EXAMPLE. $A_n = \sum_{k=1}^{n} \frac{1}{k^2}$. EXACT $A = \sum^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \simeq 1.649$

n	$N=0$ (A_n)	$N=1$	$N=2$
1	1.000	1.500	1.625
5	1.464	1.630	1.644
10	1.550	1.641	1.645

(3) ASYMPTOTIC SUMMATION

- INTEGRAL FORMULAS FOR SUMS

MANY INFINITE SERIES CAN BE WRITTEN AS AN INTEGRAL

e.g. $$\int_0^\infty \frac{t e^{-nt}}{e^t - 1}\, dt \overset{\text{BINOMIAL EXPANSION}}{=} \int_0^\infty t e^{-nt} e^{-t} \sum_{k=0}^{\infty} (e^{-t})^k \, dt$$

$$= \sum_{k=0}^{\infty} \int_0^\infty t\, e^{-(n+k+1)t}\, dt = \sum_{p=n+1}^{\infty} \int_0^\infty t e^{-pt}\, dt$$

INTEGRATE BY PARTS $$= \sum_{p=n+1}^{\infty} \left[\frac{-t}{(p)} e^{-(p)t} \Big|_0^\infty + \int_0^\infty \frac{1}{(p)} e^{-pt}\, dt \right]$$

$$= \sum_{p=n+1}^{\infty} + \frac{1}{p^2}$$

SO $$\sum_{k=n+1}^{\infty} \frac{1}{k^2} = + \int_0^\infty \frac{t e^{-nt}}{e^t - 1}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \int_0^\infty \frac{t e^{-nt}}{e^t - 1} = A_n + \underbrace{R_n}_{\text{ERROR}}$$

WE CAN EVALUATE ERROR USING WATSON'S LEMMA!

LET $$I(n) = \int_0^\infty f(t) e^{-nt}\, dt$$

WITH $$f(t) = \frac{+t}{e^t - 1} \overset{t\text{-SMALL}}{\simeq} \frac{+t}{(t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots)} = + \frac{1}{(1 + (\frac{1}{2}t + \frac{1}{6}t^2 \cdots))} \simeq + \left(1 - (\tfrac{1}{2}t + \tfrac{1}{6}t^2 + \cdots) + (\tfrac{1}{2}t + \tfrac{1}{6}t^2 + \cdots)^2 - \cdots\right)$$

$$\simeq +1 - \frac{1}{2}t + \frac{1}{12}t^2 + \cdots$$

SO AS $n \to \infty$ $$I \sim +1 \frac{\Gamma(1)}{n^1} - \frac{1}{2}\frac{\Gamma(2)}{n^2} + \frac{1}{12}\frac{\Gamma(3)}{n^3} = \frac{+1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} + O(n^{-3})$$

So FOR $A_n = \sum_{n=0}^{A} \frac{1}{k^2}$

n	A_n	$A_n + \left(+\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3}\right)$
1	1	$1+\frac{2}{3} \simeq 1.667\ldots$
2	$\frac{5}{4}=1.25$	$\frac{5}{4}+\frac{19}{48} \pm \frac{79}{48} \simeq 1.646$

MORE GENERALLY, FOR THE RIEMANN-ZETA FUNCTION

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad Re(s) > 1$$

WE CAN WRITE

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1} e^{-Nu}}{e^u - 1}\, du$$

USING WATSON'S LEMMA

$$\zeta(s) \sim \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{(s-1)N^{s-1}} - \frac{1}{2}\frac{1}{N^s} + \cdots$$

SUMMATION OF DIVERGENT SERIES

WE HAVE SEEN THAT, ALTHOUGH $\sum a_n x^n$ MAY DIVERGE IF $|x|$ IS SUFFICIENTLY LARGE, A SHANKS TRANSFORMATION CAN NONETHELESS GIVE A FINITE RESULT.

THIS AND SIMILAR TECHNIQUES ARE PARTICULARLY HANDY FOR GETTING FINITE ANSWERS TO PERTURBATION SERIES EXPANSIONS IN ε THAT DIVERGE FOR ε LARGE.

DEF$^{\underline{N}}$: EULER SUMMATION

IF $\sum_{n=0}^{\infty} a_n$ IS ALGEBRAICALLY DIVERGENT (BLOWS UP AS A POWER OF n)

THEN $f(x) = \sum_{n=0}^{\infty} a_n x^n$ CONVERGES FOR $|x| < 1$.

EULER SUM IS $S = \lim_{x \to 1^-} f(x)$ WHENEVER SUM EXISTS

EXAMPLES

1) $\sum_{n=0}^{\infty} (-1)^n$ DIVERGES (LIMIT ALTERNATES BETWEEN 1 AND 0)

BUT $\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$ SO $S = \frac{1}{2}$.

(ALTERNATELY, USE SHANKS TRANFORMATION TO GET SAME RESULT)

2) SUM $1 + 0 + (-1) + 0 + (1) + \ldots$ DIVERGES

$$S = \lim_{x \to 1^-} \sum_{n=0}^{\infty} (-x^2)^n = \lim_{x \to 1^-} \frac{1}{1+x^2} = \frac{1}{2}.$$

HERE THE SHANKS TRANSFORMATION DOESN'T WORK!

BOREL SUMMATION

SUPPOSE $\sum_{n=0}^{\infty} a_n$ DIVERGES BECAUSE a_n GROWS FASTER THAN A POWER OF n. (i.e. $a_n = 2^n$ OR $a_n = n!$)

CANNOT APPLY EULER SUMMATION BECAUSE $\sum a_n x^n$ DIVERGES EVEN FOR $|x| \lesssim 1$.

INSTEAD, DEFINE $\phi(x) \equiv \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$

AND DEFINE $B(x) = \int_0^{\infty} e^{-t} \phi(xt)\, dt$

WRITE $B(x) = \int_0^{\infty} e^{-\frac{1}{x}t} \cdot \phi(t) \frac{1}{x} dt$ AND USE WATSON'S LEMMA TO EXPAND FOR SMALL x.

$$\Rightarrow B(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} e^{-\frac{1}{x}t} t^n \frac{1}{x} dt = \sum_{n=0}^{\infty} a_n x^n$$

AS $x \to 0^+$

SO, BY CONSTRUCTION $\sum_{n=0}^{\infty} a_n x^n$ IS ASYMPTOTIC TO $B(x)$ AS $x \to 0^+$

HENCE, GIVEN $\sum_{n=0}^{\infty} a_n$, WE <u>DEFINE</u> THE BOREL SUM $\sum_{n=0}^{\infty} a_n x^n$ TO EQUAL $B(x)$.

AND SO, IN PARTICULAR, WE DEFINE $\sum_{n=0}^{\infty} a_n \equiv B(1)$

EXAMPLE $\sum_{n=0}^{\infty} (-1)^n n!$ DIVERGES

BUT $\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n = \sum_{n=0}^{\infty} (-x)^n$ CONVERGES AS $x \to 1^-$

INDEED $\phi(x) = \frac{1}{1+x}$

SO DEFINE THE BOREL SUM OF $\sum (-\cancel{x})^n n!$ TO BE

$$B(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt$$

HENCE $B(1) = \int_0^\infty \frac{e^{-t}}{1+t}\,dt$

NOTE THE BACKWARDS APPROACH TO WATSON'S LEMMA.

HERE WE START WITH SERIES AND DERIVE INTEGRAL.

PADÉ SUMMATION

APPROXIMATE $f(z) = \sum a_n z^n$ BY A RATIO OF TWO POLYNOMIALS:

$$P^N_M(z) = \frac{\sum_{n=0}^{N} A_n z^n}{\sum_{m=0}^{M} B_m z^m}$$

WHERE, WITHOUT LOSS OF GENERALITY, WE CHOOSE $B_0 = 1$

FOR GIVEN N AND M, WE FIND COEFFICIENTS A_n, B_m BY REQUIRING THE LEADING $N+M+1$ TERMS IN THE TAYLOR SERIES OF $P^N_M(z)$ TO MATCH THOSE OF $f(z)$.

IT TURNS OUT THAT, EVEN IF $\sum a_n z^n$ IS DIVERGENT, WE CAN STILL FIND $f(z) \sim P^N_M(z)$ AS $N, M \to \infty$

EXAMPLE: FIND $P^0_1(z) = \frac{A_0}{1+B_1 z}$

WANT TO FIND A_0, B_1 SUCH THAT

$$\frac{A_0}{1+B_1 z} = A_0(1 - B_1 z + \cdots) \approx a_0 + a_1 z + \cdots$$

SO $A_0 = a_0$

AND $-A_0 B_1 = a_1 \Rightarrow B_1 = -\frac{a_1}{a_0}$

SO $P^0_1(z) = \dfrac{a_0}{1 - \frac{a_1}{a_0} z}$

EXAMPLE FOR DIVERGENT SUM $\sum_{n=0}^{\infty}(-1)^n (n+1)!\, x^n$

$$B(x) = \int_0^\infty \frac{e^{-t}}{(1+xt)^2}\,dt \quad \text{IS BOREL SUM.}$$

IN PARTICULAR $B(\frac{1}{10}) \simeq 0.84367$

FOR $P_2^2(x)$ WANT $\dfrac{A_0 + A_1 x + A_2 x^2}{1 + B_1 x + B_2 x^2} = 1 - 2x + 6x^2 - 24x^3 + 120x^4 - + \cdots$

SOME ALGEBRA GIVES

$$P_2^2(x) = \frac{1+6x+2x^2}{1+8x+12x^2} \Rightarrow P_2^2(\tfrac{1}{10}) = 0.84375$$

LIKEWISE FIND

$$P_3^2 = \frac{1+10x+18x^2}{1+12x+36x^2+24x^3} \Rightarrow P_3^2(x) = 0.84365$$

NOT BAD ~~AGREEMENT~~ CONSIDERING SUM DIVERGES.

WHAT DOES IT MEAN?

THESE GIVE APPROXIMATE VALUE OF FIRST TERMS OF SUM BEFORE IT STARTS TO DIVERGE.

e.g. AT $x = \frac{1}{10}$

$$\sum(-1)^n (n+1)!\, x^n \simeq 1 - 0.2 + 0.06 - 0.024 + 0.012 - 0.0072 + 0.0054 - 0.004032 + 0.0036 - 0.0036 + \text{INCREASING TERMS}$$

$$\simeq 0.84543 \quad +$$