

Convergence results for the indifference value based on the stability of BSDEs

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This version: January 10, 2012
(*Stochastics 85 (2013), 464–488*)

Abstract

We study the exponential utility indifference value h for a contingent claim H in an incomplete market driven by two Brownian motions. The claim H depends on a nontradable asset variably correlated with the traded asset available for hedging. We provide an explicit sequence that converges to h , complementing the structural results for h known from the literature. Our study is based on a convergence result for quadratic backward stochastic differential equations. This convergence result, which we prove in a general continuous filtration under weak conditions, also yields that the indifference value in a setting with trading constraints enjoys a continuity property in the constraints.

Running title: Convergence results for the indifference value

Key words: indifference valuation, exponential utility, correlated Brownian motions, quadratic BSDE, stability of BSDEs, convergent constraints

MSC 2010 subject classification: 91G80, 60H10, 93E20

JEL classification numbers: G13, C60

1 Introduction

An important task of mathematical finance is the valuation of contingent claims. In incomplete markets, this is often done via utility indifference. The *indifference value* h_t for a contingent claim H at time t makes an investor indifferent, in terms of maximal expected utility, between not buying H and buying H for the amount h_t . We explain this definition in more detail

in Sections 3 and 4.1, where we use an exponential utility function. An overview of various aspects of indifference valuation with a long literature list is provided by the recently published book [4] edited by Carmona.

In a basic model, the financial market consists of a risk-free bank account and a stock S driven by a Brownian motion W . The contingent claim H to be valued via exponential utility indifference depends on another Brownian motion Y , which has instantaneous correlation ρ with W . If ρ is deterministic and constant in time, an explicit formula for the indifference value h_t is available from Tehranchi [14] or Chapter 4 of Carmona [4]. However, the situation is different for general ρ . With stochastic and/or time-dependent ρ , one knows for h_t only bounds and a structural formula where one parameter is not explicitly determined; see Frei and Schweizer [5]. In view of this lack of explicit results for h_t , the goal of this paper is to give an explicit sequence that converges to h_t .

Our starting point to study this problem with general ρ is the known characterisation of $(h_t)_{0 \leq t \leq T}$ via a backward stochastic differential equation (BSDE), which we present in Lemma 4.1. We deduce that if ρ is piecewise constant in time, we can obtain an explicit formula for h_t in the same way as for constant ρ , just by considering iteratively the BSDE on intervals where ρ is constant. For general ρ , the idea is to approximate ρ pointwise by a sequence $(\rho^n)_{n \in \mathbb{N}}$ of piecewise constant processes and to replace ρ by ρ^n in the BSDE so that the solutions have an explicit form. These solutions then converge to (a transform of) h_t by a convergence result for quadratic BSDEs, which we prove in a general continuous filtration to guarantee a broad framework for applications. We thus have an explicitly known sequence which converges to h_t . The only point left is whether we can approximate ρ pointwise by a sequence $(\rho^n)_{n \in \mathbb{N}}$ of piecewise constant processes. We show that the above approximation of ρ and thus that of h_t work in a general way for every deterministic ρ except for “pathological examples”. For stochastic ρ , we prove that the approximation of h_t is possible if ρ has left-continuous paths.

The paper is structured as follows. Motivated by the above approximation problem, we state in Section 2 convergence results for quadratic BSDEs in a general continuous filtration. Section 3 gives a first application of these results. We show that the indifference value in a general continuous filtration with trading constraints enjoys a continuity property in the constraints. The results on the indifference valuation in a Brownian setting are contained in Section 4. We lay out the model and prove some preliminary results in Section 4.1. We then study in Section 4.2 the approximation of the indifference value h_t by applying the convergence results of Section 2. Section 4.3 shows a continuity property of h_t in the correlation ρ . Finally, the Appendix contains the proofs of the convergence results of Section 2.

2 Convergence results

The financial applications in the subsequent sections are based on convergence results for quadratic BSDEs in the setting of Morlais [13]. We first recall this framework and then state the main convergence theorem.

We work on a finite time interval $[0, T]$ for a fixed $T > 0$, and we fix $t \in [0, T]$ throughout this section. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{0 \leq s \leq T}, P)$ be a filtered probability space satisfying the usual assumptions with $\mathcal{F} = \mathcal{F}_T$. We assume that \mathbb{F} is continuous, i.e., all local martingales are continuous. We fix an \mathbb{R}^d -valued local martingale $M = (M_s)_{0 \leq s \leq T}$ and take a nondecreasing and bounded process D (e.g., $D = \arctan(\sum_{j=1}^d \langle M^j \rangle)$) such that $d\langle M \rangle = m m' dD$ for an $\mathbb{R}^{d \times d}$ -valued predictable process m .

Let us consider, for $0 \leq s \leq T$, the BSDE

$$\Gamma_s = H + \int_s^T f(r, Z_r) dD_r + \frac{\beta}{2} (\langle N \rangle_T - \langle N \rangle_s) - \int_s^T Z_r dM_r - (N_T - N_s), \quad (2.1)$$

where $f : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ -measurable (\mathcal{P} denotes the σ -field of all predictable sets on $\Omega \times [0, T]$ and $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -field on \mathbb{R}^d), $\beta \in \mathbb{R}$ is a constant and H is a bounded random variable. A solution of (2.1) is a triple (Γ, Z, N) satisfying (2.1), where Γ is a real-valued bounded continuous semimartingale, Z is an \mathbb{R}^d -valued predictable process with $E[\int_0^T |m_s Z_s|^2 dD_s] < \infty$ and N is a real-valued square-integrable martingale null at 0 and strongly orthogonal to M .

Theorem 2.1. *Let $(f^n, \beta^n, H^n)_{n=1,2,\dots,\infty}$ be a sequence of $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ -measurable real-valued mappings, constants, and random variables bounded uniformly in L^∞ , such that*

- (i) *there exist a nonnegative predictable κ^1 with $\|\int_0^T \kappa_s^1 dD_s\|_{L^\infty} < \infty$ and a constant c^1 such that*

$$|f^n(s, z)| \leq \kappa_s^1 + c^1 |m_s z|^2 \quad \text{for all } s \in [0, T], z \in \mathbb{R}^d \quad (2.2)$$

and $n = 1, \dots, \infty$;

- (ii) *there exist a nonnegative predictable κ^2 with $\|\int_0^T |\kappa_s^2|^2 dD_s\|_{L^\infty} < \infty$ and a constant c^2 such that*

$$|f^n(s, z^1) - f^n(s, z^2)| \leq c^2 (\kappa_s^2 + |m_s z^1| + |m_s z^2|) |m_s (z^1 - z^2)|$$

for all $s \in [0, T]$, $z^1, z^2 \in \mathbb{R}^d$ and $n = 1, \dots, \infty$;

- (iii) *$\lim_{n \rightarrow \infty} \beta^n = \beta^\infty$, $\lim_{n \rightarrow \infty} H^n = H^\infty$ a.s. and for $(P \otimes D)$ -almost all $(\omega, s) \in \llbracket t, T \rrbracket$, $\lim_{n \rightarrow \infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$.*

Then there are unique solutions (Γ^n, Z^n, N^n) to the BSDE (2.1) with parameters (f^n, β^n, H^n) for $n = 1, \dots, \infty$, and Γ_t^n converges to Γ_t^∞ a.s. as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} E \left[\int_t^T |m_s(Z_s^n - Z_s^\infty)|^2 dD_s \right] = 0, \quad \lim_{n \rightarrow \infty} E [\langle N^n - N^\infty \rangle_T - \langle N^n - N^\infty \rangle_t] = 0.$$

Moreover, $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ as $n \rightarrow \infty$ in probability and in L^p , $p \in [1, \infty)$.

In the above theorem, we assumed only weak conditions for the convergence of the input data $(f^n, \beta^n, H^n)_{n=1, \dots, \infty}$. In particular, we imposed only pointwise convergence of $(f^n)_{n=1, \dots, \infty}$, which will be significant for the later applications. The price to be paid is that the solutions of the BSDE converge only in some weak sense. However, in the main application in Section 4, the generators $(f^n)_{n=1, \dots, \infty}$ of the BSDE have a specific form, and for this case, the following result states stronger convergence properties. The proofs of Theorem 2.1 and of the next corollary are presented in the Appendix.

Corollary 2.2. *Suppose in addition to the assumptions of Theorem 2.1 that*

- (iv) H^n converges to H^∞ in L^∞ as $n \rightarrow \infty$;
- (v) *there exist sequences $(\underline{a}^n)_{n \in \mathbb{N}}$ and $(\bar{a}^n)_{n \in \mathbb{N}}$ of deterministic functions which converge to 1 uniformly on $[t, T]$ (up to a $(P \otimes D)$ -nullset) such that $f^n = \underline{a}^n \underline{f} + \bar{a}^n \bar{f}$ for every $n = 1, \dots, \infty$, where \underline{f}, \bar{f} satisfy (2.2) with f^n replaced by \underline{f}, \bar{f} .*

Then we have $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ in L^∞ as $n \rightarrow \infty$ and there even exists a constant $K > 0$ such that for all $n \in \mathbb{N}$,

$$\left\| \sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \right\|_{L^\infty} \leq K (\|\underline{a}^n - 1\|_{L^\infty(P \otimes D)} + \|\bar{a}^n - 1\|_{L^\infty(P \otimes D)} + |\beta^n - \beta^\infty| + \|H^n - H^\infty\|_{L^\infty}). \quad (2.3)$$

Further, $\int Z^n dM \rightarrow \int Z^\infty dM$ and $N^n \rightarrow N^\infty$ on $[t, T]$ in BMO as $n \rightarrow \infty$.

In the literature on BSDEs, convergence results are also called stability results. The main differences between Theorem 2.8 of Kobylanski [9] and our Theorem 2.1 are the following: Kobylanski [9] works in a Brownian setting and imposes locally uniform convergence on the generators, whereas our Theorem 2.1 is stated in a general continuous filtration and for generators $(f^n)_{n=1, \dots, \infty}$ that converge only pointwise. Moreover, the generators in Kobylanski's Theorem 2.8 can unlike ours also depend on Γ^n , and the a.s. convergence of $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|$ is proved. Another convergence result in a Brownian setting is Proposition 7 of Briand and Hu [3], which gives convergence of

the moments of $\exp(\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|)$ and $(\int_t^T |Z_s^n - Z_s^\infty|^2 ds)^{1/2}$ for unbounded terminal conditions if the generators are convex. (That result needs a minor amendment as explained in Section 7 of Mocha and Westray [12].)

For a general continuous filtration, a convergence result for an exponential transformation of the BSDE (2.1) is available from Lemma 3.3 and Remark 3.4 of Morlais [13]. Lemma 3.3 serves in [13] as an auxiliary result to show existence of a solution to (2.1) with a more general generator f which can also depend on Γ . The proof of the existence result first establishes a one-to-one correspondence between solutions to (2.1) and those to a simpler BSDE which results from an exponential transformation of the original BSDE. Lemma 3.3 is then used in proving existence of a solution to the simpler BSDE. Due to the one-to-one correspondence between solutions to the original and to the simpler BSDEs, Lemma 3.3 gives also a convergence result for the original BSDE, as Morlais remarks. In particular, its application to (2.1) needs that $\exp(\beta^n H^n)$ and a certain transform of f^n are nondecreasing in n , and it yields $E[\sup_{s \in [t, T]} |e^{\beta^n \Gamma_s^n} - e^{\beta^\infty \Gamma_s^\infty}|] \rightarrow 0$, which is equivalent to $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ in L^1 for $\beta^\infty \neq 0$; the equivalence can be shown using

$$\min\{e^x, e^y\}|x - y| \leq |e^x - e^y| \leq \max\{e^x, e^y\}|x - y|, \quad x, y \in \mathbb{R},$$

and that Γ^n is uniformly bounded in $n = 1, \dots, \infty$ by Lemma 3.1 of Morlais [13]. In contrast to Morlais [13], who proves existence and uniqueness of solutions to (2.1), we focus on convergence questions and work in the proof of Theorem 2.1 directly with the BSDE (2.1) instead of doing first an exponential transformation.

For the applications, three ingredients of our stability result are essential:

- f^n are not necessarily monotonic in n ;
- f^n converge only pointwise;
- we obtain a.s.-convergence of Γ_t^n .

These three crucial points are also the reason why we cannot apply the stability results of the recent papers by Barrieu and El Karoui [2] or Mocha and Westray [12], who give existence and uniqueness results in a continuous filtration and with unbounded terminal conditions. Theorem 3.6 of [2] imposes that f^n are monotonic in n and converge locally uniformly, whereas Theorem 2.7 of [12] uses different types of convergence for f^n and Γ^n (convergence in probability of the integral $\int_t^T |f^n(s, Z_s^\infty) - f^\infty(s, Z_s^\infty)| dD_s$ and convergence of exponential moments of $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|$). While our technique does not allow for Γ^n -dependence of the generators as in [2] and [12], it is the only one which meets the above three points and hence is suitable for the applications in mathematical finance presented in Sections 3 and 4.

3 Indifference valuation with convergent constraints

We apply the convergence results in two different financial contexts. While both applications deal with indifference valuation, the underlying settings are different. In this section, we consider an investor facing trading constraints in the same fairly general framework as in the previous section. For the main application of Theorem 2.1, we restrict in Section 4 our study to a Brownian model and derive there results independently from this section.

We now work within the framework of Section 2 with a continuous filtration \mathbb{F} . Recall that M is a local martingale and $d\langle M \rangle = m m' dD$. We suppose that almost surely, the matrix m_s is invertible for every $s \in [0, T]$. The financial market consists of a risk-free bank account yielding zero interest and d risky assets whose price process $S = (S_s)_{0 \leq s \leq T}$ is given by

$$\frac{dS_s^j}{S_s^j} = dM_s^j + \sum_{i=1}^d \lambda_s^i d\langle M^j, M^i \rangle_s, \quad 0 \leq s \leq T, \quad S_0^j > 0 \quad \text{for } j = 1, \dots, d,$$

where λ is a predictable process which satisfies $\left\| \int_0^T |m_s \lambda_s|^2 dD_s \right\|_{L^\infty} < \infty$, i.e., the mean-variance tradeoff process is bounded. Let H be a bounded random variable, interpreted as a contingent claim or payoff due at time T , and let $C \subseteq \mathbb{R}^d$ be a closed set with $0 \in C$. We assume that our investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. Starting at time t with bounded \mathcal{F}_t -measurable capital x_t , she runs a self-financing strategy $\pi = (\pi_s)_{t \leq s \leq T}$ valued in C such that her wealth at time $s \in [t, T]$ is $X_s^{x_t, \pi} = x_t + \int_t^s \sum_{j=1}^d \frac{\pi_r^j}{S_r^j} dS_r^j$, where π^j represents the amount invested in S^j , $j = 1, \dots, d$. The set \mathcal{A}_t^C of C -admissible strategies on $[t, T]$ consists of all predictable \mathbb{R}^d -valued processes $\pi = (\pi_s)_{t \leq s \leq T}$ which satisfy a.s., $\pi_s \in C$ for all $s \in [t, T]$, $E\left[\int_t^T |m_s \pi_s|^2 dD_s\right] < \infty$ and are such that $\exp(-\gamma X_s^{x_t, \pi})$, $t \leq s \leq T$, is of class (D). We define $V_t^{H,C}(x_t)$ by

$$\begin{aligned} V_t^{H,C}(x_t) &:= \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t^C} E\left[U(X_T^{x_t, \pi} + H) \mid \mathcal{F}_t\right] \\ &= e^{-\gamma x_t} \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t^C} E\left[-\exp(-\gamma X_T^{0, \pi} - \gamma H) \mid \mathcal{F}_t\right] \end{aligned} \quad (3.1)$$

so that $V_t^{H,C}(x_t)$ is the maximal expected utility the investor can achieve by starting at time t with initial capital x_t , using some C -admissible strategy π , and receiving H at time T . For ease of notation, we write

$$V_t^{H,C}(x_t) = e^{-\gamma x_t} V_t^{H,C}(0) =: e^{-\gamma x_t} V_t^{H,C}. \quad (3.2)$$

Viewed over time t , $V^{H,C}$ is then the *dynamic value process* for the stochastic control problem associated to exponential utility maximisation.

The time t *indifference (buyer) value* $h_t^{H,C}(x_t)$ for H is implicitly defined by

$$V_t^{0,C}(x_t) = V_t^{H,C}(x_t - h_t^{H,C}(x_t)). \quad (3.3)$$

This says that the investor is indifferent between solely trading with initial capital x_t , versus trading with initial capital $x_t - h_t^{H,C}(x_t)$ but receiving H at T . By (3.2),

$$h_t^{H,C}(x_t) = h_t^{H,C} = \frac{1}{\gamma} \log \frac{V_t^{0,C}}{V_t^{H,C}} \quad (3.4)$$

does not depend on x_t .

The following proposition can be seen as a kind of continuity result for $V_t^{H,C}$ and $h_t^{H,C}$ in (H, C) . Its proof is based on Theorem 2.1.

Proposition 3.1. *Let H^n , $n = 1, 2, \dots, \infty$, be uniformly bounded random variables with $\lim_{n \rightarrow \infty} H^n = H^\infty$ a.s., and let C^n , $n = 1, 2, \dots, \infty$, be closed subsets of \mathbb{R}^d which contain zero and are such that $(P \otimes D)$ -a.e.,*

$$\lim_{n \rightarrow \infty} \inf_{y \in C^n} |m(y - z)| = \inf_{y \in C^\infty} |m(y - z)| \quad \text{for all } z \in \mathbb{R}^d. \quad (3.5)$$

Then $\lim_{n \rightarrow \infty} V_t^{H^n, C^n} = V_t^{H^\infty, C^\infty}$ and $\lim_{n \rightarrow \infty} h_t^{H^n, C^n} = h_t^{H^\infty, C^\infty}$ a.s., and there exist continuous versions V^{H^n, C^n} and h^{H^n, C^n} , $n = 1, \dots, \infty$, such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} |V_s^{H^n, C^n} - V_s^{H^\infty, C^\infty}| &= 0 \text{ in probab. and in } L^p, \quad 1 \leq p < \infty, \\ \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} |h_s^{H^n, C^n} - h_s^{H^\infty, C^\infty}| &= 0 \text{ in probab. and in } L^p, \quad 1 \leq p < \infty. \end{aligned} \quad (3.6)$$

Proof. Fix $n \in \{1, \dots, \infty\}$. By Theorem 4.1 of Morlais [13], there is a version V^{H^n, C^n} such that $V^{H^n, C^n} = -\exp(\gamma \Gamma^n)$, where (Γ^n, Z^n) is the solution of (2.1) with $\beta := \gamma$, H replaced by $-H^n$ and with generator f^n given by

$$f^n(s, z) := \inf_{y \in C^n} \left(\frac{\gamma}{2} \left| m_s \left(y - z - \frac{1}{\gamma} \lambda_s \right) \right|^2 \right) - (m_s z)' (m_s \lambda_s) - \frac{1}{2\gamma} |m_s \lambda_s|^2$$

for $s \in [0, T]$ and $z \in \mathbb{R}^d$. Remarks 2.3 and 2.4 of Morlais [13] and (3.5) imply that the assumptions (i)–(iii) of Theorem 2.1 are satisfied, which yields

$$\lim_{n \rightarrow \infty} \Gamma_t^n = \Gamma_t^\infty \text{ a.s. and } \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| = 0 \text{ in probab. and in } L^p. \quad (3.7)$$

Therefore, we obtain $\lim_{n \rightarrow \infty} V_t^{H^n, C^n} = \lim_{n \rightarrow \infty} -e^{\gamma \Gamma_t^n} = -e^{\gamma \Gamma_t^\infty} = V_t^{H^\infty, C^\infty}$ a.s. and analogously $\lim_{n \rightarrow \infty} V_t^{0, C^n} = V_t^{0, C^\infty}$ a.s., so $\lim_{n \rightarrow \infty} h_t^{H^n, C^n} = h_t^{H^\infty, C^\infty}$ a.s. by (3.4). Because we have

$$\begin{aligned} \sup_{s \in [t, T]} |h_s^{H^n, C^n} - h_s^{H^\infty, C^\infty}| &= \frac{1}{\gamma} \sup_{s \in [t, T]} \left| \log \frac{V_s^{0, C^n}}{V_s^{0, C^\infty}} - \log \frac{V_s^{H^n, C^n}}{V_s^{H^\infty, C^\infty}} \right| \\ &\leq \frac{1}{\gamma} \sup_{s \in [t, T]} \left| \log \frac{V_s^{0, C^n}}{V_s^{0, C^\infty}} \right| + \sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|, \end{aligned}$$

we obtain (3.6) from (3.7) and its analogue with (H^n, C^n) replaced by $(0, C^n)$. We also have $\lim_{n \rightarrow \infty} \sup_{s \in [t, T]} |e^{\gamma \Gamma_s^n} - e^{\gamma \Gamma_s^\infty}| = 0$ in probab. and in L^p , since

$$\sup_{s \in [t, T]} |e^{\gamma \Gamma_s^n} - e^{\gamma \Gamma_s^\infty}| \leq \gamma \exp \left(\gamma \left\| \sup_{n=1, \dots, \infty} \sup_{s \in [t, T]} |\Gamma_s^n| \right\|_{L^\infty} \right) \sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \quad (3.8)$$

and Γ^n is uniformly bounded by Lemma 3.1 of Morlais [13]. This concludes the proof because $V^{H^n, C^n} = -\exp(\gamma \Gamma^n)$ for a version V^{H^n, C^n} . \square

Remark 3.2. 1) The condition (3.5) can be rephrased as follows: Define a time-dependent random inner product $\langle \cdot, \cdot \rangle_m$ by $\langle x, y \rangle_m := x' m' m y$ for x, y in \mathbb{R}^d and denote by d_m the induced metric, i.e., $d_m(x, y) := \langle x - y, x - y \rangle_m$ for $x, y \in \mathbb{R}^d$. Then $\langle \cdot, \cdot \rangle_m$ is the standard scalar product on \mathbb{R}^d after a basis transformation by m^{-1} . Defining $d_m(x, C) := \inf_{y \in C} d_m(x, y)$ for a closed set $C \subseteq \mathbb{R}^d$, the condition (3.5) is equivalent to $\lim_{n \rightarrow \infty} d_m(x, C^n) = d_m(x, C^\infty)$ for all $x \in \mathbb{R}^d$. This means that the sets $(C^n)_{n \in \mathbb{N}}$ are Wijsman convergent to C^∞ with respect to the metric d_m . Beer [1] gives a survey on Wijsman convergence, which is a weaker notion than convergence in the Hausdorff metric.

2) We have used an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. By applying Theorems 4.4 and 4.7 of Morlais [13], analogous results can be derived for the value process related to power utility $U(x) = x^\gamma / \gamma$, $x > 0$, for a fixed $\gamma \in]0, 1[$, and to logarithmic utility $U(x) = \log x$, $x > 0$, when there is no claim, i.e., $H = 0$. \diamond

4 Indifference valuation in a Brownian setting

We now apply the convergence Theorem 2.1 to the indifference valuation in a Brownian setting with variable correlation. We first introduce in Section 4.1 the model and explain the problem. We then apply Theorem 2.1 in Sections 4.2 and 4.3 to give convergence results for the indifference value and the dynamic value process.

4.1 Model setup and preliminary results

We work on a finite time interval $[0, T]$ for a fixed $T > 0$, and we fix $t \in [0, T]$ throughout this section. On a complete filtered probability space $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_s)_{0 \leq s \leq T}, P)$, we have two independent one-dimensional (\mathbb{G}, P) -Brownian motions Y and Y^\perp . We denote by $\mathbb{Y} = (\mathcal{Y}_s)_{0 \leq s \leq T}$ the P -augmented filtration generated by Y . Let W be a (\mathbb{G}, P) -Brownian motion with instantaneous correlation ρ to Y so that

$$dW_s = \rho_s dY_s + \sqrt{1 - \rho_s^2} dY_s^\perp, \quad 0 \leq s \leq T. \quad (4.1)$$

Our financial market consists of a risk-free bank account yielding zero interest and a traded risky asset S with dynamics

$$dS_s = S_s \mu_s ds + S_s \sigma_s dW_s, \quad 0 \leq s \leq T, \quad S_0 > 0;$$

the drift μ and the (positive) volatility σ are \mathbb{G} -predictable. We set $\lambda := \frac{\mu}{\sigma}$ and assume that $\int_0^T \lambda_s^2 ds$ is bounded. The processes

$$\hat{W} := W + \int \lambda ds \quad \text{and} \quad \hat{Y} := Y + \int \rho \lambda ds$$

are Brownian motions under the minimal martingale measure \hat{P} given by

$$\frac{d\hat{P}}{dP} := \mathcal{E} \left(- \int \lambda dW \right)_T := \exp \left(- \int_0^T \lambda_s dW_s - \frac{1}{2} \int_0^T \lambda_s^2 ds \right). \quad (4.2)$$

In contrast to Section 3, the investor here can trade in S without constraints. He starts at time t with bounded \mathcal{G}_t -measurable capital x_t and runs a self-financing strategy $\pi = (\pi_s)_{t \leq s \leq T}$ so that his wealth at time $s \in [t, T]$ is

$$X_s^{x_t, \pi} = x_t + \int_t^s \frac{\pi_r}{S_r} dS_r = x_t + \int_t^s \pi_r \sigma_r d\hat{W}_r, \quad (4.3)$$

where π represents the amount invested in S . For a bounded random variable H , we define $V_t^H(x_t)$ like $V_t^{H,C}(x_t)$ in (3.1) with \mathcal{G}_t instead of \mathcal{F}_t and \mathcal{A}_t^C replaced by \mathcal{A}_t which consists of all \mathbb{G} -predictable real-valued processes $\pi = (\pi_s)_{t \leq s \leq T}$ which satisfy $\int_t^T \pi_s^2 \sigma_s^2 ds < \infty$ a.s. and are such that

$$\exp(-\gamma X_s^{x_t, \pi}), \quad t \leq s \leq T, \quad \text{is of class (D) on } (\Omega, \mathcal{G}_T, \mathbb{G}, P). \quad (4.4)$$

The dynamic value process V^H and the indifference value h_t^H are defined analogously to (3.2) and (3.3). From (3.4), we see that once we can calculate V_t^H and V_t^0 , we also know h_t^H . So our focus lies on studying V_t^H .

We always impose without further mention the *standing assumption* that

$$H \in L^\infty(\mathcal{Y}_T, P) \text{ and } \lambda, \rho \text{ are } \mathbb{Y}\text{-predictable.} \quad (4.5)$$

This reflects a situation where the payoff H is driven by Y , whereas hedging can only be done in S which is in general imperfectly correlated with Y . We refer to Sections 4.1 and 4.2 of Frei and Schweizer [5] for an overview of the related literature and a thorough explanation of the standing assumption (4.5), which corresponds to case (I) in [5]. (For case (II) in [5], results analogous to those in Section 4.2 can be derived if ρ is predictable for the filtration generated by \hat{Y} .)

The standing assumption (4.5) allows us to give a BSDE characterisation for V^H in the \mathbb{Y} -filtration. This BSDE is a special case of the BSDE (2.1).

Lemma 4.1. *The BSDE*

$$\Gamma_s = H - \int_s^T \left(\frac{1}{2} \gamma (1 - \rho_r^2) Z_r^2 - Z_r \rho_r \lambda_r - \frac{\lambda_r^2}{2\gamma} \right) dr + \int_s^T Z_r dY_r \quad (4.6)$$

for $0 \leq s \leq T$ has a unique solution (Γ, Z) where Γ is a real-valued bounded continuous (\mathbb{Y}, P) -semimartingale and Z is a \mathbb{Y} -predictable process such that $E_P[\int_0^T Z_s^2 ds] < \infty$. Moreover, there exists a continuous version V^H (which we always use in the sequel) such that $V^H = -\exp(-\gamma\Gamma)$, and its optimal strategy denoted by π^* is given by $\pi^* = \frac{\rho}{\sigma} Z + \frac{\lambda}{\gamma\sigma}$.

Lemma 4.1 is essentially known. In particular, Proposition 5.5 of Frei et al. [6] gives a multidimensional version. However, two assumptions of that proposition are not satisfied in our setting; firstly, \mathbb{G} is not necessarily generated by W and a Brownian motion orthogonal to W , and secondly, $|\rho|$ is not bounded away from 1. Instead of adapting the proof of Proposition 5.5 of Frei et al. [6], we give the complete argument in the following.

Proof of Lemma 4.1. Existence and uniqueness of a solution (Γ, Z) of (4.6) follow from Theorem 2.1 with $\mathbb{F} := \mathbb{Y}$, $M := -Y$ and

$$f(s, z) := -\frac{1}{2} \gamma (1 - \rho_s^2) z^2 + z \rho_s \lambda_s + \frac{\lambda_s^2}{2\gamma} \quad \text{for } s \in [0, T] \text{ and } z \in \mathbb{R}.$$

(Since any \mathbb{Y} -martingale orthogonal to Y is constant, we can choose in (2.1) $\beta \in \mathbb{R}$ arbitrarily.) Moreover, Proposition 7 of Mania and Schweizer [11] and its proof yield that $\int Z dY$ is in both $BMO(\mathbb{Y}, P)$ and $BMO(\mathbb{G}, P)$.

To establish the result, it remains to show $V_t^H = -\exp(-\gamma\Gamma_t)$. A simple calculation based on (4.3) and (4.6) yields for $\pi \in \mathcal{A}_t$ that

$$\begin{aligned} \exp(-\gamma X_s^{0,\pi}) &= \exp(\gamma\Gamma_s - \gamma\Gamma_t) \frac{\mathcal{E}\left(\int \gamma Z dY - \int \gamma\pi\sigma dW\right)_s}{\mathcal{E}\left(\int \gamma Z dY - \int \gamma\pi\sigma dW\right)_t} \\ &\quad \times \exp\left(\frac{1}{2} \int_t^s (\gamma\rho_r Z_r + \lambda_r - \gamma\pi_r\sigma_r)^2 dr\right) \\ &\geq \exp(\gamma\Gamma_s - \gamma\Gamma_t) \frac{\mathcal{E}\left(\int \gamma Z dY - \int \gamma\pi\sigma dW\right)_s}{\mathcal{E}\left(\int \gamma Z dY - \int \gamma\pi\sigma dW\right)_t}, \quad t \leq s \leq T. \end{aligned} \quad (4.7)$$

Therefore, if $\int \pi\sigma dW \in BMO(\mathbb{G}, P)$, we obtain

$$E_P[\exp(-\gamma X_T^{0,\pi} - \gamma H) | \mathcal{G}_t] \geq \exp(-\gamma\Gamma_t), \quad (4.8)$$

since the stochastic exponential of a continuous BMO -martingale is a true martingale by Theorem 2.3 of Kazamaki [7]. By a localisation argument and (4.4), we have (4.8) for every $\pi \in \mathcal{A}_t$, which implies $V_t^H \leq -\exp(-\gamma\Gamma_t)$. Equality in (4.8) holds for $\pi = \pi^* := \frac{\rho}{\sigma}Z + \frac{\lambda}{\gamma\sigma}$. Since $\exp(-\gamma X^{0,\pi^*})$ is by (4.7) the product of a bounded process and a (\mathbb{G}, P) -martingale, it is of class (D) on $(\Omega, \mathcal{G}_T, \mathbb{G}, P)$; hence $\pi^* \in \mathcal{A}_t$ and $V_t^H = -\exp(-\gamma\Gamma_t)$. \square

Although V^H is given in terms of the solution of (4.6), there is no fully explicit formula available for V_t^H unless ρ is deterministic and constant in time. While the methods in Frei and Schweizer [5] and Frei et al. [6] give bounds for V_t^H , we approximate in the subsequent sections V_t^H by approaching ρ by piecewise constant processes.

Definition 4.2. We denote by Ξ the set of all processes q of the form

$$q = q^1 \mathbf{1}_{\{\tau_0\}} + \sum_{j=1}^n q^j \mathbf{1}_{\llbracket \tau_{j-1}, \tau_j \rrbracket}, \quad \text{for } t = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n = T,$$

where τ_j is a \mathbb{Y} -stopping time and q^j is a $\mathcal{Y}_{\tau_{j-1}}$ -measurable random variable valued in $] -1, 1[$. We call $(q^j, \tau_j)_{j=1, \dots, n}$ a characterising sequence of q .

Proposition 4.3. Let q be a bounded \mathbb{Y} -predictable process. The BSDE

$$\Gamma_s^q = H - \int_s^T \left(\frac{1}{2} \gamma (1 - q_r^2) |Z_r^q|^2 - Z_r^q \rho_r \lambda_r - \frac{\lambda_r^2}{2\gamma} \right) dr + \int_s^T Z_r^q dY_r \quad (4.9)$$

for $0 \leq s \leq T$ has a unique solution (Γ^q, Z^q) (in the sense of Lemma 4.1).

1) If $q \in \Xi$ with characterising sequence $(q^j, \tau_j)_{j=1, \dots, n}$, then

$$e^{-\gamma\Gamma_t^q} = E_{\hat{P}} \left[\dots E_{\hat{P}} \left[E_{\hat{P}} \left[e^{\hat{H}(1-q^n)^2} \middle| \mathcal{Y}_{\tau_{n-1}} \right]^{\frac{1-|q^{n-1}|^2}{1-|q^n|^2}} \middle| \mathcal{Y}_{\tau_{n-2}} \right]^{\frac{1-|q^{n-2}|^2}{1-|q^{n-1}|^2}} \dots \middle| \mathcal{Y}_t \right]^{\frac{1}{1-|q^1|^2}}, \quad (4.10)$$

where $\hat{H} := -\gamma H - \frac{1}{2} \int_t^T \lambda_s^2 ds$.

2) If $|q| \geq |\rho|$ ($P \otimes \text{Leb}$)-almost everywhere, then $V^H \leq -\exp(-\gamma \Gamma^q)$.

3) If $|q| \leq |\rho|$ ($P \otimes \text{Leb}$)-almost everywhere, then $V^H \geq -\exp(-\gamma \Gamma^q)$.

If ρ itself is in Ξ , Proposition 4.3 gives explicit formulas for V_t^H and h_t^H by choosing $q = \rho$ and using (3.4). For general ρ , the idea is to find a sequence $(q^n)_{n \in \mathbb{N}}$ in Ξ which converges pointwise to ρ . The solutions $\Gamma_t^{q^n}$ of (4.9) with $q = q^n$ have the explicit form (4.10) and converge a.s. to the solution $\Gamma_t = \Gamma_t^\rho$ of (4.6) by Theorem 2.1. We thus obtain an explicitly known sequence converging a.s. to V_t^H . The only open point, which we treat in Section 4.2, is whether we can find a sequence $(q^n)_{n \in \mathbb{N}}$ in Ξ which converges pointwise to ρ .

Note that the right-hand side of (4.10) is *not* the value of V_t^H in a model with correlation q instead of ρ . Comparing (4.9) with (4.6), we see that only the ρ in front of $|Z|^2$ is replaced by q ; the ρ in the term linear in Z is kept. This implies that the measure used in the iterated expectations in (4.10) is \hat{P} , defined in (4.2). It does not depend on q —a property desired for the above-mentioned approximation of V_t^H , since we prefer to take always the same explicitly known measure in calculating the conditional expectations. If we replace ρ in (4.9) by q , the solution of the BSDE is linked to the value of V_t^H when ρ is replaced by q . In Section 4.3, we deduce from this a continuity property of V_t^H in ρ .

Parts 2) and 3) of Proposition 4.3 can be seen as a monotonicity property of V_t^H . However, since ρ still appears in (4.9), we cannot simply say that V_t^H is monotonic in $|\rho|$. This has already been pointed out in Section 5 of Frei and Schweizer [5] by saying that V_t^H is monotonic in $|\rho|$ only when the measure \hat{P} from (4.2), which depends via W on ρ , is kept fixed. Proposition 3 of Frei and Schweizer [5] gives a result analogous to parts 2) and 3) of Proposition 4.3 when $|q|$ and $|\rho|$ can be separated by a constant. Proposition 4.3 shows that this additional assumption is superfluous and generalises Proposition 3 of [5].

Proof of Proposition 4.3. Like in the proof of Lemma 4.1, (4.9) has a unique solution (Γ^q, Z^q) and $\int Z^q dY \in BMO(\mathbb{Y}, P)$. Theorem 3.6 of Kazamaki [7] yields $\int Z d\hat{Y}, \int Z^q d\hat{Y} \in BMO(\mathbb{Y}, \hat{P})$ for the solution (Γ, Z) of (4.6), and as a consequence, their stochastic exponentials are true martingales.

To prove 1), we fix $j \in \{1, \dots, n\}$ and write (4.9), for $\tau_{j-1} \leq s \leq \tau_j$ as

$$\Gamma_s^q = \Gamma_{\tau_j}^q + \frac{1}{2\gamma} \int_s^{\tau_j} \lambda_r^2 dr - \frac{1}{2} \gamma (1 - |q^j|^2) \int_s^{\tau_j} |Z_r^q|^2 dr + \int_s^{\tau_j} Z_r^q d\hat{Y}_r,$$

which implies

$$\begin{aligned} & e^{-\gamma(1-|q^j|^2)\Gamma_{\tau_{j-1}}^q} \exp\left(\gamma(1-|q^j|^2) \int_{\tau_{j-1}}^{\tau_j} Z_r^q d\hat{Y}_r - \frac{1}{2}\gamma^2(1-|q^j|^2)^2 \int_{\tau_{j-1}}^{\tau_j} |Z_r^q|^2 dr\right) \\ &= \exp\left(-\gamma(1-|q^j|^2)\Gamma_{\tau_j}^q - \frac{1-|q^j|^2}{2} \int_{\tau_{j-1}}^{\tau_j} \lambda_r^2 dr\right). \end{aligned}$$

Taking $(\mathcal{Y}_{\tau_{j-1}}, \hat{P})$ -conditional expectations and logarithms yields

$$\Gamma_{\tau_{j-1}}^q = -\frac{1}{\gamma(1-|q^j|^2)} \log E_{\hat{P}} \left[\exp\left(-\gamma\Gamma_{\tau_j}^q - \frac{1}{2} \int_{\tau_{j-1}}^{\tau_j} \lambda_r^2 dr\right)^{1-|q^j|^2} \middle| \mathcal{Y}_{\tau_{j-1}} \right].$$

Using this argument iteratively for $j = n, \dots, 1$ results in (4.10).

To prove 2), we subtract (4.6) from (4.9), which gives

$$\begin{aligned} \Gamma_s^q - \Gamma_s &= \frac{1}{2}\gamma \int_s^T ((1-\rho_r^2)|Z_r|^2 - (1-q_r^2)|Z_r^q|^2) dr + \int_s^T (Z_r^q - Z_r) d\hat{Y}_r \\ &\geq \frac{1}{2}\gamma \int_s^T (1-\rho_r^2)(|Z_r|^2 - |Z_r^q|^2) dr + \int_s^T (Z_r^q - Z_r) d\hat{Y}_r \\ &= \int_s^T (Z_r^q - Z_r) (d\hat{Y}_r - \kappa_r dr), \quad 0 \leq s \leq T, \end{aligned} \quad (4.11)$$

with $\kappa := \frac{1}{2}\gamma(1-\rho^2)(Z^q + Z)$. The $BMO(\mathbb{Y}, \hat{P})$ -property of $\int Z d\hat{Y}$ and $\int Z^q d\hat{Y}$ implies that $\int \kappa d\hat{Y}$ is in $BMO(\mathbb{Y}, \hat{P})$, and by Theorem 3.6 of Kazamaki [7], the process $\int (Z^q - Z) (d\hat{Y} - \kappa dr)$ is thus also a $BMO(\mathbb{Y}, \hat{P}')$ -martingale for the probability measure \hat{P}' given by $\frac{d\hat{P}'}{d\hat{P}} := \mathcal{E}(\int \kappa d\hat{Y})_T$. Taking $(\mathcal{Y}_s, \hat{P}')$ -conditional expectations in (4.11) yields $\Gamma_s^q \geq \Gamma_s$ for any $s \in [0, T]$, which gives $V^H = -\exp(-\gamma\Gamma) \leq -\exp(-\gamma\Gamma^q)$ by Lemma 4.1 and the continuity of Γ and Γ^q . The proof of 3) goes analogously to 2). \square

4.2 Approximating the indifference value

As explained after Proposition 4.3, the question whether V_t^H is the a.s. limit of an explicitly known sequence boils down to whether it is possible to find a sequence $(q^n)_{n \in \mathbb{N}}$ in Ξ which converges pointwise to ρ . In Section 4.2.1, we show that this is possible in a general way for every deterministic ρ except for “pathological examples”. We then give in Section 4.2.2 such a counterexample where the approximation of V_t^H indeed fails. Section 4.2.3 presents the approximation of V_t^H for general (stochastic) ρ with left-continuous paths.

4.2.1 Deterministic correlation

The approximation of ρ with piecewise constant processes is reminiscent of the construction of the Riemann integral. We recall that a bounded function $g : [t, T] \rightarrow \mathbb{R}$ is called *Riemann integrable* if there exists $J \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| J - \sum_{j=1}^n g(s_j)(t_j - t_{j-1}) \right| < \epsilon$$

for every partition (t_0, \dots, t_n) of $[t, T]$ with $\max_{1 \leq j \leq n} (t_j - t_{j-1}) < \delta$ and every choice of $s_j \in [t_{j-1}, t_j]$.

The following result, which is shown on page 29 of Lebesgue [10], is known as Lebesgue's theorem.

Lemma 4.4. *A bounded function $g : [t, T] \rightarrow \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[t, T]$.*

We now come to the convergence result for V_t^H and its optimal strategy π^* when ρ is deterministic.

Theorem 4.5. *Assume that ρ is deterministic, Riemann integrable and valued in $] -1, 1[$, and recall $\hat{H} = -\gamma H - \frac{1}{2} \int_t^T \lambda_s^2 ds$. Then for every sequence $(t_0^n, \dots, t_{\ell_n}^n)_{n \in \mathbb{N}}$ of partitions of $[t, T]$ with $\lim_{n \rightarrow \infty} (\max_{1 \leq j \leq \ell_n} (t_j^n - t_{j-1}^n)) = 0$ and every choice of $s^j \in [t_{j-1}^n, t_j^n]$ (the dependence of s^j on n is omitted for notational reasons),*

$$- E_{\hat{P}} \left[\dots E_{\hat{P}} \left[E_{\hat{P}} \left[e^{\hat{H}(1-\rho_s^2 \ell_n)} \left| \mathcal{Y}_{t_{\ell_n-1}^n} \right|^{\frac{1-\rho_s^2 \ell_n-1}{1-\rho_s^2 \ell_n}} \left| \mathcal{Y}_{t_{\ell_n-2}^n} \right|^{\frac{1-\rho_s^2 \ell_n-2}{1-\rho_s^2 \ell_n-1}} \dots \left| \mathcal{Y}_t \right|^{\frac{1}{1-\rho_s^2}} \right] \right] \right] \quad (4.12)$$

converges to V_t^H a.s. Suppose σ is uniformly bounded away from zero, and denote by (Γ^{q^n}, Z^{q^n}) the solution of (4.9) with $q = q^n := \sum_{j=1}^{\ell_n} \rho_{s^j} \mathbb{1}_{]t_{j-1}^n, t_j^n]}$ for any $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} E_P \left[\int_t^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma \sigma_s} - \pi_s^* \right|^2 ds \right] = 0$, where π^* is the optimiser for V_t^H . If $(\nu_n)_{n \in \mathbb{N}} = (t_0^n, \dots, t_{\ell_n}^n)_{n \in \mathbb{N}}$ is a sequence of partitions of $[t, T]$ with $\nu_n \subseteq \nu_{n+1}$, $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} (\max_{1 \leq j \leq \ell_n} (t_j^n - t_{j-1}^n)) = 0$, then

$$- E_{\hat{P}} \left[\dots E_{\hat{P}} \left[E_{\hat{P}} \left[e^{\hat{H}(1-\rho_{n,\ell_n}^2)} \left| \mathcal{Y}_{t_{\ell_n-1}^n} \right|^{\frac{1-\rho_{n,\ell_n}^2-1}{1-\rho_{n,\ell_n}^2}} \left| \mathcal{Y}_{t_{\ell_n-2}^n} \right|^{\frac{1-\rho_{n,\ell_n}^2-2}{1-\rho_{n,\ell_n}^2-1}} \dots \left| \mathcal{Y}_t \right|^{\frac{1}{1-\rho_{n,1}^2}} \right] \right] \right]$$

with $\rho_{n,j} := \inf_{s \in [t_{j-1}^n, t_j^n]} |\rho_s|$ (or $\rho_{n,j} := \sup_{s \in [t_{j-1}^n, t_j^n]} |\rho_s|$) is a nondecreasing (or nonincreasing) sequence which converges to V_t^H a.s.

Proof. Fix $n \in \mathbb{N}$ and let (Γ^{q^n}, Z^{q^n}) be the solution of the BSDE (4.9) with $q = q^n := \sum_{j=1}^{\ell_n} \rho_{sj} \mathbb{1}_{]t_{j-1}^n, t_j^n]}$. By 1) of Proposition 4.3, $-\exp(-\gamma \Gamma_t^{q^n})$ equals (4.12), and we show that this converges to V_t^H a.s. Because ρ is Riemann integrable, Lemma 4.4 yields $\lim_{n \rightarrow \infty} |q_s^n| = |\rho_s|$ for a.a. $s \in [t, T]$. From Lemma 4.1 and Theorem 2.1 follows that $-\exp(-\gamma \Gamma_t^{q^n})$ converges to V_t^H a.s. and also the convergence result for the optimiser π^* for V_t^H is implied.

The last part of Theorem 4.5 follows analogously, with q^n replaced by $\sum_{j=1}^{\ell_n} \rho_{n,j} \mathbb{1}_{]t_{j-1}^n, t_j^n]}$, using additionally parts 2) and 3) of Proposition 4.3. \square

Let us mention two straightforward generalisations of Theorem 4.5. The convergence still works if ρ itself is not Riemann integrable, but ρ equals Lebesgue-almost everywhere a Riemann integrable function $\tilde{\rho}$. One simply replaces ρ by $\tilde{\rho}$ in Theorem 4.5, and uses $V_t^H = -\exp(-\gamma \Gamma_t^\rho) = -\exp(-\gamma \Gamma_t^{\tilde{\rho}})$ a.s. for the solutions (Γ^ρ, Z^ρ) and $(\Gamma^{\tilde{\rho}}, Z^{\tilde{\rho}})$ of the BSDE (4.9) with $q = \rho$ and $q = \tilde{\rho}$, respectively. An example for such a pair of ρ and $\tilde{\rho}$ is $\rho = \frac{1}{2} \mathbb{1}_{\mathbb{Q} \cap [t, T]}$ and $\tilde{\rho} = 0$.

In the first part of Theorem 4.5, one can easily get rid of the restriction that ρ is valued in $] -1, 1[$. To this end, one replaces $|\rho_{sj}|$ by $|\rho_{sj}| \wedge (1 - 1/n)$ in (4.12), and uses for the proof that $\sum_{j=1}^{\ell_n} |\rho_{sj}| \wedge (1 - 1/n) \mathbb{1}_{]t_{j-1}^n, t_j^n]}$ converges pointwise to $|\rho|$ since the correlation ρ is valued in $[-1, 1]$. The same procedure works for the last part of Theorem 4.5, but the sequence of iterated expectations with $\rho_{n,j} := \sup_{s \in [t_{j-1}^n, t_j^n]} |\rho_s| \wedge (1 - 1/n)$ instead of $\rho_{n,j} := \sup_{s \in [t_{j-1}^n, t_j^n]} |\rho_s|$ is no longer nonincreasing.

Further comments on Theorem 4.5 are given in the next remark.

Remark 4.6. 1) One can show that the a.s. convergence of (4.12) to V_t^H holds uniformly with respect to the partitions. In more detail, we denote by $a_t(\Delta^n, \vec{s}^n)$ the random variable given by the iterated conditional expectation in (4.12), where the pair $(\Delta^n, \vec{s}^n) = ((t_0^n, \dots, t_{\ell_n}^n), (s^1, \dots, s^{\ell_n}))$ is called a tagged partition of $[t, T]$ with mesh $|\Delta^n|$. The first part of Theorem 4.5 yields $\lim_{n \rightarrow \infty} a_t(\Delta^n, \vec{s}^n) = V_t^H$ a.s. In the Appendix, we sketch the proof of the more general result

$$\lim_{\epsilon \searrow 0} \text{ess sup}_{(\Delta, \vec{s}) : |\Delta| < \epsilon} |a_t(\Delta, \vec{s}) - V_t^H| = 0 \quad \text{a.s.}, \quad (4.13)$$

where the essential supremum is taken over all tagged partitions (Δ, \vec{s}) of $[t, T]$ with mesh $|\Delta| < \epsilon$.

2) For q valued in $[-1, 1]$, the generator of (4.9) is concave in Z_r^q , and we can apply to $(-\Gamma^q, Z^q)$ the convergence result of Briand and Hu [3]. (Since the generators are uniformly bounded by a quadratic function, the minor amendment to that convergence result mentioned in Section 7 of Mocha and

Westray [12] is here not necessary.) Recall (Γ^{q^n}, Z^{q^n}) from the proof of Theorem 4.5, and let (Γ, Z) be the solution of (4.6). Proposition 7 of [3] implies that, for every $p \geq 1$,

$$\lim_{n \rightarrow \infty} E_P \left[\exp \left(\sup_{s \in [t, T]} |\Gamma_s^{q^n} - \Gamma_s| \right)^p + \left(\int_t^T |Z_s^{q^n} - Z_s|^2 ds \right)^{p/2} \right] = 0.$$

If σ is uniformly bounded away from zero, this yields similarly to Theorem 4.5 that, for every $p \geq 1$, $\lim_{n \rightarrow \infty} E_P \left[\left(\int_t^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma \sigma_s} - \pi_s^* \right|^2 ds \right)^{p/2} \right] = 0$. \diamond

Under a slightly more restrictive assumption on ρ , we can prove a stronger convergence result than Theorem 4.5.

Proposition 4.7. *Assume that ρ is deterministic and the one-sided limits $\lim_{r \nearrow s} \rho_r$ for all $s \in]t, T]$ and $\lim_{r \searrow s} \rho_r$ for all $s \in [t, T[$ exist. Then there exist partitions $(t_0^n, \dots, t_{\ell_n}^n)_{n \in \mathbb{N}}$ of $[t, T]$ with $\lim_{n \rightarrow \infty} (\max_{1 \leq j \leq \ell_n} (t_j^n - t_{j-1}^n)) = 0$ such that for every choice of $s^j \in]t_{j-1}^n, t_j^n[$, $(b^n)_{n \in \mathbb{N}}$ given for $r \in [t_{j-1}^n, t_j^n[$ by*

$$b_r^n := -E_{\hat{P}} \left[\dots E_{\hat{P}} \left[E_{\hat{P}} \left[e^{\hat{H}(1-\rho_s^{2\ell_n})} \middle| \mathcal{Y}_{t_{\ell_n-1}^n} \right]^{\frac{1-\rho_s^{2\ell_n-1}}{1-\rho_s^{2\ell_n}}} \middle| \mathcal{Y}_{t_{\ell_n-2}^n} \right]^{\frac{1-\rho_s^{2\ell_n-2}}{1-\rho_s^{2\ell_n-1}}} \dots \middle| \mathcal{Y}_r \right]^{\frac{1}{1-\rho_s^2}}$$

satisfies $\lim_{n \rightarrow \infty} \left\| \sup_{s \in [t, T]} |b_s^n - V_s^H| \right\|_{L^\infty(P)} = 0$. For σ uniformly bounded away from zero, we have $\lim_{n \rightarrow \infty} \sup_{\tau} \left\| E_P \left[\int_{\tau}^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma \sigma_s} - \pi_s^* \right|^2 ds \middle| \mathcal{G}_{\tau} \right] \right\|_{L^\infty(P)} = 0$, where the supremum is taken over all \mathbb{G} -stopping times τ valued in $[t, T]$, and (Γ^{q^n}, Z^{q^n}) is the solution of (4.9) with $q = q^n := \sum_{j=1}^{\ell_n} \rho_{s^j} \mathbb{1}_{]t_{j-1}^n, t_j^n]}$ for $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and define by

$$t_0^n := t, \quad t_j^n := \inf \left\{ s > t_{j-1}^n : \left| \rho_s - \lim_{r \searrow t_{j-1}^n} \rho_r \right| > 1/n \right\} \wedge T, \quad j \in \mathbb{N},$$

a partition of $[t, T]$, noting that there is $\ell_n \in \mathbb{N}$ such that $t_{\ell_n}^n = T$ by a compactness argument. For every $s^j \in]t_{j-1}^n, t_j^n[$, $q^n = \sum_{j=1}^{\ell_n} \rho_{s^j} \mathbb{1}_{]t_{j-1}^n, t_j^n]}$ converges to ρ in $L^\infty(\text{Leb}, [t, T])$ as $n \rightarrow \infty$. Except for the point mentioned next, the remainder of the proof goes like in Theorem 4.5, using Corollary 2.2 instead of Theorem 2.1 and additionally the idea of (3.8). Corollary 2.2 only yields $\lim_{n \rightarrow \infty} \sup_{\tau} \left\| E_P \left[\int_{\tau}^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma \sigma_s} - \pi_s^* \right|^2 ds \middle| \mathcal{Y}_{\tau} \right] \right\|_{L^\infty(P)} = 0$, where the supremum is taken over all \mathbb{Y} -stopping times τ . But the proof of Corollary 2.2 shows that we also have $\lim_{n \rightarrow \infty} \sup_{\tau} \left\| E_P \left[\int_{\tau}^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma \sigma_s} - \pi_s^* \right|^2 ds \middle| \mathcal{G}_{\tau} \right] \right\|_{L^\infty(P)} = 0$ in this situation, where the supremum is taken over all \mathbb{G} -stopping times τ . \square

4.2.2 A counterexample

We have seen in Theorem 4.5 that V_t^H is the a.s. limit of an explicitly known sequence if ρ equals almost everywhere a Riemann integrable function. In particular, the choice of a nondecreasing sequence of partitions in Theorem 4.5 allows us to approximate V_t^H from above and below. We give here an example of a correlation process which is not almost everywhere equal to a Riemann integrable function and where indeed the approximations of V_t^H in the sense of Theorem 4.5 from above and below are not possible.

We take for simplicity $t = 0$, $T = 1$, $\gamma = 1$, $\mu \equiv 0$ and $\sigma \equiv 1$. Let $C \subseteq [0, 1]$ be the “fat” Cantor set with Lebesgue measure $1/2$. This set, which is also known as Smith-Volterra-Cantor set, is constructed iteratively as follows: Start by removing $]3/8, 5/8[$ from the interval $[0, 1]$; in the n -th step, remove subintervals of width $1/2^{2n}$ from the middle of each of the 2^{n-1} intervals. If we continue like this, C consists of all points in $[0, 1]$ that are never removed. Because C is the complement of a countable union of open intervals, it is Borel measurable. Moreover, it is well known that C is nowhere dense, yet has Lebesgue measure $1/2$. We assume that the correlation ρ is given by $\rho = \frac{1}{2}\mathbf{1}_{C \cap [0, 1/2]} + \frac{1}{2}\mathbf{1}_{C^c \cap [1/2, 1]}$ and $H := Y_1$. Since Y_1 is not bounded, we have to adjust slightly the definition of admissible strategies: Instead of (4.4), we impose on $\pi \in \mathcal{A}_0$ that $(\exp(-X_s^{0, \pi} - Y_1))_{0 \leq s \leq T}$ is of class (D). (Alternatively, one could approximate Y_1 by bounded random variables like in the example in Section 5 of Frei and Schweizer [5].) We claim that

$$\sup_{q \in \Xi, |q| \leq \rho} \Gamma_0^q \leq -15/32 < -\log(-V_0^H) = -7/16 < -13/32 \leq \inf_{q \in \Xi, |q| \geq \rho} \Gamma_0^q,$$

where Γ^q is the solution of (4.9). This means that ρ cannot be approximated by piecewise constant processes from above and below such that the corresponding values converge to V_0^H . We first show $V_0^H = -\exp(7/16)$. For any $\pi \in \mathcal{A}_0$ with bounded $\int_0^T \pi_s^2 ds$, we have

$$\begin{aligned} E_P[U(X_1^{0, \pi} + H)] &= -E_P \left[\exp \left(- \int_0^1 \pi_s dW_s - Y_1 \right) \right] \\ &= -E_P \left[\exp \left(- \int_0^1 \pi_s dW_s - Y_1 - \frac{1}{2} \left\langle \int \pi dW + Y \right\rangle_1 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \int_0^1 ((\pi_s + \rho_s)^2 + 1 - \rho_s^2) ds \right) \right] \\ &\leq -E_P \left[\exp \left(- \int_0^1 \pi_s dW_s - Y_1 - \frac{1}{2} \left\langle \int \pi dW + Y \right\rangle_1 \right) \right] \\ &\quad \times \exp \left(\frac{1}{2} \int_0^1 (1 - \rho_s^2) ds \right) \end{aligned} \quad (4.14)$$

and hence

$$E_P[U(X_1^{0,\pi} + H)] \leq -\exp\left(\frac{1}{2} \int_0^1 (1 - \rho_s^2) ds\right) = -\exp(7/16) \quad (4.15)$$

since $\text{Leb}(C \cap [0, 1/2]) = \text{Leb}(C^c \cap [1/2, 1]) = 1/4$. Equality in (4.15) holds for $\pi = -\rho \in \mathcal{A}_0$. Because of the class (D) condition on $(\exp(-X_s^{0,\pi} - Y_1))_{0 \leq s \leq T}$ for any $\pi \in \mathcal{A}_0$, we obtain $V_0^H = -\exp(7/16)$ by a localisation argument. To prove $\sup_{q \in \Xi, |q| \leq \rho} \Gamma_0^q \geq -15/32$, we note that $q \in \Xi$, $|q| \leq \rho$ implies $q \equiv 0$ on $[0, 1/2]$ since C does not contain any nontrivial intervals. By 3) of Proposition 4.3 with ρ replaced by $\tilde{\rho} := \rho \mathbf{1}_{[1/2, 1]} = \frac{1}{2} \mathbf{1}_{C^c \cap [1/2, 1]}$, we have

$$\sup_{q \in \Xi, |q| \leq \rho} \Gamma_0^q \leq \Gamma_0^{\tilde{\rho}},$$

and a calculation similar to (4.15) shows $\Gamma_0^{\tilde{\rho}} = -15/32$, using that by Lemma 4.1, $-\exp(-\Gamma_0^{\tilde{\rho}})$ equals V_0^H with ρ replaced by $\tilde{\rho}$. Similarly, we obtain

$$\inf_{q \in \Xi, |q| \geq \rho} \Gamma_0^q \geq \Gamma_0^{\hat{\rho}} = -13/32,$$

where $\hat{\rho} := \rho \mathbf{1}_{[0, 1/2]} + \frac{1}{2} \mathbf{1}_{[1/2, 1]} = \frac{1}{2} \mathbf{1}_{C \cap [0, 1/2]} + \frac{1}{2} \mathbf{1}_{[1/2, 1]}$.

4.2.3 Stochastic correlation

When ρ is stochastic, we cannot approximate V_t^H from above and below like in Theorem 4.5. However, we still have a convergence result for V_t^H if ρ is left-continuous.

Theorem 4.8. *Assume that ρ is on $\llbracket t, T \rrbracket$ left-continuous and valued in $]-1, 1[$. Then for every sequence $(t = \tau_0^n \leq \dots \leq \tau_{\ell_n}^n = T)_{n \in \mathbb{N}}$ of $[t, T]$ -valued \mathbb{Y} -stopping times with $\lim_{n \rightarrow \infty} (\max_{1 \leq j \leq \ell_n} (\tau_j^n - \tau_{j-1}^n)) = 0$ a.s.,*

$$-E_{\hat{P}} \left[\dots E_{\hat{P}} \left[E_{\hat{P}} \left[e^{\hat{H}(1 - \rho_{\tau_{\ell_n-1}^n}^2)} \middle| \mathcal{Y}_{\tau_{\ell_n-1}^n} \right]^{\frac{1 - \rho_{\tau_{\ell_n-2}^n}^2}{1 - \rho_{\tau_{\ell_n-1}^n}^2}} \middle| \mathcal{Y}_{\tau_{\ell_n-2}^n} \right]^{\frac{1 - \rho_{\tau_{\ell_n-3}^n}^2}{1 - \rho_{\tau_{\ell_n-2}^n}^2}} \dots \middle| \mathcal{Y}_t \right]^{\frac{1}{1 - \rho_t^2}} \quad (4.16)$$

converges to V_t^H a.s. Suppose σ is uniformly bounded away from zero, and denote by (Γ^{q^n}, Z^{q^n}) the solution of (4.9) with $q = q^n := \sum_{j=1}^{\ell_n} \rho_{\tau_{j-1}^n} \mathbf{1}_{\llbracket \tau_{j-1}^n, \tau_j^n \rrbracket}$ for any $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} E_P \left[\int_t^T \left| \frac{\rho_s}{\sigma_s} Z_s^{q^n} + \frac{\lambda_s}{\gamma_{\sigma_s}} - \pi_s^* \right|^2 ds \right] = 0$, where π^* is the optimiser for V_t^H .

Proof. Fix $n \in \mathbb{N}$ and let (Γ^{q^n}, Z^{q^n}) be the solution of the BSDE (4.9) with $q = q^n := \sum_{j=1}^{\ell_n} \rho_{\tau_{j-1}^n} \mathbb{1}_{\llbracket \tau_{j-1}^n, \tau_j^n \rrbracket}$, which is \mathbb{Y} -predictable. By 1) of Proposition 4.3, $-\exp(-\gamma \Gamma_t^{q^n})$ equals (4.16). We have $\lim_{n \rightarrow \infty} q_s^n(\omega) = \rho_s(\omega)$ for a.a. $(\omega, s) \in \llbracket t, T \rrbracket$ by the left-continuity of ρ , and the result follows from Lemma 4.1 and Theorem 2.1. \square

In the same way as Theorem 4.5, one can slightly generalise Theorem 4.8 to the case where ρ equals $(P \otimes \text{Leb})$ -a.e. a $(P \otimes \text{Leb})$ -a.e. left-continuous process, and one can get rid of the assumption that ρ is valued in $] -1, 1[$.

Remark 4.9. The assumption from Section 4.1 that $\int_0^T \lambda_s^2 ds$ is bounded can be slightly weakened. Theorem 4.8 still holds if $\int \lambda dW \in BMO(\mathbb{G}, P)$ and

$$\sup_{s \in [0, T]} \left\| E_P \left[\exp \left(\int_s^T (1 + \rho_r^2) \lambda_r^2 dr \right) \middle| \mathcal{G}_s \right] \right\|_{L^\infty} < \infty. \quad (4.17)$$

By the John-Nirenberg inequality (see Theorem 2.2 of Kazamaki [7]), (4.17) is satisfied if, for example, the $BMO_2(\mathbb{G}, P)$ -norm of $\int \lambda dW$ is less than $1/\sqrt{2}$. In the Appendix, we sketch the proof of this slight generalisation of Theorem 4.8. \diamond

4.3 Continuity of the value process in the correlation

This short section exploits the convergence Theorem 2.1 to show a continuity property of V^H in ρ .

Let us introduce more precise notations by writing (4.1) as

$$dW_s(\tilde{\rho}) = \tilde{\rho}_s dY_s + \sqrt{1 - \tilde{\rho}_s^2} dY_s^\perp, \quad 0 \leq s \leq T$$

for a \mathbb{G} -predictable process $\tilde{\rho}$ denoting the instantaneous correlation between the (\mathbb{G}, P) -Brownian motions $W(\tilde{\rho})$ and Y so that $W = W(\rho)$. We replace in all definitions W by $W(\tilde{\rho})$ and write $\hat{W}(\tilde{\rho})$, $V^H(\tilde{\rho})$, etc. $V^H(\tilde{\rho})$ is then the dynamic value process for a stochastic control problem when the correlation between the underlying Brownian motions $W(\tilde{\rho})$ and Y is $\tilde{\rho}$. Note that if we change $\tilde{\rho}$, only $W(\tilde{\rho})$ and all expressions depending on it will change. This is reasonable; clearly H and \mathbb{Y} should not be affected.

Proposition 4.10. *Let $(\rho^n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{Y} -predictable $[-1, 1]$ -valued processes which converge pointwise to ρ on $\llbracket t, T \rrbracket$. Then $V_t^H(\rho^n)$ converges to $V_t^H = V_t^H(\rho)$ P -a.s. as $n \rightarrow \infty$. Moreover, $\sup_{s \in [t, T]} |V_s^H(\rho^n) - V_s^H| \rightarrow 0$ as $n \rightarrow \infty$ in P -probability and in $L^p(P)$, $1 \leq p < \infty$.*

Proof. This follows from Lemma 4.1 and Theorem 2.1, using additionally the same argument as in (3.8) to show the second statement. \square

Proposition 4.10 can be generalised to a multidimensional setting where W and Y are stochastically correlated multidimensional Brownian motions. But we give no details since this provides no essential new insights.

A Appendix: Proofs of the convergence results

Proof of Theorem 2.1. By Theorems 2.5 and 2.6 of Morlais [13], there exist unique solutions (Γ^n, Z^n, N^n) to (2.1) with parameters (f^n, β^n, H^n) for $n = 1, \dots, \infty$. Moreover, Lemma 3.1 of Morlais [13] implies that Γ^n and the $BMO(P)$ -norms of $\int Z^n dM$ and N^n are bounded uniformly in $n = 1, \dots, \infty$. (Theorems 2.5, 2.6 and Lemma 3.1 of [13] do not use the assumption in Section 2.1 of [13] that a.s., the matrix $m_s m'_s$ is invertible for every $s \in [0, T]$.)

We now subtract (2.1) with parameters $(f^\infty, \beta^\infty, H^\infty)$ from that with parameters (f^n, β^n, H^n) for a fixed $n \in \mathbb{N}$ to obtain, for $0 \leq s \leq T$,

$$\begin{aligned} \Gamma_s^n - \Gamma_s^\infty &= H^n - H^\infty + \int_s^T (f^n(r, Z_r^n) - f^\infty(r, Z_r^\infty)) dD_r - \int_s^T (Z_r^n - Z_r^\infty) dM_r \\ &\quad + \frac{\beta^n}{2} (\langle N^n \rangle_T - \langle N^n \rangle_s) - \frac{\beta^\infty}{2} (\langle N^\infty \rangle_T - \langle N^\infty \rangle_s) - \int_s^T d(N^n - N^\infty)_r, \end{aligned} \tag{A.1}$$

which implies

$$\begin{aligned} \Gamma_s^n - \Gamma_s^\infty &= \frac{\beta^n - \beta^\infty}{2} (\langle N^\infty \rangle_T - \langle N^\infty \rangle_s) - \int_s^T (Z_r^n - Z_r^\infty) (dM_r - d\langle M \rangle_r g_r^n) \\ &\quad - \int_s^T \left(d(N^n - N^\infty)_r - \frac{\beta^n}{2} d\langle N^n - N^\infty, N^n + N^\infty \rangle_r \right) \\ &\quad + H^n - H^\infty + \int_s^T (f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)) dD_r, \end{aligned} \tag{A.2}$$

where g^n is defined for $0 \leq s \leq T$ by

$$g_s^n := \begin{cases} \frac{f^n(s, Z_s^n) - f^n(s, Z_s^\infty)}{|m(Z_s^n - Z_s^\infty)|^2} (Z_s^n - Z_s^\infty) & \text{if } |m(Z_s^n - Z_s^\infty)| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Due to the assumption (ii) of the theorem, $\int g^n dM$ is in $BMO(P)$ and its $BMO(P)$ -norm is uniformly bounded since the $BMO(P)$ -norm of $\int Z^n dM$

is bounded uniformly in $n = 1, \dots, \infty$. Therefore, taking conditional expectations in (A.2) under the probability measure Q^n given by

$$\frac{dQ^n}{dP} := \mathcal{E} \left(\int g^n dM + \frac{\beta^n}{2}(N^n + N^\infty) \right)_T$$

yields

$$\begin{aligned} \Gamma_s^n - \Gamma_s^\infty &= \frac{\beta^n - \beta^\infty}{2} E_{Q^n} [\langle N^\infty \rangle_T - \langle N^\infty \rangle_s | \mathcal{F}_s] + E_{Q^n} [H^n - H^\infty | \mathcal{F}_s] \\ &\quad + E_{Q^n} \left[\int_s^T (f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)) dD_r \middle| \mathcal{F}_s \right]. \end{aligned} \quad (\text{A.3})$$

Because the convergent sequence $(\beta^n)_{n=1, \dots, \infty}$ is bounded, the $BMO(P)$ -norm of $\tilde{M}^n := \int g^n dM + \frac{\beta^n}{2}(N^n + N^\infty)$, which equals the stochastic logarithm of the P -density process of Q^n , is uniformly bounded in $n = 1, \dots, \infty$. Therefore, Theorem 3.6 of Kazamaki [7] and continuity in s of $E_{Q^n}[\langle N^\infty \rangle_T | \mathcal{F}_s]$ and $\langle N^\infty \rangle_s$ imply

$$\left\| \sup_{n=1, \dots, \infty} \sup_{s \in [0, T]} E_{Q^n} [\langle N^\infty \rangle_T - \langle N^\infty \rangle_s | \mathcal{F}_s] \right\|_{L^\infty(P)} < \infty, \quad (\text{A.4})$$

and hence

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \left| \frac{\beta^n - \beta^\infty}{2} E_{Q^n} [\langle N^\infty \rangle_T - \langle N^\infty \rangle_s | \mathcal{F}_s] \right| = 0 \quad \text{in } L^\infty(P). \quad (\text{A.5})$$

Since the $BMO(P)$ -norm of \tilde{M}^n is uniformly bounded in n , there exist by Theorem 3.1 of Kazamaki [7] $p > 1$ and a constant C_p , both independent of n , such that for all $n = 1, \dots, \infty$

$$E_{Q^n} \left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{1/(p-1)} \middle| \mathcal{F}_s \right] = E_P \left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{p/(p-1)} \middle| \mathcal{F}_s \right] \leq C_p. \quad (\text{A.6})$$

Recall the constant c^1 from the assumption (i) of the theorem and set

$$\alpha := \frac{1}{2pc^1 \|\int Z^\infty dM\|_{BMO_2(P)}^2 + 1}. \quad (\text{A.7})$$

Applying $\alpha x \leq e^{\alpha x} - 1$ for $x \in \mathbb{R}$, the Hölder inequality (with exponents p and $p/(p-1)$) and (A.6) yields

$$\begin{aligned} &E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \\ &\leq \frac{1}{\alpha} E_{Q^n} \left[\left(\frac{\mathcal{E}(\tilde{M}^n)_T}{\mathcal{E}(\tilde{M}^n)_s} \right)^{1/p} \left(\frac{\mathcal{E}(\tilde{M}^n)_s}{\mathcal{E}(\tilde{M}^n)_T} \right)^{1/p} \left(e^{\alpha \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right) \middle| \mathcal{F}_s \right] \\ &\leq \frac{1}{\alpha} |C_p|^{(p-1)/p} E_P \left[\left(e^{\alpha \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right)^p \middle| \mathcal{F}_s \right]^{1/p}. \end{aligned} \quad (\text{A.8})$$

By the assumption (i), we have

$$\begin{aligned}
& \left(e^{\alpha \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right)^p \\
& \leq \exp \left(p\alpha \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \right) \\
& \leq \exp \left(2p\alpha \left\| \int_0^T \kappa_r^1 dD_r \right\|_{L^\infty(P)} \right) \exp \left(2pc^1\alpha \int_0^T |m_r Z_r^\infty|^2 dD_r \right).
\end{aligned}$$

Using the definition (A.7) of α , the last expression is P -integrable by the John-Nirenberg inequality; see Theorem 2.2 of Kazamaki [7]. Therefore, dominated convergence and (A.8) imply that for $s \in [t, T]$,

$$\lim_{n \rightarrow \infty} E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] = 0 \quad P\text{-a.s.} \quad (\text{A.9})$$

Similarly to (A.8), we obtain

$$E_{Q^n} [|H^n - H^\infty| | \mathcal{F}_s] \leq |C_p|^{(p-1)/p} E_P \left[(e^{|H^n - H^\infty|} - 1)^p \middle| \mathcal{F}_s \right]^{1/p}, \quad (\text{A.10})$$

which again converges P -a.s. to zero by dominated convergence. Therefore, (A.3), (A.5) and (A.9) give $\lim_{n \rightarrow \infty} |\Gamma_s^n - \Gamma_s^\infty| = 0$ P -a.s. for every $s \in [t, T]$.

We now prove $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|$ converges in P -probability to zero. From (A.8) and the martingale maximum inequality we obtain that, for any $\epsilon \geq 0$,

$$\begin{aligned}
& \epsilon^p P \left[\sup_{s \in [t, T]} E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \geq \frac{1}{\alpha} |C_p|^{(p-1)/p} \epsilon \right] \\
& \leq \epsilon^p P \left[\sup_{s \in [t, T]} E_P \left[\left(e^{\alpha \int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right)^p \middle| \mathcal{F}_s \right]^{1/p} \geq \epsilon \right] \\
& \leq \epsilon^p P \left[\sup_{s \in [t, T]} E_P \left[\left(e^{\alpha \int_t^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right)^p \middle| \mathcal{F}_s \right] \geq \epsilon^p \right] \\
& \leq E_P \left[\left(e^{\alpha \int_t^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r} - 1 \right)^p \right], \quad (\text{A.11})
\end{aligned}$$

which converges to zero as $n \rightarrow \infty$ by dominated convergence. Analogously, we obtain from (A.10) that

$$\epsilon^p P \left[\sup_{s \in [t, T]} E_{Q^n} [|H^n - H^\infty| | \mathcal{F}_s] \geq |C_p|^{(p-1)/p} \epsilon \right] \leq E_P \left[\left(e^{|H^n - H^\infty|} - 1 \right)^p \right], \quad (\text{A.12})$$

which also converges to zero as $n \rightarrow \infty$ by dominated convergence. All in all, (A.3), (A.5), (A.11) and (A.12) show that $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|$ converges

in P -probability to zero, and also convergence in $L^p(P)$, $1 \leq p < \infty$, follows since Γ^n is bounded uniformly in n .

To prove the convergence statements for Z^n and N^n , we apply Itô's formula between t and T , and use (A.1) to obtain

$$\begin{aligned}
& (H^n - H^\infty)^2 - (\Gamma_t^n - \Gamma_t^\infty)^2 \\
&= \int_t^T 2(\Gamma_s^n - \Gamma_s^\infty)(Z_s^n - Z_s^\infty) dM_s + \int_t^T 2(\Gamma_s^n - \Gamma_s^\infty) d\langle N^n - N^\infty \rangle_s \\
&\quad - \int_t^T (\Gamma_s^n - \Gamma_s^\infty)(\beta^n d\langle N^n \rangle_s - \beta^\infty d\langle N^\infty \rangle_s) \\
&\quad - \int_t^T 2(\Gamma_s^n - \Gamma_s^\infty)(f^n(s, Z_s^n) - f^\infty(s, Z_s^\infty)) dD_s \\
&\quad + \int_t^T (|m_s(Z_s^n - Z_s^\infty)|^2 dD_s + d\langle N^n - N^\infty \rangle_s),
\end{aligned}$$

which implies by taking expectations that

$$\begin{aligned}
& E_P \left[\int_t^T (|m_s(Z_s^n - Z_s^\infty)|^2 dD_s + d\langle N^n - N^\infty \rangle_s) \right] \\
&\leq E_P [(H^n - H^\infty)^2] + E[(\Gamma_t^n - \Gamma_t^\infty)^2] \\
&\quad + E_P \left[\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \left(|\beta^n| (\langle N^n \rangle_T - \langle N^n \rangle_t) + |\beta^\infty| (\langle N^\infty \rangle_T - \langle N^\infty \rangle_t) \right) \right] \\
&\quad + 2E_P \left[\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \int_t^T |f^n(s, Z_s^n) - f^\infty(s, Z_s^\infty)| dD_s \right]. \quad (\text{A.13})
\end{aligned}$$

Because $(H^n)_{n=1, \dots, \infty}$ and $(\Gamma_t^n)_{n=1, \dots, \infty}$ are a.s.-convergent bounded sequences, we have $\lim_{n \rightarrow \infty} E_P[(H^n - H^\infty)^2] = 0$ and $\lim_{n \rightarrow \infty} E_P[(\Gamma_t^n - \Gamma_t^\infty)^2] = 0$ by dominated convergence. Hölder's inequality and the assumption (i) of the theorem imply

$$\begin{aligned}
& E_P \left[\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \int_t^T |f^n(s, Z_s^n) - f^\infty(s, Z_s^\infty)| dD_s \right]^2 \\
&\leq E_P \left[\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty|^2 \right] E_P \left[\left(\int_t^T (2\kappa_s^1 + c^1 |m_s Z_s^n|^2 + c^1 |m_s Z_s^\infty|^2) dD_s \right)^2 \right].
\end{aligned}$$

This converges to zero since $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ in $L^2(P)$ and the second

term is bounded uniformly in n , which can be seen as follows: We have

$$\begin{aligned} & E_P \left[\left(\int_t^T (2\kappa_s^1 + c^1 |m_s Z_s^n|^2 + c^1 |m_s Z_s^\infty|^2) dD_s \right)^2 \right] \\ & \leq E_P \left[3 \left(\int_t^T 2\kappa_s^1 dD_s \right)^2 + 3|c^1|^2 \left\langle \int Z^n dM \right\rangle_T^2 + 3|c^1|^2 \left\langle \int Z^\infty dM \right\rangle_T^2 \right], \end{aligned}$$

and this is bounded uniformly in n because $\| \int_0^T \kappa_s^1 dD_s \|_{L^\infty(P)} < \infty$, the $BMO(P)$ -norms of $\int Z^n dM$ are uniformly bounded and we can take $j = 2$ in the energy inequalities

$$E_P \left[\left\langle \int Z^n dM \right\rangle_T^j \right] \leq j! \left\| \int Z^n dM \right\|_{BMO_2(P)}^{2j} < \infty, \quad j \in \mathbb{N}; \quad (\text{A.14})$$

see the corollary to Theorem 4 of Kikuchi [8]. Analogously, the term

$$E_P \left[\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| (|\beta^n| (\langle N^n \rangle_T - \langle N^n \rangle_t) + |\beta^\infty| (\langle N^\infty \rangle_T - \langle N^\infty \rangle_t)) \right]$$

converges to zero. By (A.13), we obtain that

$$\lim_{n \rightarrow \infty} E_P \left[\int_t^T (|m_s(Z_s^n - Z_s^\infty)|^2 dD_s + d\langle N^n - N^\infty \rangle_s) \right] = 0,$$

which concludes the proof. \square

Remark A.1. To prove (A.9), one can also apply directly the energy inequalities instead of using the John-Nirenberg inequality. In fact, taking $\ell \in \mathbb{N}$ with $\ell \geq p$, we obtain from (A.6) and the Hölder inequality that

$$\begin{aligned} & E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \\ & \leq |C_p|^{(p-1)/p} E_P \left[\left(\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \right)^\ell \middle| \mathcal{F}_s \right]^{1/\ell}. \quad (\text{A.15}) \end{aligned}$$

By the assumption (i), we have

$$\begin{aligned} & \left(\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \right)^\ell \\ & \leq \left(2 \int_0^T \kappa_r^1 dD_r + 2c^1 \int_0^T |m_r Z_r^\infty|^2 dD_r \right)^\ell \\ & = 2^\ell \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\int_0^T \kappa_r^1 dD_r \right)^{\ell-j} |c^1|^j \left\langle \int Z^\infty dM \right\rangle_T^j, \end{aligned}$$

which is P -integrable because of (A.14) and $\left\| \int_0^T \kappa_r^1 dD_r \right\|_{L^\infty(P)} < \infty$. Dominated convergence and (A.15) now imply (A.9). \diamond

Proof of Corollary 2.2. To show (2.3), it is by (A.3) and (A.4) enough to prove the existence of a constant $K > 0$ such that

$$\begin{aligned} & \sup_{s \in [t, T]} E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \\ & \leq K \left(\|\underline{a}^n - 1\|_{L^\infty(P \otimes D)} + \|\bar{a}^n - 1\|_{L^\infty(P \otimes D)} \right), \quad n \in \mathbb{N}. \end{aligned}$$

But the assumption (v) implies

$$\begin{aligned} & \sup_{s \in [t, T]} E_{Q^n} \left[\int_s^T |f^n(r, Z_r^\infty) - f^\infty(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \\ & \leq \|\underline{a}^n - 1\|_{L^\infty(P \otimes D)} \sup_{s \in [t, T]} E_{Q^n} \left[\int_s^T |\underline{f}(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right] \\ & \quad + \|\bar{a}^n - 1\|_{L^\infty(P \otimes D)} \sup_{s \in [t, T]} E_{Q^n} \left[\int_s^T |\bar{f}(r, Z_r^\infty)| dD_r \middle| \mathcal{F}_s \right], \end{aligned}$$

and the conditional expectations are bounded in $L^\infty(P)$ uniformly in $n \in \mathbb{N}$ and $s \in [t, T]$ by an argument similar to (A.4). So (2.3) is established, and since its right-hand side converges to zero by the assumptions (iii)–(v), we have $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ in $L^\infty(P)$.

To show that $\int Z^n dM \rightarrow \int Z^\infty dM$, $N^n \rightarrow N^\infty$ on $[[t, T]]$ in $BMO(P)$, we derive similarly to (A.13) that, for any stopping time τ valued in $[[t, T]]$,

$$\begin{aligned} & E_P \left[\int_\tau^T (|m_s(Z_s^n - Z_s^\infty)|^2 dD_s + d\langle N^n - N^\infty \rangle_s) \middle| \mathcal{F}_\tau \right] \\ & \leq \|H^n - H^\infty\|_{L^\infty(P)}^2 + \|\Gamma_\tau^n - \Gamma_\tau^\infty\|_{L^\infty(P)}^2 \\ & \quad + \left\| \sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \right\|_{L^\infty(P)} \left(2E_P \left[\int_\tau^T |f^n(s, Z_s^n) - f^\infty(s, Z_s^\infty)| dD_s \middle| \mathcal{F}_\tau \right] \right. \\ & \quad \left. + E_P [|\beta^n|(\langle N^n \rangle_T - \langle N^n \rangle_\tau) + |\beta^\infty|(\langle N^\infty \rangle_T - \langle N^\infty \rangle_\tau) \middle| \mathcal{F}_\tau] \right). \end{aligned}$$

Since the $BMO(P)$ -norms of $\int Z^n dM$ as well as N^n are bounded uniformly in n and $\sup_{s \in [t, T]} |\Gamma_s^n - \Gamma_s^\infty| \rightarrow 0$ in $L^\infty(P)$, it follows that

$$\lim_{n \rightarrow \infty} \sup_\tau \left\| E_P \left[\int_\tau^T (|m_s(Z_s^n - Z_s^\infty)|^2 dD_s + d\langle N^n - N^\infty \rangle_s) \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty(P)} = 0,$$

and hence $\int Z^n dM \rightarrow \int Z^\infty dM$ and $N^n \rightarrow N^\infty$ on $[[t, T]]$ in $BMO(P)$ as $n \rightarrow \infty$. \square

Sketch of the proof of (4.13) in Remark 4.6.1. To check (4.13), one first deduces from Lemma 4.4 that $\lim_{\epsilon \searrow 0} \sup_{(\Delta, \vec{s}): |\Delta| < \epsilon} |q_r^{(\Delta, \vec{s})} - \rho_r| = 0$ for Leb-a.a. $r \in [t, T]$, where one sets $q^{(\Delta, \vec{s})} := \sum_{j=1}^{\ell} \rho_{s^j} \mathbb{1}_{]t_{j-1}, t_j]}$ for any tagged partition $(\Delta, \vec{s}) = ((t_0, \dots, t_\ell), (s^1, \dots, s^\ell))$. Then one slightly generalises Theorem 2.1 in the sense that this uniform convergence of $q^{(\Delta, \vec{s})}$ in (Δ, \vec{s}) implies that the corresponding solutions $\Gamma_t^{q^{(\Delta, \vec{s})}}$ of (4.9) converge a.s. to $\Gamma_t^\rho = -\frac{1}{\gamma} \log(-V_t^H)$ uniformly in (Δ, \vec{s}) . In fact, one needs only to generalise (A.9), which goes similarly to (A.8) by dominated convergence. Now one deduces (4.13) from

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \operatorname{ess\,sup}_{(\Delta, \vec{s}): |\Delta| < \epsilon} \left| -\frac{1}{\gamma} \log(-a_t(\Delta, \vec{s})) + \frac{1}{\gamma} \log(-V_t^H) \right| \\ &= \lim_{\epsilon \searrow 0} \operatorname{ess\,sup}_{(\Delta, \vec{s}): |\Delta| < \epsilon} \left| \Gamma_t^{q^{(\Delta, \vec{s})}} + \frac{1}{\gamma} \log(-V_t^H) \right| = 0 \quad \text{a.s.} \end{aligned}$$

similarly to the last part of the proof of Proposition 3.1, using that $-a_t(\Delta, \vec{s})$ is bounded away from zero by $e^{-\|\hat{H}\|_{L^\infty(P)}}$ uniformly in (Δ, \vec{s}) . \square

Sketch of the proof of Remark 4.9. From Proposition 3 of Briand and Hu [3] and Proposition 7 and Theorem 8 of Mania and Schweizer [11], one deduces that for a $[-1, 1]$ -valued \mathbb{Y} -predictable process q , the BSDE (4.9) still has a unique solution (Γ^q, Z^q) where Γ^q is a real-valued bounded continuous (\mathbb{Y}, P) -semimartingale and Z^q is a \mathbb{Y} -predictable process such that $E_P[\int_0^T |Z_s^q|^2 ds] < \infty$. Furthermore, $\int Z^q dY$ is in both $BMO(\mathbb{Y}, P)$ and $BMO(\mathbb{G}, P)$, and the BMO -norms are bounded uniformly with respect to the $[-1, 1]$ -valued q . Now one can proceed like in Lemma 4.1 and Proposition 4.3 to obtain $V^H = -\exp(-\gamma\Gamma^\rho)$ and (4.10). The argument is finished by applying Theorem 2.1, using that, under the assumption of uniform boundedness of the $BMO(\mathbb{F}, P)$ -norms of $\int Z^n dM$ and N^n , the convergence result can also be shown if in the assumptions (i) and (ii), one only has $\sup_\tau \|E_P[\int_\tau^T \kappa_s^1 dD_s | \mathcal{F}_\tau]\|_{L^\infty} < \infty$ and $\sup_\tau \|E_P[\int_\tau^T |\kappa_s^2|^2 dD_s | \mathcal{F}_\tau]\|_{L^\infty} < \infty$ instead of $\|\int_0^T \kappa_s^1 dD_s\|_{L^\infty} < \infty$ and $\|\int_0^T |\kappa_s^2|^2 dD_s\|_{L^\infty} < \infty$, where the suprema are taken over all \mathbb{F} -stopping times τ . \square

Acknowledgments

It is a pleasure to thank Martin Schweizer for many valuable comments and Mike Tehranchi for an interesting discussion on explicit formulas for the indifference value when the correlation is piecewise constant. Financial support by the Natural Sciences and Engineering Research Council of Canada and the National Centre of Competence in Research ‘‘Financial Valuation and Risk

Management” (NCCR FINRISK), Project D1 (Mathematical Methods in Financial Risk Management) is gratefully acknowledged. The NCCR FINRISK is a research instrument of the Swiss National Science Foundation.

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