BIFURCATION THEORY [LOGAN #7.1]

RECALL POPULATION MODEL.

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \quad p(0) = p_0$$

WHERE $p(t)$ IS POPULATION AT TIME t, r IS GROWTH RATE AND K IS "CARRYING CAPACITY".

LET'S STUDY STABILITY OF THE POPULATION

EQUILIBRIUM POINTS OCCUR WHERE $\frac{dp}{dt} = 0$

$$\Rightarrow p = 0 \text{ OR } p = K.$$

- IS POINT $p = K$ STABLE?

  PERTURB $p(t) = K + p'(t)$ $\qquad |p'| \ll K$

  SUBSTITUTE INTO IVP

$$\Rightarrow \frac{d}{dt}(K + p') = r(K + p')\left(1 - \frac{1}{K}(K + p')\right)$$

$$\Rightarrow \frac{dp'}{dt} = rK + rp' - \frac{r}{K}\left(K^2 + 2Kp' + p'^2\right)$$

$$= -rp' \cancel{} - \frac{r}{K}p'^2$$

$$\simeq -rp'$$

  SO $p'(t) \propto e^{-rt}$ WHICH DECREASES TO ZERO IN TIME

  $\Rightarrow p = K$ IS STABLE

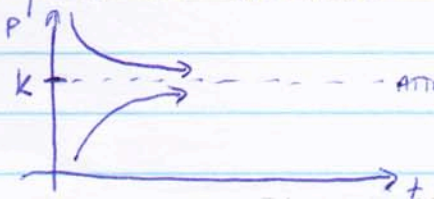
- IS POINT $p = 0$ STABLE?

  PERTURB $p(t) = 0 + p'(t)$

  SUBSTITUTE $\Rightarrow \frac{dp'}{dt} = rp'\left(1 - \frac{p'}{K}\right) \approx rp'$

$$\Rightarrow p'(t) \propto e^{rt}$$ WHICH INCREASES IN TIME

  $\Rightarrow p = 0$ IS UNSTABLE



LET'S KICK IT UP A NOTCH!

SUPPOSE WE "HARVEST" A FRACTION OF THE POPULATION A CONSTANT RATE h

$$\Rightarrow \quad \frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - h$$

$$r, K, h > 0$$

FIRST NON-DIMENSIONALISE $\quad \tilde{p} = \frac{p}{K}, \quad \tilde{t} = rt$

$$\Rightarrow \quad \frac{d\tilde{p}}{d\tilde{t}} = \tilde{p}(1-\tilde{p}) - \mu$$

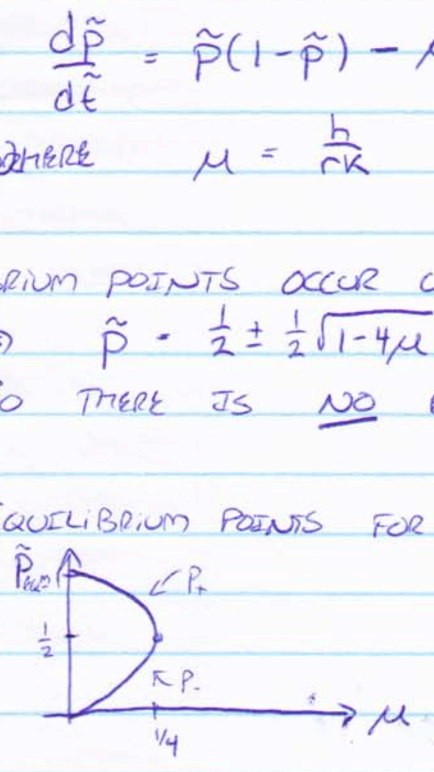
WHERE $\quad \mu = \frac{h}{rK} \quad$ IS DIMENSIONLESS "HARVESTING" PARAMETER

EQUILIBRIUM POINTS OCCUR WHERE $\quad \frac{d\tilde{p}}{d\tilde{t}} = 0$

$$\Rightarrow \quad \tilde{p} = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1-4\mu}$$

SO THERE IS NO EQUILIBRIUM IF $\mu > 1/4$

EQUILIBRIUM POINTS FOR $0 \leq \mu \leq 1/4$ CAN BE GRAPHED



ARE THESE POINTS STABLE OR UNSTABLE?

DENOTE $\quad P_+ = \frac{1}{2} + \frac{1}{2}\sqrt{1-4\mu} \qquad P_- = \frac{1}{2} - \frac{1}{2}\sqrt{1-4\mu}.$

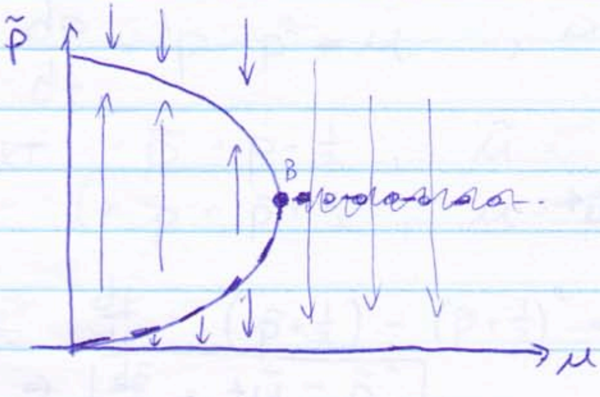
RATHER THAN EXPLICITLY GOING THROUGH PERTURBATION PROCEDURE, NOTE FOR $\mu < \frac{1}{4}$

IF $\tilde{p} > P_+$ THEN $\frac{d\tilde{p}}{d\tilde{t}} < 0 \quad$ (e.g. TRY $\tilde{p} = 1 \Rightarrow \frac{d\tilde{p}}{d\tilde{t}} = -\mu < 0$)

IF $P_- < \tilde{p} < P_+$ THEN $\frac{d\tilde{p}}{dt} > 0 \quad$ (eg TRY $\tilde{p} = \frac{1}{2} \Rightarrow \frac{d\tilde{p}}{d\tilde{t}} = \frac{1}{4} - \mu > 0$)

IF $\tilde{p} < P_-$ THEN $\frac{d\tilde{p}}{dt} < 0 \quad$ (e.g TRY $\tilde{p} = 0 \Rightarrow \frac{d\tilde{p}}{dt} = -\mu < 0$)

SO WE CAN REPRESENT STABILITY WITH THE "BIFURCATION DIAGRAM"



μ - IS CALLED A "BIFURCATION PARAMETER"
THE POINT $B \equiv (\tilde{\mu}=\frac{1}{4}, \tilde{p}=\frac{1}{2})$ IS CALLED A "BIFURCATION POINT"
(BIFURCATION COMES FROM LATIN MEANING TWO-FORKED).

IN BIFURCATION THEORY ONE CONSIDERS BIFURCATION DIAGRAMS THAT RESULT FROM DIFFERENTIAL EQUATIONS

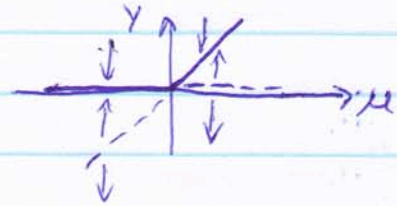
$$\frac{dy}{dx} = F(y;\mu)$$

- ONE FINDS EQ$^{\underline{m}}$ POINTS u AS A FUNCTION OF BIFURCATION PARAMETER μ BY SOLVING $F(y;\mu) = 0$. THERE MAY BE MORE THAN ONE CURVE
- THEN ONE DETERMINES WHERE THE CURVES CROSS TO FIND BIFURCATION POINTS. (y_c, μ_c)
- FINALLY ONE DETERMINES STABILITY NEAR BIFURCATION POINTS (USUALLY BY SHIFTING COORDINATES TO $\tilde{y} = y - u_c$, $\tilde{\mu} = \mu - \mu_c$ SO THAT BIFURCATION POINT IS AT THE ORIGIN, AND TAYLOR SERIES REVEALS A SMALL CLASS OF "NORMAL FORMS" THAT REPRESENT DIFFERENT TYPES OF BIFURCATIONS)

SOME BIFURCATION DIAGRAMS ARE

① "TRANSCRITICAL" BIFURCATION

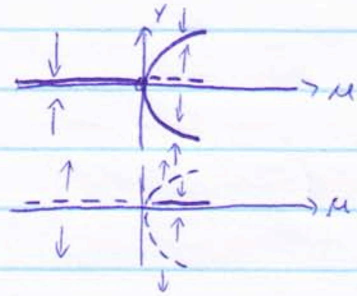
$$\frac{dy}{dx} = \mu y - y^2$$



② PITCHFORK BIFURCATION

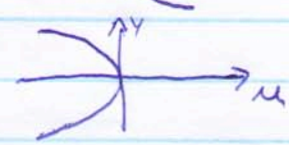
$$\frac{dy}{dx} = \mu y - y^3$$

$$\frac{dy}{dx} = -\mu y + y^3$$



THESE CAN BE SUB-CLASSIFIED AS SUPERCRITICAL

OR SUBCRITICAL



$$\frac{dy}{dx} = \mu - y^2 \; ; \; \frac{dy}{dx} = -\mu + y^2$$

ALSO SOMETIMES CALLED BITCHFORK BIFURCATION

③ HOPF BIFURCATION (COMPLEX PLANE $Z(x) = Z_r + iZ_i = r(x)\, e^{i\theta(x)}$

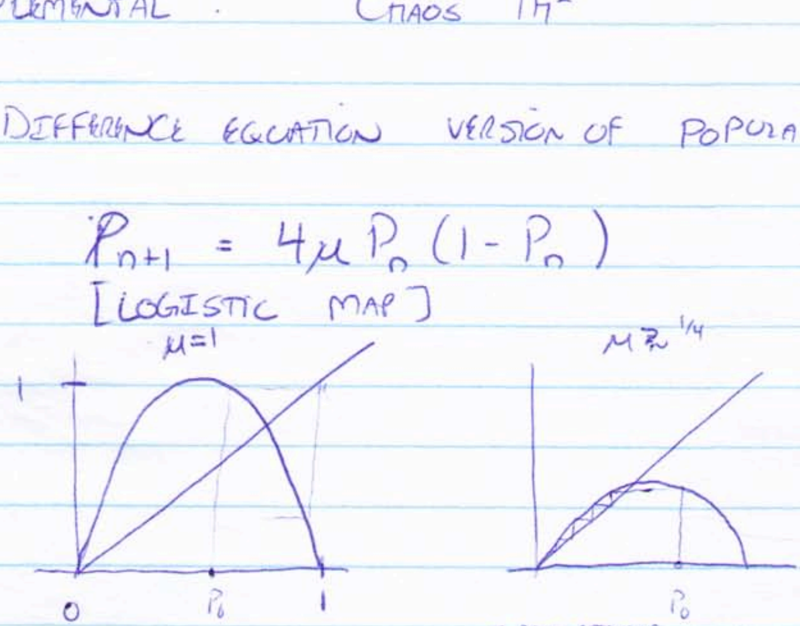
$$\frac{dz}{dx} = \mu z - z^3$$

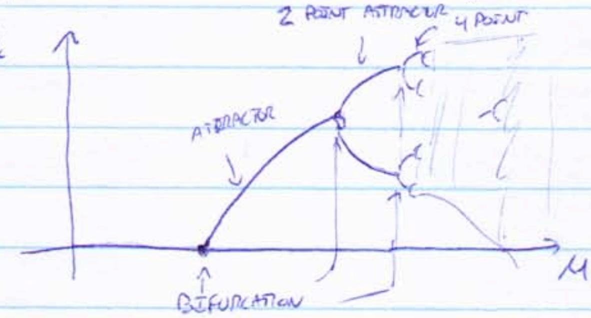
[SUPPLEMENTAL : CHAOS TH^{Y}

DIFFERENCE EQUATION VERSION OF POPULATION MODEL.

$$P_{n+1} = 4\mu P_n (1 - P_n) \quad , \quad P_0 = \bar{P}_0 \text{ GIVEN WITH } 0 < P_0 < 1 \text{ AND } 0 < \mu < 1$$

[LOGISTIC MAP]





~~FOR~~ FOR RANGE OF μ HAVE A ATTRACTOR WITH ∞ # POINTS (IT IS A FRACTAL SET SO UNCOUNTABLY MANY POINTS BUT WITH DIMENSION LESS THAN 1)

COMPARE WITH CONVECTION EXPERIMENT [LIBSCHABER - 1970s]