ v(T,s) = H(s), \quad s \in \mathcal{O}, \end{cases} \tag{6.9}$$

or, more explicitly, for $(t,s) \in [0,T] \times \mathcal{O}$,

$$\begin{cases} \frac{\partial v}{\partial t}(t,s) + \frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s) = \kappa(t,s)\frac{\partial v}{\partial s}(t,s) + x_2 r^l_t B^l_t + r^{1,b}_t\big(-s\frac{\partial v}{\partial s}(t,s)\big)^+ \\ \qquad - r^l_t\Big(-v(t,s) + x_2 B^l_t + \big(-s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^+ \\ \qquad + r^b_t\Big(-v(t,s) + x_2 B^l_t + \big(-s\frac{\partial v}{\partial s}(t,s)\big)^-\Big)^-, \\ v(T,s) = H(s), \quad s \in \mathcal{O}. \end{cases} \tag{6.10}$$

Conversely, if a function $v \in C^{1,2}([0,T] \times \mathcal{O})$ solves PDE (6.10) such that $v(u,s)$ and $\frac{\partial v}{\partial s}(u,s)$ have polynomial growth in $s$, then $(v(u,S_u), \sigma(u,S_u)\frac{\partial v}{\partial s}(u,S_u))$ is the unique solution to BSDE (6.8) on $u \in [t,T]$ where we write $S = S^{s,t}$. Consequently, the pair $(v(u,S_u), B^l_u\frac{\partial v}{\partial s}(u,S_u))$ solves BSDE (6.7). We conclude that the unique replicating strategy for the hedger equals $\varphi = \big(\xi, \psi^l, \psi^b, \psi^{1,b}\big)$ where, for every $u \in [t,T]$,

$$\begin{aligned} \xi_u &= -\frac{\partial v}{\partial s}(u,S_u), \quad \psi^{1,b}_u = -(B^{1,b}_u)^{-1}\Big(-S_u\frac{\partial v}{\partial s}(u,S_u)\Big)^+, \\ \psi^l_u &= (B^l_u)^{-1}\Big(-v(u,S_u) + x_2 B^l_u + \big(-S_u\frac{\partial v}{\partial s}(u,S_u)\big)^-\Big)^+, \\ \psi^b_u &= -(B^b_u)^{-1}\Big(-v(u,S_u) + x_2 B^l_u + \big(-S_u\frac{\partial v}{\partial s}(u,S_u)\big)^-\Big)^-. \end{aligned} \tag{6.11}$$

We are now in a position to formulate the following proposition, which summarizes pricing and hedging results for a European claim. We denote by $v^h$ and $v^c$ the solutions to the hedger's and counterparty's pricing PDEs, respectively.

**Proposition 6.1** *(i) Let $v^h(t,s) \in C^{1,2}([0,T]\times\mathcal{O})$ be the solution to the quasi-linear PDE (6.5), such that $v^h(t,s)$ and $\frac{\partial v^h}{\partial s}(t,s)$ have a polynomial growth in $s$. Then the hedger's price of the European contingent claim $H(S_T)$ is given by $v^h(t,S_t)$ and the unique replicating strategy $\varphi = \big(\xi,\psi^l,\psi^b,\psi^{1,b}\big)$ for the hedger is given by (6.6) with $v=v^h$.*
*(ii) Let $v^c(t,s) \in C^{1,2}([0,T]\times\mathcal{O})$ be the solution to the quasi-linear PDE (6.10), such that $v^c(t,s)$ and $\frac{\partial v^c}{\partial s}(t,s)$ have a polynomial growth in $s$. Then the counterparty's price of the European contingent claim $H(S_T)$ is given by $v^c(t,S_t)$ and the unique replicating strategy $\varphi = \big(\xi,\psi^l,\psi^b,\psi^{1,b}\big)$ for the counterparty is given by (6.11) with $v=v^c$.*

We notice that PDE (6.5) depends on the initial endowment $x_1$. In the special case where $r^l=r^b=r$, it reduces to the following PDE independent of $x_1$

$$
(6.12)\qquad \begin{cases} \frac{\partial v}{\partial t}(t,s)+\frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s)=\kappa(t,s)\frac{\partial v}{\partial s}(t,s)-r^{1,b}_t\big(s\frac{\partial v}{\partial s}(t,s)\big)^+ \\ \qquad\qquad +\,r_t v(t,s)+r_t\big(s\frac{\partial v}{\partial s}(t,s)\big)^-, \quad (t,s)\in[0,T]\times\mathcal{O}, \\ v(T,s)=H(s), \quad s\in\mathcal{O}. \end{cases}
$$

Note that PDE (6.12) characterizes the price and the hedging strategy for a European contingent claim when the borrowing rate and the lending rates are equal, but the funding rate for the risky asset may differ from $r$. If we further assume that $r^{1,b}=r$, then PDE (6.12) becomes

$$
(6.13)\quad \begin{cases} \frac{\partial v}{\partial t}(t,s)+\frac{1}{2}\sigma^2(t,s)\frac{\partial^2 v}{\partial s^2}(t,s)+(r_t s-\kappa(t,s))\frac{\partial v}{\partial s}(t,s)-r_t v(t,s)=0, \quad (t,s)\in[0,T]\times\mathcal{O}, \\ v(T,s)=H(s), \quad s\in\mathcal{O}. \end{cases}
$$

Obviously, PDE (6.13) is simply the classical Black and Scholes PDE. We mentioned in Remark 3.10 that the market model with partial netting does not cover the standard Bergman's model with different borrowing and lending rates when $r^{i,b}=r^b>r^l$ and trading is unrestricted. However, when the equalities $r^{i,b}=r^b=r^l$ are postulated, then the associated PDEs for a European contingent claim are identical so, as expected, the prices and hedging strategies coincide as well.

One can also obtain Proposition 6.1 by applying classical arguments rather than a BSDE approach as was done, for instance, in Bergman (1995). In essence, both methods hinge on the same mathematical tool, namely, the non-linear Feynman-Kac formula. If a solution of a PDE under study is not smooth, then a BSDE approach gives a probabilistic representation for the viscosity solution of a PDE.

# 7 Initial Endowments of Opposite Signs

In the final section, we examine a particular instance of valuation when the initial endowments of the hedger and the counterparty have opposite signs. Our goal is to show that the range of bilaterally profitable prices $\mathcal{R}^p_0(x_1,x_2)$ (see Definition 3.12) may be non-empty, due to the asymmetry in initial endowments of counterparties.

As usual, we work under Assumptions (M.1)–(M.3). For simplicity, we assume that the lending and borrowing rates $r^l$ and $r^b$ are deterministic and satisfy $r^l_t<r^b_t$ for all $t\in[0,T]$. Our goal is to find a contract $(A,C)$ and a date $\widehat{t}\in[0,T]$ such that the inequality $P^c_{\widehat{t}}(x_2,-A,-C)\le P^h_{\widehat{t}}(x_1,A,C)$ fails to hold when $x_1>0$ and $x_2<0$. To this end, we consider an uncollateralized contract (hence $C=0$) with the following cash flows

$$
A_t = p\,\mathbb{1}_{[0,T]}(t)-\alpha\mathbb{1}_{[t_0,T]}(t)+\alpha e^{\int_{t_0}^T r_u\,du}\mathbb{1}_{[T]}(t)
$$

where $t_0 \in (0,T)$ is a fixed date and the auxiliary function $r$ satisfies $r_u \in (r^l_u, r^b_u)$ for all $u \in [0,T]$. Moreover, a constant $\alpha > 0$ is chosen in such a way that the following inequalities hold

$$x_1 B^l_{t_0} - \alpha e^{\int_{t_0}^T (r_u - r^l_u)\,du} \geq 0, \quad x_1 + \alpha (B^l_{t_0})^{-1} - \alpha (B^l_T)^{-1} e^{\int_{t_0}^T r_u\,du} \geq 0,$$
$$x_2 B^b_{t_0} + \alpha e^{\int_{t_0}^T (r_u - r^b_u)\,du} \leq 0, \quad x_2 - \alpha (B^b_{t_0})^{-1} + \alpha (B^b_T)^{-1} e^{\int_{t_0}^T r_u\,du} \leq 0,$$

which in turn is equivalent to: $\alpha > 0$ and

$$x_2 \kappa_2^{-1} \leq \alpha \leq \min\Big\{ x_1 B^l_{t_0} e^{-\int_{t_0}^T (r_u - r^l_u)\,du},\ -x_2 B^b_{t_0} e^{-\int_{t_0}^T (r_u - r^b_u)\,du},\ x_1 \kappa_1^{-1} \Big\} \tag{7.1}$$

where

$$\kappa_1 := -(B^l_{t_0})^{-1} + (B^l_T)^{-1} e^{\int_{t_0}^T r_u\,du}, \quad \kappa_2 := (B^b_{t_0})^{-1} - (B^b_T)^{-1} e^{\int_{t_0}^T r_u\,du}.$$

Note that $\kappa_1 > 0$ and $\kappa_2 > 0$, since from the inequalities $r^l < r < r^b$ we obtain

$$-\int_0^{t_0} r^l_u\,du - \Big( \int_{t_0}^T r_u\,du - \int_0^T r^l_u\,du \Big) = -\int_{t_0}^T (r_u - r^l_u)\,du < 0.$$

and

$$-\int_0^{t_0} r^b_u\,du - \Big( \int_{t_0}^T r_u\,du - \int_0^T r^b_u\,du \Big) = -\int_{t_0}^T (r_u - r^b_u)\,du > 0,$$

Therefore, a constant $\alpha > 0$ satisfying (7.1) exists and for the number $x$ given by

$$x := x_1 + \alpha (B^l_{t_0})^{-1} - \alpha (B^l_T)^{-1} e^{\int_{t_0}^T r_u\,du} \geq 0,$$

we have that

$$x B^l_{t_0} - \alpha = x_1 B^l_{t_0} - \alpha e^{\int_{t_0}^T (r_u - r^l_u)\,du} \geq 0.$$

We define the strategy $\varphi = \big(\xi^1, \dots, \xi^d, \psi^{1,b}, \dots, \psi^{d,b}, \psi^l, \psi^b, \eta^b, \eta^l\big)$ where $\xi^i = \psi^{i,b} = \psi^b = \eta^b = \eta^l = 0$ for $i = 1, 2, \dots, d$ and

$$\psi^l_t = x \mathbb{1}_{[0,t_0)} + (B^l_{t_0})^{-1} \big( x B^l_{t_0} - \alpha \big) \mathbb{1}_{[t_0,T)} + (B^l_T)^{-1} \Big( x B^l_T - \alpha e^{\int_{t_0}^T r^l_u\,du} + \alpha e^{\int_{t_0}^T r_u\,du} \Big) \mathbb{1}_{[T,T]}.$$

Then the hedger's wealth process satisfies

$$\begin{aligned} V_T(x, \varphi, A, C) &= \big( x B^l_{t_0} - \alpha \big) e^{\int_{t_0}^T r^l_u\,du} + \alpha e^{\int_{t_0}^T r_u\,du} = x B^l_T - \alpha e^{\int_{t_0}^T r^l_u\,du} + \alpha e^{\int_{t_0}^T r_u\,du} \\ &= \Big( x_1 + \alpha (B^l_{t_0})^{-1} - \alpha (B^l_T)^{-1} e^{\int_{t_0}^T r_u\,du} \Big) B^l_T - \alpha e^{\int_{t_0}^T r^l_u\,du} + \alpha e^{\int_{t_0}^T r_u\,du} \\ &= x_1 B^l_T = V^0_T(x_1). \end{aligned}$$

It is thus clear that the self-financing strategy $(x, \varphi, A, C)$ replicates $(A, C)$ on $[0,T]$. Moreover, from the uniqueness of a solution to the pricing BSDE, we know that the replicating strategy is unique. From Definition 3.8, it now follows that

$$P^h_0(x_1, A, C) = x - x_1 = \alpha (B^l_{t_0})^{-1} - \alpha (B^l_T)^{-1} e^{\int_{t_0}^T r_u\,du} = -\alpha \kappa_1 < 0.$$

Let us now focus on the counterparty's valuation problem. If we set

$$\widetilde{x} := x_2 - \alpha (B^b_{t_0})^{-1} + \alpha (B^b_T)^{-1} e^{\int_{t_0}^T r_u\,du} \leq 0,$$

then we obtain

$$\widetilde{x} B^b_{t_0} + \alpha = x_2 B^b_{t_0} + \alpha e^{\int_{t_0}^T (r_u - r^b_u)\,du} \leq 0.$$

We define the strategy $\widetilde{\varphi} = \big(\widetilde{\xi}^1, \dots, \widetilde{\xi}^d, \widetilde{\psi}^{1,b}, \dots, \widetilde{\psi}^{d,b}, \widetilde{\psi}^l, \widetilde{\psi}^b, \widetilde{\eta}^b, \widetilde{\eta}^l\big)$ where $\widetilde{\xi}^i = \widetilde{\psi}^{i,b} = \widetilde{\psi}^l = \widetilde{\eta}^b = \widetilde{\eta}^l = 0$ for $i = 1, 2, \dots, d$ and

$$\widetilde{\psi}^b_t = \widetilde{x} \mathbb{1}_{[0,t_0)} + (B^b_{t_0})^{-1} \big( \widetilde{x} B^b_{t_0} + \alpha \big) \mathbb{1}_{[t_0,T)} + (B^b_T)^{-1} \Big( \widetilde{x} B^b_T + \alpha e^{\int_{t_0}^T r^b_u\,du} - \alpha e^{\int_{t_0}^T r_u\,du} \Big) \mathbb{1}_{[T,T]}.$$

Then we have

$$\begin{aligned} V_T(\widetilde{x},\widetilde{\varphi},-A,-C) &= \big(\widetilde{x}B^b_{t_0}+\alpha\big)e^{\int_{t_0}^T r^b_u\,du}-\alpha e^{\int_{t_0}^T r_u\,du}=\widetilde{x}B^b_T+\alpha e^{\int_{t_0}^T r^b_u\,du}-\alpha e^{\int_{t_0}^T r_u\,du}\\ &= \Big(x_2-\alpha(B^b_{t_0})^{-1}+\alpha(B^b_T)^{-1}e^{\int_{t_0}^T r_u\,du}\Big)B^b_T+\alpha e^{\int_{t_0}^T r^b_u\,du}-\alpha e^{\int_{t_0}^T r_u\,du}\\ &= x_2B^b_T=V^0_T(x_2). \end{aligned}$$

It is clear that $(\widetilde{x},\widetilde{\varphi},-A,-C)$ is the unique self-financing strategy replicating $(-A,-C)$ on $[0,T]$. From the definition of the counterparty's ex-dividend price, we obtain

$$P^c_0(x_2,-A,-C)=x_2-\widetilde{x}=\alpha(B^b_{t_0})^{-1}-\alpha(B^b_T)^{-1}e^{\int_{t_0}^T r_u\,du}=\alpha\kappa_2>0.$$

We have thus established the following inequalities

$$P^c_0(x_2,-A,-C)>\alpha\kappa_2>0>-\alpha\kappa_1=P^h_0(x_1,A,C),$$

so that the range of bilaterally profitable prices $\mathcal{R}^p_0(x_1,x_2)=[P^h_0(x_1,A,C),P^c_0(x_2,-A,-C)]$ is indeed non-empty. Observe that here both parties are willing to pay a strictly positive cash amount to the other party when the contract is initiated at time 0. This is not surprising, since both parties are able to make profits after entering the contract at the expense of the counterparty's funder who will suffer a loss due to reduced servicing costs for the counterparty's debt. To summarize, limited arbitrages opportunity of type (A.3) (see Section 3.2) arise for the hedger and his counterparty, but their gains are offset by the losses of the counterparty's funder.

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