E-companion to 'Systemic Influences on Optimal Equity-Credit Investment'

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A Model Construction and Properties

Recall that $\mathbf{S}(t) = \big(S_1(t), \dots, S_M(t)\big)$ is the vector of stock prices at t given by (2.1), in which $\mathbf{W}(t) = \big(W_1(t), \dots, W_d(t)\big)^\top$ is a standard Brownian motion. The default indicator process is denoted by $\mathbf{H}(t) = \big(H_1(t), \dots, H_M(t)\big)$. We assume that between two subsequent default times, each of the K components of the factor process $\mathbf{F}(t)$ follows an Itô diffusion process whose drift and volatility functions are uniformly Lipschitz continuous and whose driving Brownian motions can be correlated with each other and with $\mathbf{W}(t)$. For given trajectories of $\mathbf{S}(u) = \big(S_1(u), \dots, S_M(u)\big)$ and $\mathbf{F}(u) = \big(F_1(u), \dots, F_K(u)\big)$, $u \leq t$,

$$\begin{aligned}&\mathbf{H}(t) \text{ follows a time-inhomogeneous continuous time Markov chain on } \{0,1\}^M \text{ transitioning}\\ &\text{to a state } \mathbf{H}^{(i)}(t) \text{ with } H_j^{(i)}(t) = H_j(t), j \neq i, \text{ and } H_i^{(i)}(t) = 1 \text{ at rate } \mathbf{1}_{H_i(t)=0} h_i(t, \mathbf{S}(t), \mathbf{F}(t)).\end{aligned} \tag{A.1}$$

Since the stock prices are observable, the market information is given by the filtration generated by $\mathbf{S}(t)$. Under suitable assumptions on h_i, one can prove mathematically that our model exists. We assume that for every $n \in \mathbb{N}$, $h_i(t, s_1, \dots, s_M, f_1, \dots, f_K)$ are nonnegative, uniformly Lipschitz continuous and bounded (the bound can depend on n) for $t \geq 0$, $s_i \geq 1/n$, $s_j \geq 0$ with $j \neq i$ and $f_\ell \in \mathbb{R}$.

Lemma A.1. *For given K-dimensional factor process $\mathbf{F}(t)$, M-dimensional Brownian motion $\mathbf{W}(t)$ and a sequence of standard exponentially distributed random variables triggering the default events, there exists a unique $(\mathbf{S}(t), \mathbf{H}(t))$ satisfying (2.1) and (A.1).*

Proof. We first show that, for $i = 1, \dots, M$, the SDEs

$$\mathrm{d}\tilde{S}_i(t) = \tilde{S}_i(t)\big(\mu_i + h_i\big(t, \tilde{\mathbf{S}}(t), \mathbf{F}(t)\big)\big)\,\mathrm{d}t + \tilde{S}_i(t)\boldsymbol{\Sigma}_i\,\mathrm{d}\mathbf{W}(t), \quad \tilde{S}_i(0) > 0 \tag{A.2}$$

have a unique solution. Note that at this point, the vector of pre-default prices $\tilde{\mathbf{S}}(t)$ occurs as an argument in h_i because we have not yet constructed the default times, and thus we cannot define defaultable prices $\mathbf{S}(t)$ at this time. By considering $X_i = \log\big(\tilde{S}_i(t)\big)$, we see that (A.2) is equivalent to

$$\mathrm{d}X_i(t) = \big(\mu_i - \boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_i^\top/2 + h_i\big(t, \mathrm{e}^{X_1(t)}, \dots, \mathrm{e}^{X_M(t)}, F_1(t), \dots, F_K(t)\big)\big)\,\mathrm{d}t + \boldsymbol{\Sigma}_i\,\mathrm{d}\mathbf{W}(t).$$

By Theorem II.5.2 of Kunita (1984), this has a unique solution up to a possibly finite explosion time. To see that this solution is non-exploding, we show that X_i can be bounded from below and above by non-exploding processes. Because h_i is nonnegative, we have $\mathrm{d}X_i(t) \geq \big(\mu_i - \mathbf{\Sigma}_i\mathbf{\Sigma}_i^\top/2\big)\,\mathrm{d}t + \mathbf{\Sigma}_i\,\mathrm{d}\mathbf{W}(t)$ so that $X_i(t) \geq X_i(0) + (\mu_i - \mathbf{\Sigma}_i\mathbf{\Sigma}_i^\top/2)t + \mathbf{\Sigma}_i\mathbf{W}(t)$, which shows that $X_i(t)$ does not explode to $-\infty$. Due to the assumptions on h_i, there is a constant c_i with $\mathbf{1}_{x_i \geq X_i(0)} h_i(t, \mathrm{e}^{x_1}, \dots, \mathrm{e}^{x_M}, f_1, \dots, f_K) \leq c_i$, which implies $X_i(t) \leq |X_i(0)| + (|\mu_i| + c_i)t + \max\{0, \mathbf{\Sigma}_i\mathbf{W}(t)\}$ because the trajectories of the process on the right-hand side stay always above the trajectories of $X_i(t)$ as they have a bigger drift and never become negative. Hence, $X_i(t)$ does not explode to ∞, and $X_i(t)$ and $S_i(t) = \exp(X_i(t))$ are well defined for all t.

We construct the pair $(\mathbf{S}(t), \mathbf{H}(t))$ iteratively, using exponentially distributed random variables similarly to Section 4 of Lando (1998). Let $(E_{i,j})_{i,j=1,\dots,M}$ be independent standard exponentially distributed random variables, which are also independent of $\mathbf{F}(t)$ and the Brownian motion $\mathbf{W}(t)$. In the first step, we start with zero defaults, which means to consider (A.2) up to the stopping time τ_1, where

$$\tau_1 = \min_{j=1,\dots,M} \nu_{1,j}, \quad \nu_{1,j} = \inf\left\{t \geq 0 : \int_0^t h_j\big(u, \tilde{\mathbf{S}}(u), \mathbf{F}(u)\big)\,\mathrm{d}u \geq E_{1,j}\right\}. \tag{A.3}$$

From the first part of the proof, we get a unique, strictly positive solution $\tilde{\mathbf{S}}(t)$, and we set $\mathbf{S}(t) = \tilde{\mathbf{S}}(t)$ up to time τ_1. We then continue in the same manner, just by setting $\tilde{S}_{j_1^*}(t) = 0$ after time τ_1 where $j_1^* = \underset{j=1,\dots,M}{\operatorname{argmin}}\, \nu_{1,j}$. We can apply again the first part of the proof, which gives us the construction of $\mathbf{S}(t)$ from τ_1 up to some stopping time τ_2, defined analogously to (A.3) with

$$\tau_2 = \min_{j \in \{1,\dots,M\}\setminus\{j_1^*\}} \nu_{2,j}, \quad \nu_{2,j} = \inf\left\{t \geq \tau_1 : \int_{\tau_1}^t h_j\big(u, \tilde{\mathbf{S}}(u), \mathbf{F}(u)\big)\,\mathrm{d}u \geq E_{2,j}\right\}.$$

By iteratively continuing like this until all M names have defaulted, we conclude the proof. $\square$

We assume that the risk-neutral default intensities λ_i are such that

$$\begin{aligned}&\lambda_i(t, s_1, \dots, s_M) \text{ are nonnegative, uniformly Lipschitz continuous, bounded for } t \geq 0,\ s_i \geq 1/n,\\ &s_j \geq 0 \text{ with } j \neq i \text{ and the ratios } \frac{h_i(t, \mathbf{S}(t), \mathbf{F}(t))}{\lambda_i(t, \mathbf{S}(t))} \text{ are bounded away from zero and infinity.}\end{aligned} \tag{A.4}$$

To later study the dynamics of the wealth process, we first derive the dynamics of the CDS prices.

Lemma A.2. *The CDS prices are such that* $\mathrm{d}C_i(t) + \mathrm{d}D_i(t) - rC_i(t)\,\mathrm{d}t$ *equals to*

$$\sum_{n \in \mathbb{M}(t)} \big(\Phi_i(t, \boldsymbol{S}^{(n)}(t-)) - \Phi_i(t, \boldsymbol{S}(t-))\big)\big(\mathrm{d}H_n(t) - \lambda_n(t, \mathbf{S}(t))\,\mathrm{d}t\big) + L_i\big(\mathrm{d}H_i(t) - \lambda_i(t, \mathbf{S}(t))\big)$$

$$+\sum_{n\,\in\,\mathbb{M}(t)}\frac{\partial\Phi_i}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t)\big(\boldsymbol{\Sigma}_n\,\mathrm{d}\mathbf{W}(t)+\big(\mu_n-r+h_n(t,\mathbf{S}(t),\mathbf{F}(t))-\lambda_n(t,\mathbf{S}(t))\big)\,\mathrm{d}t\big).$$

Proof. We first note that under a risk-neutral probability measure $\mathbb{Q}$, we have

$$\mathrm{d}S_i(t)=S_i(t-)\big(r\,\mathrm{d}t+\boldsymbol{\Sigma}_i\,\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t)-\big(\mathrm{d}H_i(t)-\lambda_i(t,\mathbf{S}(t))\,\mathrm{d}t\big)\big) \tag{A.5}$$

because the default intensity is $\lambda_i(t,\mathbf{S}(t))$ and the drift needs to be r as $\mathbb{Q}$ is a risk-neutral probability measure, where $\mathbf{W}^{\mathbb{Q}}(t)$ is a Brownian motion under $\mathbb{Q}$. Comparing (A.5) with (2.1) implies

$$\boldsymbol{\Sigma}\,\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t)=\boldsymbol{\Sigma}\,\mathrm{d}\mathbf{W}(t)+\big(\mu_i-r+h_i(t,\mathbf{S}(t),\mathbf{F}(t))-\lambda_i(t,\mathbf{S}(t))\big)_{i=1,\ldots,M}\mathrm{d}t. \tag{A.6}$$

To apply Itô's formula to $\Phi_i(t,\boldsymbol{S}(t))$, we first show that $\Phi_i(t,s_1,\ldots,s_M)$ is continuously differentiable in $t\geq 0$ and twice continuously differentiable in $s_1>0,\ldots,s_M>0$. We use (2.2) and (2.3) to write

$$\Phi_i(t,\boldsymbol{S}(t))=L_i\Phi_i^{(1)}(t,\boldsymbol{S}(t))-L_i\mathbf{1}_{S_i(t)=0}\Phi_i^{(2)}(t,\boldsymbol{S}(t))-\nu_i\Phi_i^{(3)}(t,\boldsymbol{S}(t)).$$

For $\boldsymbol{s}=(s_1,\ldots,s_M)$, consider now the three partial differential equations

$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)f_i^{(1)}(t,\boldsymbol{s})=r\mathbf{1}_{s_i>0}f_i^{(1)}(t,\boldsymbol{s}),\qquad f_i^{(1)}(T_i,\boldsymbol{s})=\mathbf{1}_{s_i=0}, \tag{A.7}$$
$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)f_i^{(2)}(t,\boldsymbol{s})=r\mathbf{1}_{s_i>0}f_i^{(2)}(t,\boldsymbol{s}),\qquad f_i^{(2)}(T_i,\boldsymbol{s})=1, \tag{A.8}$$
$$\left(\frac{\partial}{\partial t}+\mathcal{A}\right)f_i^{(3)}(t,\boldsymbol{s})+\mathbf{1}_{s_i>0}=rf_i^{(3)}(t,\boldsymbol{s}),\quad f_i^{(3)}(T_i,\boldsymbol{s})=0, \tag{A.9}$$

where $\mathcal{A}f(t,\boldsymbol{s})$ equals

$$\sum_{j=1}^{M}\frac{\partial f}{\partial s_j}(t,\boldsymbol{s})s_j\left(r+h_j(t,\mathbf{s})\right)+\frac{1}{2}\sum_{i,j=1}^{M}\frac{\partial^2 f}{\partial s_i\partial s_j}(t,\boldsymbol{s})\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{\top}s_is_j+\sum_{j=1}^{M}\Big(f\big(t,\boldsymbol{s}^{(j)}\big)-f(t,\boldsymbol{s})\Big)\mathbf{1}_{s_j>0}h_j(t,\mathbf{s}),$$

and $\boldsymbol{s}^{(j)}\in\mathbb{R}^M$ is equal to $\boldsymbol{s}$ except for the j-th component which is zero. Thanks to (A.4), we can apply Proposition 2.3 of Becherer and Schweizer (2005) iteratively, first when all names have defaulted, then when all except one name have defaulted, and so on. This yields that the differential equations (A.7)–(A.9) have unique solutions being continuously differentiable in t and twice continuously differentiable in $s_1,\ldots,s_M$. An application of the Feynman-Kac formula yields that $f_i^{(j)}(t,\boldsymbol{s})=\Phi_i^{(j)}(t,\boldsymbol{s})$ for $j=1,2,3$ so that $\Phi_i(t,\boldsymbol{s}) = L_i\Phi_i^{(1)}(t,\boldsymbol{s}) - L_i\mathbf{1}_{s_i=0}\Phi_i^{(2)}(t,\boldsymbol{s}) - \nu_i\Phi_i^{(3)}(t,\boldsymbol{s})$ indeed satisfies the differentiability property.

On the one hand, we have $C_i(t) = \Phi_i(t,\boldsymbol{S}(t))$ and can apply Itô's formula to $\Phi_i(t,\boldsymbol{S}(t))$, which yields

$$\mathrm{d}C_i(t) = \sum_{n\,\in\,\mathbb{M}(t)} \frac{\partial\Phi_i}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t)\boldsymbol{\Sigma}_n\,\mathrm{d}\mathbf{W}(t) + \sum_{n\,\in\,\mathbb{M}(t)} \big(\Phi_i(t,\boldsymbol{S}^{(n)}(t-)) - \Phi_i(t,\boldsymbol{S}(t-))\big)\,\mathrm{d}H_n(t) + (\dots)\,\mathrm{d}t,$$

where we sum over $n\in\mathbb{M}(t)$ as only the nondefaulted stocks matter, and we suppress the dt-term because its precise form will be irrelevant. Combining this with (2.2) gives

$$\begin{aligned}\mathrm{d}C_i(t) + \mathrm{d}D_i(t) - rC_i(t)\,\mathrm{d}t = &\sum_{n\,\in\,\mathbb{M}(t)} \frac{\partial\Phi_i}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t)\boldsymbol{\Sigma}_n\,\mathrm{d}\mathbf{W}(t) + L_i\,\mathrm{d}H_i(t) \\ &+ \sum_{n\,\in\,\mathbb{M}(t)} \big(\Phi_i(t,\boldsymbol{S}^{(n)}(t-)) - \Phi_i(t,\boldsymbol{S}(t-))\big)\mathrm{d}H_n(t) + (\dots)\,\mathrm{d}t.\end{aligned} \tag{A.10}$$

On the other hand, from (2.3), $\mathrm{e}^{-rt}C_i(t) + \int_0^t \mathrm{e}^{-ur}\mathrm{d}D_i(u) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{T_i}\mathrm{e}^{-ur}\mathrm{d}D_i(u)\middle|\boldsymbol{S}(t)\right]$ is a martingale under $\mathbb{Q}$, thus $\mathrm{d}C_i(t) + \mathrm{d}D_i(t) - rC_i(t)\,\mathrm{d}t$ has zero drift under $\mathbb{Q}$ and we deduce from (A.10) that

$$\begin{aligned}\mathrm{d}C_i(t) + \mathrm{d}D_i(t) - rC_i(t)\,\mathrm{d}t = &\sum_{n\,\in\,\mathbb{M}(t)} \frac{\partial\Phi_i}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t)\boldsymbol{\Sigma}_n\,\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) + L_i\big(\mathrm{d}H_i(t) - \lambda_i(t,\mathbf{S}(t))\big) \\ &+ \sum_{n\,\in\,\mathbb{M}(t)} \big(\Phi_i(t,\boldsymbol{S}^{(n)}(t-)) - \Phi_i(t,\boldsymbol{S}(t-))\big)\big(\mathrm{d}H_n(t) - \lambda_n(t,\mathbf{S}(t))\big).\end{aligned}$$

This concludes the proof in view of (A.6). $\square$

We next give a useful interpretation of the matrix $\boldsymbol{\Theta}(t,\boldsymbol{S}(t))$ in terms of hedging errors, mentioned in Section 4.1. Let Ξ_n be the compounded payment stream of the n-th CDS. This is given by

$$\Xi_n = \int_0^T \mathrm{e}^{r(T-t)}\big(\mathrm{d}C_n(t) - rC_n(t)\,\mathrm{d}t + \mathrm{d}D_n(t)\big), \tag{A.11}$$

where $\mathrm{d}C_n(t)$ is the change in the price of the n-th CDS, $-rC_n(t)\,\mathrm{d}t$ is the interest cost for a unit investment in the n-th CDS, and $\mathrm{d}D_n(t)$ is the dividend income related to the n-th CDS.

Lemma A.3. *If an investor hedges the n-th CDS with a strategy in stocks to eliminate market risk, the residual risk (credit risk) between the terminal value $V^S(T)$ of the stock portfolio and Ξ_n equals*

$$\mathbb{E}^{\mathbb{Q}}\big[(V^S(T)-\Xi_n)^2\big] = \sum_{j=1}^{M}\int_0^T \mathrm{e}^{2r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Theta_{n,j}^2(t,\boldsymbol{S}(t))]\,\mathrm{d}t. \tag{A.12}$$

Proof. A self-financing strategy in stock yields wealth dynamics

$$\mathrm{d}V^S(t) = \Big(V^S(t) - \sum_{j\,\in\,\mathbb{M}(t)}\alpha_j(t)S_j(t)\Big)r\,\mathrm{d}t + \sum_{j\,\in\,\mathbb{M}(t)}\alpha_j(t)\,\mathrm{d}S_j(t),$$

where $\alpha_j(t)$ denotes the units invested in stock j. Then the hedging error under the risk-neutral probability measure can be written as

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\big[(V^S(T)-\Xi_n)^2\big]\\
&= \big(\mathrm{e}^{rT}V^S(0)-\mathbb{E}^{\mathbb{Q}}[\Xi_n]\big)^2 + \int_0^T \mathrm{e}^{2r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left\|\boldsymbol{\Sigma}^\top\Big(S_j(t)\alpha_j(t)-S_j(t)\frac{\partial\Phi_n}{\partial s_j}(t,\boldsymbol{S}(t))\Big)_{j\,\in\,\mathbb{M}(t)}\right\|\right]\mathrm{d}t\\
&\quad+\int_0^T \mathrm{e}^{2r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\sum_{j\neq n}\big(\alpha_j(t)S_j(t)+\Phi_n(t,\boldsymbol{S}^{(j)}(t))-C_n(t)\big)^2\right]\mathrm{d}t\\
&\quad+\int_0^T \mathrm{e}^{2r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[\big(\alpha_n(t)S_n(t)+L_n-C_n(t)\big)^2\big]\,\mathrm{d}t
\end{aligned}$$

by (A.11), (A.5), Lemma A.2 and the isometry property of stochastic integration. Recall that $\boldsymbol{\Sigma}$ denotes the reduced matrix corresponding to untriggered CDSs as described after Theorem 4.1. Choosing $V_0^S = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}[\Xi_n]$ and $\alpha_j(t) = \frac{\partial\Phi_n}{\partial s_j}(t,\boldsymbol{S}(t))$ minimizes diffusion risk, and the residual risk equals (A.12). □

B Proof of the Main Theorem 4.1

As in Theorem 4.1, we omit in the following the argument $(t,\mathbf{S}(t))$ in h_i, λ_i and $\boldsymbol{\Theta}$. Before giving the proof of Theorem 4.1, we state a mathematically precise definition of admissible strategies.

Definition B.1. *A strategy $(\boldsymbol{\pi},\boldsymbol{\psi})$ is* admissible *if the semimartingale decomposition of its relative wealth process is of the form $\int_0^s \frac{\mathrm{d}V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t)}{V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t-)} = \int_0^s \alpha(t)\,\mathrm{d}t + \sum_{j=1}^d\int_0^s\beta_j(t)\,\mathrm{d}W_j(t) + \sum_{j=1}^M\int_0^s\gamma_j(t)\big(\mathrm{d}H_j(t)-h_j\,\mathrm{d}t\big)$ for an adapted α with well defined integral $\int_0^T\alpha(t)\,\mathrm{d}t$, predictable β_j with $\mathbb{E}\big[\int_0^T|\beta_j(t)|^2\,\mathrm{d}t\big]<\infty$ for all $j=1,\dots,d$, and predictable γ_j which are bounded away from -1 and ∞ for all $j=1,\dots,M$.*

Remark B.2. *The technical condition* $\mathbb{E}\big[\int_0^T |\beta_j(t)|^2\,\mathrm{d}t\big] < \infty$ *ensures that unrealistic strategies are not admissible. We will see later in* (B.1) *that if there were no default risk, we would have* $\boldsymbol{\beta} = \boldsymbol{\Sigma}^\top \boldsymbol{\pi}$ *so that the condition* $\mathbb{E}\big[\int_0^T |\beta_j(t)|^2\,\mathrm{d}t\big] < \infty$ *for all* $j = 1, \dots, d$ *is equivalent to* $\mathbb{E}\big[\int_0^T \big|\boldsymbol{\Sigma}^\top \boldsymbol{\pi}(t)\big|^2\,\mathrm{d}t\big] < \infty$*, which indeed has been used in Definition 18 of Hu et al. (2005) in a different Brownian setting without defaults. The condition that* γ_j *are bounded away from* -1 *guarantees that the wealth process stays strictly positive.*

For an admissible strategy $(\boldsymbol{\pi}, \boldsymbol{\psi})$, the change in relative wealth is given by

$$\begin{aligned}\frac{\mathrm{d}V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t)}{V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t-)} &= \sum_{n\,\in\,\mathbb{M}(t)} \pi_n(t-)\frac{\mathrm{d}S_n(t)}{S_n(t-)} + \sum_{n\,\in\,\mathbb{M}(t)} \psi_n(t-)\,\mathrm{d}C_n(t) + \sum_{n\,\in\,\mathbb{M}(t)} \psi_n(t-)\,\mathrm{d}D_n(t)\\ &\quad + \Bigg(1 - \sum_{n\,\in\,\mathbb{M}(t)} \pi_n(t) - \sum_{n\,\in\,\mathbb{M}(t)} \psi_n(t) C_n(t)\Bigg) r\,\mathrm{d}t.\end{aligned}$$

Using the dynamics of stocks and CDS in (2.1) and Lemma A.2, we can rewrite this as

$$\begin{aligned}\frac{\mathrm{d}V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t)}{V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t-)} &= \Bigg[r + \sum_{n\,\in\,\mathbb{M}(t)} \Bigg(\sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\frac{\partial \Phi_j}{\partial s_n}(t, \boldsymbol{S}(t))S_n(t) + \pi_n(t)\Bigg)(\mu_n - r)\\ &\qquad + \sum_{j,n\,\in\,\mathbb{M}(t)} \psi_j(t)\Theta_{j,n}(h_n - \lambda_n)\Bigg]\,\mathrm{d}t\\ &\quad + \sum_{n\,\in\,\mathbb{M}(t)} \Bigg(\sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\frac{\partial \Phi_j}{\partial s_n}(t, \boldsymbol{S}(t))S_n(t) + \pi_n(t)\Bigg)\boldsymbol{\Sigma}_n\,\mathrm{d}\mathbf{W}(t)\\ &\quad + \sum_{n\,\in\,\mathbb{M}(t)} \Bigg(\sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t-)\big(\Phi_j\big(t, \boldsymbol{S}^{(n)}(t-)\big) - C_j(t-)\big) + \psi_n(t-)L_n - \pi_n(t-)\Bigg)\\ &\qquad\qquad \times \big(\mathrm{d}H_n(t) - h_n\,\mathrm{d}t\big).\end{aligned} \tag{B.1}$$

This is of the form $\frac{\mathrm{d}V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t)}{V^{\boldsymbol{\pi},\boldsymbol{\psi}}(t-)} = \mathrm{d}X(t)$, which implies

$$\log\big(V^{\boldsymbol{\pi},\boldsymbol{\psi}}(T)\big) = \log(V(0)) + X(T) - \frac{1}{2}[X]^c(T) + \sum_{0<t\leq T} \big(\log(1 + \Delta X(t)) - \Delta X(t)\big)$$

by Theorem II.37 of Protter (2006). Therefore, we can write

$$\log\big(V^{\boldsymbol{\pi},\boldsymbol{\psi}}(T)\big) = \log(V(0)) + N^c(T) + N^J(T) + \int_0^T \alpha(t)\,\mathrm{d}t,$$

where we define

$$
\begin{aligned}
N^c(t) &= \int_0^t \sum_{n \,\in\, \mathbb{M}(u)} \bigg(\sum_{j \,\in\, \mathbb{M}(u)} \psi_j(u) \frac{\partial \Phi_j}{\partial s_n}(u, \boldsymbol{S}(u)) S_n(u) + \pi_n(u) \bigg) \boldsymbol{\Sigma}_n \,\mathrm{d}\mathbf{W}(u), \\
N^J(t) &= \int_0^t \sum_{n \,\in\, \mathbb{M}(u)} \log \Bigg(1 + \sum_{j \,\in\, \mathbb{M}(t)} \psi_j(u-) \big(\Phi_j\big(u, \boldsymbol{S}^{(n)}(u-)\big) - C_j(u-) \big) \\
&\qquad\qquad + \psi_n(u-) L_n - \pi_n(u-) \Bigg) \big(\mathrm{d}H_n(u) - h_n(u, \mathbf{S}(u)) \,\mathrm{d}u \big), \\
\alpha(t) &= -\frac{1}{2} \bigg| \boldsymbol{\Sigma}^\top \bigg(\sum_{j \,\in\, \mathbb{M}(t)} \psi_j(t) \frac{\partial \Phi_j}{\partial s_n}(t, \boldsymbol{S}(t)) S_n(t) + \pi_n(t) \bigg)_{n \,\in\, \mathbb{M}(t)} \bigg|^2 \\
&\quad + r + \sum_{j,n \,\in\, \mathbb{M}(t)} \psi_j(t) \Theta_{j,n} (h_n - \lambda_n) \\
&\quad + \sum_{n \,\in\, \mathbb{M}(t)} \bigg(\sum_{j \,\in\, \mathbb{M}(t)} \psi_j(t) \frac{\partial \Phi_j}{\partial s_n}(t, \boldsymbol{S}(t)) S_n(t) + \pi_n(t) \bigg) (\mu_n - r) \\
&\quad - \sum_{n \,\in\, \mathbb{M}(t)} \bigg(\sum_{j \,\in\, \mathbb{M}(t)} \psi_j(t) \big(\Phi_j\big(t, \boldsymbol{S}^{(n)}(t)\big) - C_j(t) \big) + \psi_n(t) L_n - \pi_n(t) \bigg) h_n \\
&\quad + \sum_{n \,\in\, \mathbb{M}(t)} h_n \log \bigg(1 + \sum_{j \,\in\, \mathbb{M}(t)} \psi_j(t) \Big(\Phi_j\big(t, \boldsymbol{S}^{(n)}(t)\big) - C_j(t) \Big) + \psi_n(t) L_n - \pi_n(t) \bigg),
\end{aligned}
$$

where $\boldsymbol{\Sigma}$ denotes the reduced matrix consisting of rows corresponding to untriggered CDSs as described after Theorem 4.1. From (B.1) and the Definition B.1 of admissible strategies, it follows that

$$
\mathbb{E}\bigg[\int_0^T \bigg| \boldsymbol{\Sigma}^\top \bigg(\sum_{j \,\in\, \mathbb{M}(t)} \psi_j(t) \frac{\partial \Phi_j}{\partial s_n}(t, \boldsymbol{S}(t)) S_n(t) + \pi_n(t) \bigg)_{n \,\in\, \mathbb{M}(t)} \bigg|^2 \mathrm{d}t \bigg] < \infty,
$$

and the jumps of $\sum_{j=1}^M \psi_j(u-)\big(\Phi_j\big(u, \boldsymbol{S}^{(n)}(u-)\big) - C_j(u-)\big) + \psi_n(u-) L_n - \pi_n(u-)$ are bounded away from -1 and ∞. This implies that $N^c(t)$ and $N^J(t)$ are martingales with zero expectations, hence

$$
\mathbb{E}\big[\log \big(V^{\boldsymbol{\pi}, \boldsymbol{\psi}}(T) \big) \big] = \log \big(V(0) \big) + \int_0^T \mathbb{E}[\alpha(t)] \,\mathrm{d}t, \tag{B.2}
$$

and the maximizer is found by maximizing $\alpha(t)$. To find the optimal strategy, we could use the first-order conditions, but this leads here to a tedious calculation. It is easier to analyze $\alpha(t)$ for fixed t by writing it as $\alpha(t) = r + f(\boldsymbol{x}) + \sum_{n \,\in\, \mathbb{M}(t)} h_n g_n(y_n)$, where we define

$$
f(\boldsymbol{x}) = -\frac{1}{2} \big| \boldsymbol{\Sigma}^\top \big(\boldsymbol{x} - \big(\boldsymbol{\Sigma} \boldsymbol{\Sigma}^\top \big)^{-1} (\mu_n - r + h_n - \lambda_n)_{n \,\in\, \mathbb{M}(t)} \big) \big|^2
$$

$$
\begin{aligned}
&+\frac{1}{2}\left|\boldsymbol{\Sigma}^\top\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\,\in\,\mathbb{M}(t)}\right|^2,\\
g_n(y_n) &= \log(1+y_n) - \frac{\lambda_n}{h_n} y_n,\\
\boldsymbol{x} &= \left(\sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\frac{\partial \Phi_j}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t) + \pi_n(t)\right)_{n\,\in\,\mathbb{M}(t)},\\
y_n &= \sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\Big(\Phi_j\big(t,\boldsymbol{S}^{(n)}(t)\big) - C_j(t)\Big) + \psi_n(t)L_n - \pi_n(t).
\end{aligned}
$$

The functions f and g_n are maximized by $\boldsymbol{x} = \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}\left(\mu_n - r + h_n - \lambda_n\right)_{n\,\in\,\mathbb{M}(t)}$ and $y_n = \frac{h_n - \lambda_n}{\lambda_n}$, respectively. Therefore, if the system

$$
\begin{aligned}
\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}\left(\mu_n - r + h_n - \lambda_n\right)_{n\,\in\,\mathbb{M}(t)} &= \left(\sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\frac{\partial \Phi_j}{\partial s_n}(t,\boldsymbol{S}(t))S_n(t) + \pi_n(t)\right)_{n\,\in\,\mathbb{M}(t)},\\
\frac{h_n - \lambda_n}{\lambda_n} &= \sum_{j\,\in\,\mathbb{M}(t)} \psi_j(t)\Big(\Phi_j\big(t,\boldsymbol{S}^{(n)}(t)\big) - C_j(t)\Big) + \psi_n(t)L_n - \pi_n(t)
\end{aligned}
\tag{B.3}
$$

has a solution $\psi_n(t)$ and $\pi_n(t)$ for $n \in \mathbb{M}(t)$, then this is the maximizer of $\alpha(t)$. Thanks to Assumption A, this system indeed has a solution, which is explicitly given by (4.2) and (4.3). To show that this strategy satisfies the admissibility condition of Definition B.1, we use (B.3) to write (B.1) as

$$
\frac{\mathrm{d}V_t(\boldsymbol{\phi})}{V_{t-}(\boldsymbol{\phi})} = (\dots)\,\mathrm{d}t + \left(\mu_n - r + h_n - \lambda_n\right)^\top_{n\,\in\,\mathbb{M}(t)}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}\boldsymbol{\Sigma}\,\mathrm{d}\mathbf{W}(t) + \sum_{n\,\in\,\mathbb{M}(t)} \frac{h_n - \lambda_n}{\lambda_n}(\mathrm{d}H_n(t) - h_n\,\mathrm{d}t), \tag{B.4}
$$

from which it can be seen that the conditions of Definition B.1 are satisfied by using (A.4).

It remains to show (4.4). To this end, we note that for the optimal strategy, we have

$$
\begin{aligned}
\alpha(t) &= r + f(\boldsymbol{x}) + \sum_{n\,\in\,\mathbb{M}(t)} h_n g_n(y_n)\\
&= r + \frac{1}{2}\left|\boldsymbol{\Sigma}^\top\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top\right)^{-1}(\mu_n - r + h_n - \lambda_n)_{n\,\in\,\mathbb{M}(t)}\right|^2 + \sum_{n\,\in\,\mathbb{M}(t)}\left(h_n \log\left(\frac{h_n}{\lambda_n}\right) - h_n + \lambda_n\right).
\end{aligned}
$$

Using (B.2), we then deduce (4.4) and conclude the proof of Theorem 4.1. □

Proof of Corollary 4.2. This follows from (B.1) and (B.4), using that the sensitivity in H_n is the integrand of the $\mathrm{d}H_n$-integral and that $\boldsymbol{S}(t)$ and $C_j(t)$ are equal to $\boldsymbol{S}(t-)$ and $C_j(t-)$ almost everywhere. □