

ASSⁿ 2

2) REGULAR PERTⁿ

$$\epsilon \rightarrow 0 \Rightarrow X^2 + 2X = 0 \quad \text{ROOTS } X_1^{(0)} = 0, X_2^{(0)} = -2$$

SUPPOSE $X_1 = 0 + \epsilon X_1^{(1)} + \epsilon^2 X_1^{(2)} + \dots$
 $\Rightarrow \epsilon^2 (\epsilon X_1^{(1)} + \dots)^3 + (\epsilon X_1^{(1)} + \dots)^2 + 2(\epsilon X_1^{(1)} + \dots) + \epsilon = 0$

$$O(\epsilon): 2X_1^{(1)} + 1 = 0 \Rightarrow X_1^{(1)} = -\frac{1}{2}$$

$$O(\epsilon^2): (X_1^{(1)})^2 + 2X_1^{(2)} = 0 \Rightarrow X_1^{(2)} = -\frac{1}{2} \left(-\frac{1}{2}\right)^2 = -\frac{1}{8}$$

$$\text{SO } X_1 \approx -\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2$$

SUPPOSE $X_2 = -2 + \epsilon X_2^{(1)} + \dots$

$$O(\epsilon): -4X_2^{(1)} + 2(X_2^{(0)}) + 1 = 0 \Rightarrow X_2^{(1)} = \frac{1}{2}$$

$$\text{SO } X_2 \approx -2 + \frac{1}{2}\epsilon$$

SINGULAR PERTⁿ

$$\text{LET } X_3 = \frac{X}{\delta} \Rightarrow \frac{\epsilon^2}{\delta^3} X^3 + \frac{1}{\delta^2} X^2 + \frac{2}{\delta} X + \epsilon$$

$$\textcircled{1} + \textcircled{4} \Rightarrow \frac{\epsilon^2}{\delta^3} \sim \epsilon \Rightarrow \delta \sim \epsilon^{1/3} \quad \text{BUT } O(\textcircled{2}) < O(\textcircled{4}) \quad \times$$

$$\textcircled{1} + \textcircled{3} \Rightarrow \frac{\epsilon^2}{\delta^3} \sim \frac{1}{\delta} \Rightarrow \delta \sim \epsilon \quad \text{BUT } O(\textcircled{2}) < O(\textcircled{1}) \quad \times$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \frac{\epsilon^2}{\delta^3} \sim \frac{1}{\delta^2} \Rightarrow \delta \sim \epsilon^2 \quad \text{WORKS!}$$

$$\text{SO LET } X = \frac{X}{\epsilon^2} \Rightarrow X^3 + X^2 + 2\epsilon^2 X + \epsilon^5 = 0$$

SUPPOSE $X = X^{(0)} + \epsilon X^{(1)} + \dots$

$$O(\epsilon^0): X^{(0)} = 0, 0, -1$$

$$O(\epsilon^1): -3X^{(1)} - 2X^{(1)} = 0 \Rightarrow X^{(1)} = 0$$

$$O(\epsilon^2): +3X^{(2)} - 2X^{(2)} + 2(-1) = 0 \Rightarrow X^{(2)} = +2$$

$$\text{SO } X = -1 + 2\epsilon^2$$

$$\text{SO } X_3 \approx -\frac{1}{\epsilon^2} + 2$$

ASSⁿ 2)

3) FOR SMALL x , $\sin\left(\frac{\pi}{4} + x\right) = \sin\frac{\pi}{4} + \left(\cos\frac{\pi}{4}\right)x - \frac{1}{2}\left(\sin\frac{\pi}{4}\right)x^2 - \frac{1}{6}\cos\left(\frac{\pi}{4}\right)x^3 + \dots$
 $= \frac{1}{\sqrt{2}}\left(1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots\right)$

SO APPROXIMATE EQⁿ IS

$$-\frac{1}{6}x^3 + \dots = -\frac{1}{6}\epsilon \quad \epsilon \rightarrow 0 \Rightarrow x^3 \approx 0 \quad \text{A TRIPLE ZERO ROOT}$$

SO GUESS $X = \epsilon^{1/3}x_1 + \epsilon^{2/3}x_2 + \dots$

$$\Rightarrow \sqrt{2} \sin\left[\frac{\pi}{4} + \epsilon^{1/3}(x_1 + \epsilon^{1/3}x_2 + \dots)\right] - 1 - \epsilon^{1/3}(x_1 + \dots) + \frac{1}{2}\epsilon^{2/3}(x_1 + \dots)^2 = -\frac{1}{6}\epsilon$$

$$\Rightarrow \sqrt{2}\left(\sin\frac{\pi}{4} + \left(\cos\frac{\pi}{4}\right)\epsilon^{1/3}(x_1 + \dots) - \left(\sin\frac{\pi}{4}\right)\frac{1}{2}\epsilon^{2/3}(x_1 + \dots)^2 - \left(\cos\frac{\pi}{4}\right)\frac{1}{6}\epsilon(x_1 + \dots)^3 + \dots\right) - 1 - \epsilon^{1/3}(x_1 + \dots) + \frac{1}{2}\epsilon^{2/3}(x_1 + \dots)^2 = -\frac{1}{6}\epsilon$$

$$\Rightarrow -\frac{1}{6}\epsilon(x_1 + \dots)^3 + \frac{1}{24}\epsilon^{4/3}(x_1 + \dots)^4 + \dots = -\frac{1}{6}\epsilon$$

$$\Rightarrow (x_1 + \epsilon^{1/3}x_2 + \dots)^3 - \frac{1}{4}\epsilon^{1/3}(x_1 + \dots)^4 + \dots = 1$$

$$O(\epsilon^0) \Rightarrow x_1^3 = 1 \Rightarrow x_1 = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$O(\epsilon^{1/3}) \Rightarrow 3x_1^2x_2 - \frac{1}{4}x_1^4 = 0 \Rightarrow x_2 = +\frac{1}{12}x_1^2$$

$$= +\frac{1}{12}, +\frac{1}{12}\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

SO $x_1 \approx \epsilon^{1/3} + \frac{1}{12}\epsilon^{2/3}$

AND $\approx \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\epsilon^{1/3} + \left(\frac{-1}{24} - i\frac{\sqrt{3}}{24}\right)\epsilon^{2/3}$

AND $\approx \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\epsilon^{1/3} + \left(\frac{-1}{24} + i\frac{\sqrt{3}}{24}\right)\epsilon^{2/3}$

Ass 2

4) $\underline{A}\underline{x} = \mu\underline{x} + \varepsilon\underline{F}$
 with $\underline{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\mu = \underbrace{2}_{\mu_0} + \varepsilon\mu_1 + \varepsilon^2\mu_2$, $\underline{F} = \begin{pmatrix} a^2+b \\ a+b \end{pmatrix}$, $\underline{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$O(\varepsilon^0)$: $\underline{A}\underline{x}_0 = \mu_0\underline{x}_0 \Rightarrow \underline{x}_0 = \begin{pmatrix} 0 \\ B \end{pmatrix}$ B-FIXED
 LEFT EIGENVECTOR OF $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ IS $\underline{x}_0^+ = (0, 1)$

SUPPOSE $\underline{x} = \begin{pmatrix} 0 \\ B \end{pmatrix} + \varepsilon\underline{x}_1 + \dots$,

$O(\varepsilon^1)$: $(\underline{A} - \mu_0)\underline{x}_1 = \mu_1\underline{x}_0 + \underline{F}(\underline{x}_0)$
 $\Rightarrow 0 = \underbrace{(0, 1)}_{\underline{x}_0^+} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \underline{x}_1 = \mu_1 \underline{x}_0^+ \underline{x}_0 + \underline{x}_0^+ \underline{F}(\underline{x}_0)$
 $= \mu_1 (B) + (0, 1) \begin{pmatrix} B \\ B \end{pmatrix}$
 $= \mu_1 B + B$

SO $\mu_1 = -1$

$\Rightarrow (\underline{A} - \mu_0)\underline{x}_1 = -1 \begin{pmatrix} 0 \\ B \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$

$\Rightarrow \underline{x}_1 = \begin{pmatrix} -B \\ 0 \end{pmatrix}$

$O(\varepsilon^2)$: $(\underline{A} - \mu_0)\underline{x}_2 = \mu_2\underline{x}_0 + \mu_1\underline{x}_1 + \underbrace{\underline{F}(\underline{x}_0)}_{\varepsilon\underline{F} \text{ AT } O(\varepsilon^1)}$
 $\Rightarrow 0 = \mu_2 \underline{x}_0^+ \underline{x}_0 + \mu_1 \underline{x}_0^+ \underline{x}_1 + \underline{x}_0^+ \begin{pmatrix} 0 \\ -B \end{pmatrix}$
 $= \mu_2 B - B$

$\Rightarrow \mu_2 = 1$

$\Rightarrow (\underline{A} - \mu_0)\underline{x}_2 = (1) \begin{pmatrix} 0 \\ B \end{pmatrix} + (-1) \begin{pmatrix} -B \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -B \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$

$\Rightarrow \underline{x}_2 = \begin{pmatrix} -B \\ 0 \end{pmatrix}$

SO $\begin{pmatrix} a \\ b \end{pmatrix} \approx \underline{x} \approx \begin{pmatrix} 0 \\ B \end{pmatrix} + \varepsilon \begin{pmatrix} -B \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -B \\ 0 \end{pmatrix}$
 $\mu \approx 2 - \varepsilon + \varepsilon^2$

ASS² 2

$$5) a) \tilde{p} = \frac{p}{p_1} = 1 + \varepsilon r, \quad \tilde{h} = \frac{h}{h_0}, \quad \tilde{\eta} = \frac{\eta}{h_0}, \quad \tilde{t} = t / \left(\frac{h_0 A}{\alpha} \right)$$

$$(1) \Rightarrow \tilde{\eta} = (1 + \varepsilon r) \tilde{h} \quad (4)$$

$$(2) \Rightarrow \frac{d}{d\tilde{t}} (\tilde{\eta} + \tilde{h}) = -1 \quad (5)$$

$$(3) \Rightarrow \frac{d}{d\tilde{t}} (\tilde{\eta} + (1 + \varepsilon r) \tilde{h}) = -(1 + \varepsilon r) \quad (6)$$

$$b) (4) + (5) \Rightarrow \frac{d}{d\tilde{t}} ((2 + \varepsilon r) \tilde{h}) = -1$$

$$\Rightarrow (2 + \varepsilon r) \tilde{h} = -\tilde{t} + \mathcal{C}$$

$$h(0) = h_0 \Rightarrow \tilde{h}(0) = 1; \quad p(0) = p_0 = p_1(1 + \varepsilon) \Rightarrow r(0) = 1$$

$$\text{so } \mathcal{C} = 2 + \varepsilon$$

$$\Rightarrow \tilde{h} = (2 + \varepsilon - \tilde{t}) / (2 + \varepsilon r) \quad (7)$$

$$(4), (7) + (6) \Rightarrow \frac{d}{d\tilde{t}} \left[2(1 + \varepsilon r) \frac{(2 + \varepsilon - \tilde{t})}{2 + \varepsilon r} \right] = -(1 + \varepsilon r) \quad (8)$$

$$r(0) = 1$$

$$c) \text{ USE } \frac{1}{2 + \varepsilon r} = \frac{1}{2} (1 - \frac{1}{2} \varepsilon r + \dots)$$

$$(8) \Rightarrow \frac{d}{d\tilde{t}} \left[(2 + \varepsilon - \tilde{t}) (1 + \varepsilon r) (1 - \frac{1}{2} \varepsilon r + \dots) \right] = -(1 + \varepsilon r)$$

$$O(\varepsilon^0): \frac{d}{d\tilde{t}} [2 - \tilde{t}] = -1 \quad \text{CONSISTENT. WHEW!}$$

$$\text{WRITE } r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \dots, \text{ so } r(0) = 1 \Rightarrow r_0(0) = 1, r_1(0) = 0, \text{ etc}$$

$$O(\varepsilon^1): \frac{d}{d\tilde{t}} \left[1 + (2 - \tilde{t}) \frac{1}{2} r_0 \right] = -r_0$$

$$\Rightarrow (1 - \frac{1}{2} \tilde{t}) \frac{dr_0}{d\tilde{t}} - \frac{1}{2} r_0 = -r_0$$

$$\Rightarrow \frac{dr_0}{d\tilde{t}} = -\frac{1}{2} r_0 / (1 - \frac{1}{2} \tilde{t})$$

$$\Rightarrow r_0 = \mathcal{C} [1 - \frac{1}{2} \tilde{t}]$$

$$r_0(0) = 1 \Rightarrow \mathcal{C} = 1$$

$$\text{So } r_0(\tilde{t}) = 1 - \frac{1}{2} \tilde{t}$$

NON-ZERO, SO THIS IS LEADING ORDER

$$\Rightarrow p(t) \approx p_1 (1 + \varepsilon r_0(t / \frac{h_0 A}{\alpha}))$$

$$= p_1 \left[1 + \frac{p_0 - p_1}{p_1} \left(1 - \frac{1}{2} \frac{\alpha}{h_0 A} t \right) \right]$$

$$= p_0 - \frac{1}{2} (p_0 - p_1) \frac{\alpha}{h_0 A} t$$