# 2] PERTURBATION METHODS



(0) OFTEN AN EQUATION CANNOT BE SOLVED EXACTLY, BUT PROGRESS CAN BE MADE BY ASSUMING SOME TERMS IN THE EQUATION ARE SMALL, AS REPRESENTED BY A PARAMETER $\varepsilon$, WHICH ONE TAKES TO BE SMALL BY ASSUMING $\varepsilon \ll 1$.

(IN TYPICAL PROBLEM $\varepsilon \lesssim 0.1$ IS GOOD ENOUGH FOR MOST APPROXIMATIONS OF INTEREST. SOMETIMES, VALID RANGES FOR $\varepsilon$ CAN BE DETERMINED EXPLICITLY).

SOME PROBLEMS INCLUDE

(1) ALGEBRAIC PROBLEMS

e.g. FIND ROOTS OF $\varepsilon x^3 + x^2 - 3x + 2 = 0$ ; $\varepsilon \ll 1$

(2) EIGENVALUE PROBLEMS

e.g. FIND EIGENVALUES & EIGENVECTORS OF $A\underline{x} = \mu \underline{x} + \varepsilon \underline{F}(\underline{x})$

(3) DIFFERENTIAL EQUATIONS

e.g. SOLVE $\dfrac{d^2h}{dt^2} = -\dfrac{1}{(1+\varepsilon h)^2}$ ; $h(0) = 0$, $h'(0) = 1$

IN EACH CASE, IT IS EASY TO FIND SOLUTIONS IF $\varepsilon = 0$. EXPECT SIMILAR SOLUTIONS, BUT ADJUSTED BY SMALL AMOUNT $\varepsilon$ FOR $\varepsilon \ll 1$

NOTE: TEXT BY LOGAN DISCUSSES CASES (1) (3). WE'LL DO ALL THREE. AND DO CASE (1) IN MORE DETAIL.

# (1) ALGEBRAIC PROBLEMS

## REGULAR PERTURBATION [FROM NOTES BY M. PROCTER] (BUT SEE EX. 2.2 SEC 2.2 OF LOGAN TEXT)

e.g. SOLVE $x^2 + ax + b + \varepsilon f(x) = 0$ (*)
IN WHICH $f(x)$ IS A POLYNOMIAL IN $x$ AND $\varepsilon \ll 1$

IF $\varepsilon = 0$ HAVE ROOTS OF $x^2 + ax + b$
i) 2 DISTINCT REAL ROOTS, $x_1, x_2$, FOR $a^2 > 4b$
ii) " COMPLEX " " " $a^2 < 4b$
iii) EQUAL ROOTS FOR $a^2 = 4b$. $(x = -a/2)$

CONSIDER CASES i) & ii) FOR $0 < \varepsilon \ll 1$

SUPPOSE WE CAN WRITE ROOTS $x_i(\varepsilon) = x_i^{(0)} + \sum_{n=1}^{\infty} \varepsilon^n x_i^{(n)}$ $i = 1, 2.$
(A POWER SERIES IN $\varepsilon$ FOR EACH ROOT, $x_1, x_2$.
NOTE $x_1(0) = x_1^{(0)}$, $x_2(0) = x_2^{(0)}$ ARE ROOTS OF $x^2 + ax + b$)

WISH TO FIND $x_i^{(n)}$ FOR $i = 1, 2$ AND $n = 1, 2, \ldots$

TO FIND $x_i^{(1)}$, SUBSTITUTE POWER SERIES INTO (*)
AND PULL OUT TERMS OF $O(\varepsilon)$ ("ORDER EPSILON")
[NOTATION $x = O(y), y \to 0 \Rightarrow \frac{x}{y}$ IS BOUNDED AS $y \to 0$; $x = o(y), y \to 0 \Rightarrow \frac{x}{y} \to 0$ AS $y \to 0$ BUT NOT $x \neq O(y)$]

$$\Rightarrow \left( x_i^{(0)} + \varepsilon x_i^{(1)} + \varepsilon^2 x_i^{(2)} + \cdots \right)^2 + a\left( x_i^{(0)} + \varepsilon x_i^{(1)} + \varepsilon^2 x_i^{(2)} + \cdots \right) + b + \varepsilon f\left(x_i^{(0)} + \varepsilon x_i^{(1)} + \cdots\right)$$

$O(\varepsilon^0)$: $\left(x_i^{(0)}\right)^2 + a\left(x_i^{(0)}\right) + b = 0 \Rightarrow x_i^{(0)}$ ARE ROOTS OF $x^2 + ax + b$

$O(\varepsilon^1)$: $2 x_i^{(0)} x_i^{(1)} + a x_i^{(1)} + f(x_i^{(0)}) = 0$

$$\Rightarrow x_i^{(1)} = -\frac{f(x_i^{(0)})}{2x_i^{(0)} + a}, \quad i = 1, 2.$$

NO SINGULARITY PROBLEM BECAUSE $x_i^{(0)} = -a/2$ ONLY FOR DOUBLE ROOT

NOW FIND $X_i^{(2)}$ FROM $O(\varepsilon^2)$ TERMS

RECALL, $\overset{\text{NEAR } x=A}{f(x)} \approx f(A) + (x-A)f'(A)$ SO $f_1(x^{(0)} + \varepsilon\Delta) \approx f(x^{(0)}) + \varepsilon\Delta f'(x^{(0)})$

$$O(\varepsilon^2): \quad 2X_i^{(0)}X_i^{(2)} + \left(X_i^{(1)}\right)^2 + aX_i^{(2)} + X_i^{(1)}f'(X_i^{(0)}) = 0$$

$$\Rightarrow X_i^{(2)} = \frac{1}{2X_i^{(0)}+a}\left[-X_i^{(1)}f'(X_i^{(0)}) - \left(X_i^{(1)}\right)^2\right]$$

IN GENERAL, FIND $n^{th}$ TERM IN PERTURBATION EXPANSION IS

$$\varepsilon^n X_1^{(n)} \propto \frac{\varepsilon^n}{(2X_i^{(0)}+a)^n} \equiv \frac{\varepsilon^n}{(X_1^{(0)}-X_2^{(0)})^n} \qquad \left(\text{RECALLING } a = -X_1^{(0)} - X_2^{(0)}\right)$$

$$\varepsilon^n X_2^{(n)} \propto \frac{\varepsilon^n}{(X_2^{(0)}-X_1^{(0)})^n}$$

WHICH DEMONSTRATES RADIUS OF CONVERGENCE OF SERIES:

$$|\varepsilon| < |X_1^{(0)} - X_2^{(0)}|$$

GET NO CONVERGENCE IF $X_1^{(0)} = X_2^{(0)}$ (EQUAL ROOTS : CASE iii)

CASE iii) FOR $0 < \varepsilon \ll 1$

HERE $a^2 = 4b$ SO $(*) \Rightarrow \quad X^2 + aX + \left(\frac{1}{4}a^2 + \varepsilon f(x)\right) = 0$

RECURSION FORMULA FOUND FROM QUADRATIC EQUATION:

$$X = \frac{1}{2}\left[-a \pm \sqrt{-4\varepsilon f(x)}\right] = -\frac{1}{2}a \pm O(\varepsilon^{1/2})$$

SO EXPECT SERIES EXPANSION

$$X = X^{(0)} + \varepsilon^{1/2}X^{(1)} + \varepsilon X^{(2)} + \cdots$$

(DROPPING SUBSCRIPT "i" FOR NOW)

SUBSTITUTING INTO $x^2 + ax + \overbrace{b}^{a^2/4} + \varepsilon f(x) = 0$ AND KEEPING TERMS UP TO $\varepsilon^{3/2}$ GIVES

$$
\begin{aligned}
&(x^{(0)})^2 + \varepsilon (x^{(1)})^2 + 2\varepsilon^{1/2} x^{(0)} x^{(1)} + 2\varepsilon x^{(0)} x^{(2)} + 2\varepsilon^{3/2} x^{(0)} x^{(3)} + 2\varepsilon^{3/2} x^{(1)} x^{(2)} \\
&+ a x^{(0)} + a\varepsilon^{1/2} x^{(1)} + a\varepsilon x^{(2)} + a\varepsilon^{3/2} x^{(3)} \\
&+ a^2/4 \\
&+ \varepsilon f(x^{(0)}) + \varepsilon^{3/2} x^{(1)} f'(x^{(0)})
\end{aligned}
$$

$O(\varepsilon^0)$: $(x^{(0)})^2 + a x^{(0)} + a^2/4 = 0 \Rightarrow x^{(0)} = -a/2$ (DOUBLE ROOT)

$O(\varepsilon^{1/2})$: $2x^{(0)}x^{(1)} + a x^{(1)} = 0 \Rightarrow$ ALWAYS TRUE, $x^{(1)}$ STILL UNDETERMINED

$O(\varepsilon)$: $\cancel{2x^{(0)}x^{(2)}} + (x^{(1)})^2 + \cancel{a x^{(2)}} + f(x^{(0)}) = 0 \Rightarrow x^{(1)} = \pm\sqrt{-f(x^{(0)})}$

IF $f(x^{(0)}) \neq 0$ $x^{(1)}$ HAS TWO SOLUTIONS AND SO AGAIN FIND TWO SERIES SOLUTIONS. [$f(x^{(0)}) = 0$ IS A DEGENERATE CASE WE WON'T CONSIDER]

$O(\varepsilon^{3/2})$: $\cancel{2x^{(0)}x^{(3)}} + 2x^{(1)}x^{(2)} + \cancel{a x^{(3)}} + x^{(1)} f'(x^{(0)}) = 0 \Rightarrow x^{(2)} = -\frac{1}{2} f'(x^{(0)})$

etc.

NOTES:

- THE ABOVE CASE IS A "SINGULAR PERTURBATION PROBLEM", MEANING A SERIES SOLUTION IN POWERS OF $\varepsilon$ CANNOT BE FOUND.

- IT IS TYPICAL OF SINGULAR PERTURBATION PROBLEM TO HAVE A REDUNDANT STEP ~~SUCH THAT~~ $x^{(1)}$ IS NOT DETERMINED IN THE EQUATION IN WHICH IT FIRST APPEARS (e.g. $O(\varepsilon^{1/2})$ ABOVE).

THE ~~SERIES~~ EXPANSIONS FOR CASES i, ii, iii FIND APPROXIMATIONS TO ROOTS CLOSE TO ROOTS OF $x^2 + ax + b = 0$.

BUT IF $f(x)$ HAS DEGREE HIGHER THAN 2, EXPECT EVEN MORE ROOTS.

FOR EXAMPLE, SUPPOSE $f(x) \equiv x^3$: WANT TO FIND THE THIRD ROOT OF $\varepsilon x^3 + x^2 + ax + b = 0$ WHICH IS NOT CLOSE TO ROOTS OF $x^2 + ax + b = 0$

SINCE $\varepsilon \ll 1$, MUST HAVE $x$ LARGE ENOUGH SO $\varepsilon x^3$ IS BALANCED BY $x^2 + ax + b$ TERM (e.g. $10^{-6}x^3 + x^2 - 2x + 3 = 0$ )

e.g. SUPPOSE $\varepsilon x^3 = O(b) \equiv O(1)$

THEN $x = O(\varepsilon^{-1/3})$ AND $x^2 = O(\varepsilon^{-2/3})$

THIS DOESN'T WORK BECAUSE $x$ & $x^2$ TERMS ARE LARGER THAN $O(1)$

SUPPOSE $\varepsilon x^3 = O(x^2)$

THEN $x = O(\varepsilon^{-1})$, $x^2 = O(\varepsilon^{-2})$

WHICH DOES WORK BECAUSE $x^2$ TERM IS LARGEST

SO, RESCALE EQUATION: DEFINE $X = \varepsilon x$.

SINCE $x = O(\varepsilon^{-1})$, $X = O(1)$

SUBSTITUTE INTO $\varepsilon x^3 + x^2 + ax + b = 0$

$$\Rightarrow X^3 + X^2 + \varepsilon a X + \varepsilon^2 b = 0$$

AND LOOK FOR $O(1)$ SOLUTION AS $\varepsilon \to 0$.

SUPPOSE $X = X^{(0)} + \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \cdots$

$O(\varepsilon^0)$ : $\left(X^{(0)}\right)^3 + \left(X^{(0)}\right)^2 = 0$

$\Rightarrow X^{(0)} = -1, 0, 0$

ONLY FIRST OF THESE ROOTS IS $O(1)$

SO $X = -1 + \varepsilon X^{(1)} + \varepsilon^2 X^{(2)} + \cdots$

NOW FIND $X^{(1)}, X^{(2)}$ etc THROUGH USUAL REGULAR PERTURBATION METHOD.

FINALLY $x = \varepsilon^{-1} X = -\frac{1}{\varepsilon} + X^{(1)} + \varepsilon X^{(2)} + \cdots$

EXAMPLE. $\varepsilon x^3 + x^2 - 2x + 3 = 0$

FIND ROOTS NEAR $X = 1, 2$ USING REGULAR PERT$^{\underline{n}}$ TH$^{\underline{Y}}$