Ass$^{\underline{n}}$ 2

1) $$\frac{\partial u}{\partial t} = \varkappa \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) = \varkappa\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho}\right)$$ SINCE SPHERICALLY SYMM

DIM$^{\underline{n}}$ ANALYSIS $\Rightarrow$ $u = u_0 f\Big(\overbrace{\frac{\rho}{\sqrt{\varkappa t}}}^{\xi}\Big)$

SO PDE $\Rightarrow$ $\frac{\rho}{\sqrt{\varkappa}}\left(-\frac{1}{2}t^{-3/2}\right)u_0 f' = \varkappa\left(\frac{u_0}{\varkappa t} f'' + \frac{2}{\rho}\frac{u_0}{\sqrt{\varkappa t}} f'\right)$

$$\Rightarrow \quad f' = -2\underbrace{\frac{\sqrt{\varkappa}}{\rho} t^{1/2}}_{1/\xi} f'' - 4\underbrace{\frac{1}{\rho^2}(\varkappa t)}_{1/\xi^2} f'$$

$$\Rightarrow \quad +\frac{2}{\xi} f'' + \left(1 + \frac{4}{\xi^2}\right) f' = 0$$

$$\Rightarrow \quad f'' + \left(\frac{\xi}{2} + \frac{2}{\xi}\right) f' = 0$$

LET $g \equiv f'$ $\Rightarrow$ $g' + \left(\frac{\xi}{2} + \frac{2}{\xi}\right) g = 0$

$$\Rightarrow \quad \ln g = -\frac{1}{4}\xi^2 - 2\ln\xi + \mathcal{C}$$

$$\Rightarrow \quad g = \mathcal{C}\frac{1}{\xi^2} e^{-\xi^2/4}$$

$$\text{SO} \quad f(\xi) = C_1 \int^{\xi} \frac{1}{\tilde{\xi}^2} e^{-\tilde{\xi}^2/4}\, d\tilde{\xi} + C_2$$

INTEGRATION BY PARTS

$$= C_1\left[-\frac{1}{\xi} e^{-\xi^2/4} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi/2)\right] + C_3$$

IN WHICH $\operatorname{erf}(\xi/2) = \frac{2}{\sqrt{\pi}}\int_0^{\xi/2} e^{-\tilde{\xi}^2} d\tilde{\xi}$ IS 'ERROR FUNCTION'

$$\left(= \frac{1}{\sqrt{\pi}}\int_0^{\xi} e^{-\tilde{\xi}^2/4}\, d\tilde{\xi}\right)$$

$$\Rightarrow u(\rho, t) = \mathcal{C}_1\left[\frac{\sqrt{\varkappa t}}{\rho} e^{-\rho^2/4\kappa t} + \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{\rho}{2\sqrt{\varkappa t}}\right)\right] + \mathcal{C}_2$$

ASS$^{\underline{\text{N}}}$ 2

2) REGULAR PERT$^{\underline{\text{N}}}$

$\varepsilon \to 0 \Rightarrow X^2 + 2X = 0$ ROOTS $X_1^{(0)} = 0,\ X_2^{(0)} = -2$

SUPPOSE $X_1 = 0 + \varepsilon X_1^{(1)} + \varepsilon^2 X_1^{(2)} + \cdots$

$\Rightarrow \varepsilon^2(\varepsilon X_1^{(1)} + \cdots)^3 + (\varepsilon X_1^{(1)} + \cdots)^2 + 2(\varepsilon X_1^{(1)} + \cdots) + \varepsilon = 0$

$O(\varepsilon)$: $2X_1^{(1)} + 1 = 0 \Rightarrow X_1^{(1)} = -\frac{1}{2}$

$O(\varepsilon^2)$: $(X_1^{(1)})^2 + 2X_1^{(2)} = 0 \Rightarrow X_1^{(2)} = -\frac{1}{2}\left(-\frac{1}{2}\right)^2 = -\frac{1}{8}$

SO $X_1 \simeq -\frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2$

SUPPOSE $X_2 = -2 + \varepsilon X_2^{(1)} + \cdots$

$O(\varepsilon)$: $-4X_2^{(1)} + 2(X_2^{(1)}) + 1 = 0 \Rightarrow X_2^{(1)} = \frac{1}{2}$

So $X_2 \simeq -2 + \frac{1}{2}\varepsilon$

SINGULAR PERT$^{\underline{\text{N}}}$

LET $X_3 = \frac{X}{\delta} \Rightarrow \underset{①}{\frac{\varepsilon^2}{\delta^3}X^3} + \underset{②}{\frac{1}{\delta^2}X^2} + \underset{③}{\frac{2}{\delta}X} + \underset{④}{\varepsilon}$

① & ④ $\Rightarrow \frac{\varepsilon^2}{\delta^3} \sim \varepsilon \Rightarrow \delta \sim \varepsilon^{1/3}$ BUT $O(②) < O(④)$ ✗

① & ③ $\Rightarrow \frac{\varepsilon^2}{\delta^3} \sim \frac{1}{\delta} \Rightarrow \delta \sim \varepsilon$ BUT $O(②) < O(①)$ ✗

① & ② $\Rightarrow \frac{\varepsilon^2}{\delta^3} \sim \frac{1}{\delta^2} \Rightarrow \delta \sim \varepsilon^2$ WORKS!

SO LET $x = \frac{X}{\varepsilon^2} \Rightarrow X^3 + X^2 + 2\varepsilon^2 X + \varepsilon^5 = 0$

SUPPOSE $X = X^{(0)} + \varepsilon X^{(1)} + \cdots$

$O(\varepsilon^0)$: $X^{(0)} = \underbrace{0, 0}_{\text{IGNORE}}, -1$

$O(\varepsilon^1)$: $-3X^{(1)} - 2X^{(1)} = 0 \Rightarrow X^{(1)} = 0$

$O(\varepsilon^2)$: $+3X^{(2)} - 2X^{(2)} + 2(-1) = 0 \Rightarrow X^{(2)} = +2$

So $X = -1 + 2\varepsilon^2$

SO $X_3 \simeq -\frac{1}{\varepsilon^2} + 2$

ASS$^{N}$ 21

3) FOR SMALL $x$, $\sin\left(\frac{\pi}{4}+x\right) = \sin\frac{\pi}{4} + \left(\cos\frac{\pi}{4}\right)x - \frac{1}{2}\left(\sin\frac{\pi}{4}\right)x^2 - \frac{1}{6}\cos\left(\frac{\pi}{4}\right)x^3 + \cdots$

$$= \frac{1}{\sqrt{2}}\left(1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right)$$

SO APPROXIMATE EQ$^{N}$ IS

$$-\frac{1}{6}x^3 + \cdots = -\frac{1}{6}\varepsilon \quad \overset{\varepsilon\to 0}{\Rightarrow} \quad x^3 \simeq 0 \qquad \text{A TRIPLE ZERO ROOT}$$

SO GUESS $\quad x = \varepsilon^{1/3}x_1 + \varepsilon^{2/3}x_2 + \cdots$

$$\Rightarrow \sqrt{2}\sin\left[\frac{\pi}{4} + \varepsilon^{1/3}(x_1 + \varepsilon^{1/3}x_2 + \cdots)\right] - 1 - \varepsilon^{1/3}(x_1 + \cdots) + \frac{1}{2}\varepsilon^{2/3}(x_1+\cdots)^2 = -\frac{1}{6}\varepsilon$$

$$= \sqrt{2}\left(\sin\frac{\pi}{4} + \left(\cos\frac{\pi}{4}\right)\varepsilon^{1/3}(x_1+\cdots) - \left(\sin\frac{\pi}{4}\right)\frac{1}{2}\varepsilon^{2/3}(x_1+\cdots)^2 - \left(\cos\frac{\pi}{4}\right)\frac{1}{6}\varepsilon(x_1+\cdots)^3 + \cdots\right)$$

$$\Rightarrow -\frac{1}{6}\varepsilon(x_1+\cdots)^3 + \frac{1}{24}\varepsilon^{4/3}(x_1+\cdots)^4 + \cdots = -\frac{1}{6}\varepsilon$$

$$\Rightarrow (x_1 + \varepsilon^{1/3}x_2 + \cdots)^3 - \frac{1}{4}\varepsilon^{1/3}(x_1+\cdots)^4 + \cdots = 1$$

$$O(\varepsilon^0) \Rightarrow x_1^3 = 1 \qquad \Rightarrow x_1 = 1,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$O(\varepsilon^{1/3}) \Rightarrow 3x_1^2x_2 - \frac{1}{4}x_1^4 = 0 \Rightarrow x_2 = +\frac{1}{12}x_1^2$$

$$= +\frac{1}{12},\ +\frac{1}{12}\left(-\frac{1}{2} \mp i\frac{\sqrt{3}}{2}\right)$$

SO $\quad x \simeq \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3}$

AND $\simeq \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\varepsilon^{1/3} + \left(\frac{-1}{24} - i\frac{\sqrt{3}}{24}\right)\varepsilon^{2/3}$

AND $\simeq \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\varepsilon^{1/3} + \left(\frac{-1}{24} + i\frac{\sqrt{3}}{24}\right)\varepsilon^{2/3}$

Ass$^t$ 2

4). $\underline{\underline{A}}\,\underline{x} = \mu \underline{x} + \varepsilon \underline{F}$

with $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\mu = \overbrace{2}^{\mu_0} + \varepsilon \mu_1 + \varepsilon^2 \mu_2$, $\underline{F} = \begin{pmatrix} a^2+b \\ a+b \end{pmatrix}$, $\underline{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$O(\varepsilon^0)$: $\underline{\underline{A}}\,\underline{x}_0 = \overset{2}{\mu_0}\underline{x}_0 \Rightarrow \underline{x}_0 = \begin{pmatrix} 0 \\ B \end{pmatrix}$ B – FIXED

LEFT EIGENVECTOR OF $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ IS $x_0^+ = (0, 1)$

SUPPOSE $\underline{x} = \begin{pmatrix} 0 \\ B \end{pmatrix} + \varepsilon \underline{x}_1 + \cdots$,

$O(\varepsilon^1)$: $(\underline{A} - \mu_0)\underline{x}_1 = \mu_1 \underline{x}_0 + \underline{F}(x_0)$

$$\Rightarrow 0 = \underbrace{(0, 1)}_{x_0^+} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \underline{x}_1 = \mu_1 \underline{x}_0^+ \underline{x}_0 + \underline{x}_0^+ \underline{F}(\underline{x}_0)$$

$$= \mu_1 (B) + (0, 1)\begin{pmatrix} B \\ B \end{pmatrix}$$

$$= \mu_1 B + B$$

SO $\mu_1 = -1$

$$\Rightarrow (A - \mu_0)\underline{x}_1 = -1\begin{pmatrix} 0 \\ B \end{pmatrix} + \begin{pmatrix} B \\ B \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{x}_1 = \begin{pmatrix} -B \\ 0 \end{pmatrix}$$

$O(\varepsilon^2)$: $(\underline{A} - \mu_0)\underline{x}_2 = \mu_2 \underline{x}_0 + \mu_1 \underline{x}_1 + \overbrace{\begin{pmatrix} 0 \\ -B \end{pmatrix}}^{\varepsilon \underline{F} \text{ AT } O(\varepsilon^2)}$

$$\Rightarrow 0 = \mu_2 \underline{x}_0^+ x_0 + \mu_1 \underline{x}_0^+ \underline{x}_1 + \underline{x}_0^+ \begin{pmatrix} 0 \\ -B \end{pmatrix}$$

$$= \mu_2 B - B$$

$$\Rightarrow \mu_2 = 1$$

$$\Rightarrow (\underline{A} - \mu_0)\underline{x}_2 = (1)\overbrace{\begin{pmatrix} 0 \\ B \end{pmatrix}}^{x_0} + (-1)\overbrace{\begin{pmatrix} -B \\ 0 \end{pmatrix}}^{x_1} + \begin{pmatrix} 0 \\ -B \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{x}_2 = \begin{pmatrix} -B \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} a \\ b \end{pmatrix} = \underline{x} \approx \begin{pmatrix} 0 \\ B \end{pmatrix} + \varepsilon \begin{pmatrix} -B \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -B \\ 0 \end{pmatrix}$

$\mu \simeq 2 - \varepsilon + \varepsilon^2$

ASS= 2

5) a) $\tilde{\rho} = \frac{\rho}{\rho_1} = 1 + \varepsilon r$ , $\tilde{h} = \frac{h}{h_0}$ , $\tilde{\eta} = \frac{\eta}{h_0}$ , $\tilde{t} = t/(\frac{h_0 A}{Q})$

$$(1) \Rightarrow \quad \tilde{\eta} = (1+\varepsilon r)\tilde{h} \qquad (4)$$
$$(2) \Rightarrow \quad \frac{d}{d\tilde{t}}(\tilde{\eta} + \tilde{h}) = -1 \qquad (5)$$
$$(3) \Rightarrow \quad \frac{d}{d\tilde{t}}(\tilde{\eta} + (1+\varepsilon r)\tilde{h}) = -(1+\varepsilon r) \qquad (6)$$

b) $(4)+(5) \Rightarrow \quad \frac{d}{d\tilde{t}}((2+\varepsilon r)\tilde{h}) = -1$

$$\Rightarrow \quad (2+\varepsilon r)\tilde{h} = -\tilde{t} + \mathcal{C}$$

$h(0) = h_0 \Rightarrow \tilde{h}(0) = 1$ ; $\rho(0) = \rho_0 = \rho_1(1+\varepsilon) \Rightarrow r(0) = 1$

so $\mathcal{C} = 2 + \varepsilon$

$$\Rightarrow \quad \tilde{h} = (2+\varepsilon-\tilde{t})/(2+\varepsilon r) \qquad (7)$$

$$(4),(7)+(6) \Rightarrow \quad \frac{d}{d\tilde{t}}\left[2(1+\varepsilon r)\frac{(2+\varepsilon-\tilde{t})}{2+\varepsilon r}\right] = -(1+\varepsilon r) \qquad (8)$$

$r(0) = 1$

c) USE $\frac{1}{2+\varepsilon r} \simeq \frac{1}{2}(1 - \frac{1}{2}\varepsilon r + \cdots)$

$$(8) \Rightarrow \frac{d}{d\tilde{t}}\left[(2+\varepsilon-\tilde{t})(1+\varepsilon r)(1-\tfrac{1}{2}\varepsilon r + \cdots)\right] = -(1+\varepsilon r)$$

$O(\varepsilon^0)$: $\frac{d}{d\tilde{t}}[(2-\tilde{t})] = -1$ CONSISTENT. WHEW!

WRITE $r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 + \cdots$, so $r(0) = 1 \Rightarrow r_0(0) = 1$, $r_1(0) = 0$, etc

$O(\varepsilon^1)$: $\frac{d}{d\tilde{t}}[1 + (2-\tilde{t})\frac{1}{2}r_0] = -r_0$

$$\Rightarrow \quad (1-\tfrac{1}{2}\tilde{t})\frac{dr_0}{d\tilde{t}} - \tfrac{1}{2}r_0 = -r_0$$
$$\Rightarrow \quad \frac{dr_0}{d\tilde{t}} = -\tfrac{1}{2}r_0/(1-\tfrac{1}{2}\tilde{t})$$
$$\Rightarrow \quad r_0 = \mathcal{C}[1 - \tfrac{1}{2}\tilde{t}]$$

$r_0(0) = 1 \Rightarrow \mathcal{C} = 1$

So $r_0(\tilde{t}) = 1 - \frac{1}{2}\tilde{t}$ NON-ZERO, SO THIS IS LEADING ORDER

$$\Rightarrow \rho(t) \simeq \rho_1(1 + \varepsilon r_0(t/\tfrac{h_0 A}{Q}))$$
$$= \rho_1\left[1 + \frac{\rho_0-\rho_1}{\rho_1}(1 - \frac{1}{2}\frac{Q}{h_0 A}t)\right]$$
$$= \rho_0 - \frac{1}{2}(\rho_0-\rho_1)\frac{Q}{h_0 A}t$$