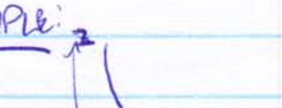
NOTE (6) ANOTHER ADVANTAGE OF DIMENSIONAL ANALYSIS IS THAT IT HELPS TO REDUCE A PARTIAL DIFFERENTIAL EQUATION TO AN ORDINARY DIFF. EQ$^{\underline{n}}$.

(A) <u>EXAMPLE</u>:



AFTER TIME $t=0$
A FLAT PLATE IS TOWED AT SPEED U_0 UNDERNEATH AN INITIALLY STATIONARY FLUID WITH VISCOSITY ν. WHAT IS SPEED OF FLUID, $u(z,t)$ IN SPACE AND TIME?

<u>SOL$^{\underline{n}}$</u>

INITIAL - BOUNDARY VALUE PROBLEM IS GIVEN BY DIFFUSION EQ$^{\underline{n}}$.

PDE $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial z^2}$

IC $u(z,0) = 0$,

BC $u(0,t) = U_0$, u - BOUNDED

5 PITHE PARAM. u, z, t, U_0, ν 3 DIMENSIONLESS VARIABLES: $\frac{u}{U_0}, \frac{z}{\sqrt{\nu t}}, \frac{z/t}{U_0}$, LAST DOES NOT RELATE TO DIFFUSION IGNORE $\frac{z/t}{U_0}$ - SO IGNORE

(SEE B.d) [DIMENSIONAL ANALYSIS REVEALS] $u(z,t) = U_0\, f\left(\frac{z}{\sqrt{\nu t}}\right)$

SUBSTITUTE INTO ABOVE EQ$^{\underline{n}}$S. LETTING $\eta = \frac{z}{\sqrt{\nu t}}$

PDE $\Rightarrow U_0\, f'(\eta)\, \frac{z}{\sqrt{\nu}}\left(-\tfrac{1}{2} t^{-3/2}\right) = \nu U_0 f''(\eta)\, \frac{1}{\nu t}$

$\Rightarrow -\tfrac{1}{2}\eta f' = f''$

$\Rightarrow f(\eta) = \int_0^{\eta} C_1\, e^{-\frac{1}{4}\tilde{\eta}^2}\, d\tilde{\eta} + C_2.$

BC $\Rightarrow f(0) = 1$

$\Rightarrow C_2 = 1$

IC $\Rightarrow f(\eta) \to 0$ AS $\eta \to \infty$ (CONSISTENT WITH u-BOUNDED)

$\Rightarrow C_1 = -\frac{1}{\sqrt{\pi}}$ [CHECK]

SO $f(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int_0^{\eta} e^{-\frac{1}{4}\tilde{\eta}^2}\, d\tilde{\eta} \equiv 1 - \operatorname{erf}\left(\frac{\eta}{2}\right)$ ERROR F$^{\underline{n}}$

$\Rightarrow u(z,t) = U_0 \left[1 - \operatorname{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right]$ [SHOW OVERHEAD]

PHYSICAL PARAMS: u, z, t, ν, U_0 $\quad m=5$ $\quad \Rightarrow m-r=3$ NONDIM.

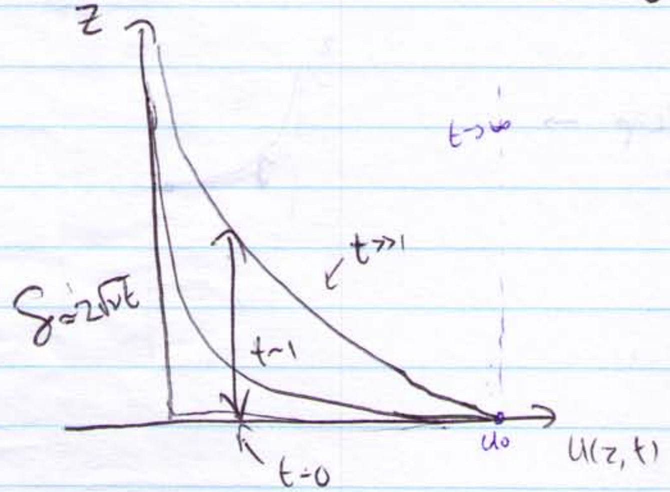
FUNDAMENTAL UNITS: $L/T \quad L \quad T \quad L^2/T \quad L/T \quad \leftarrow r=2$

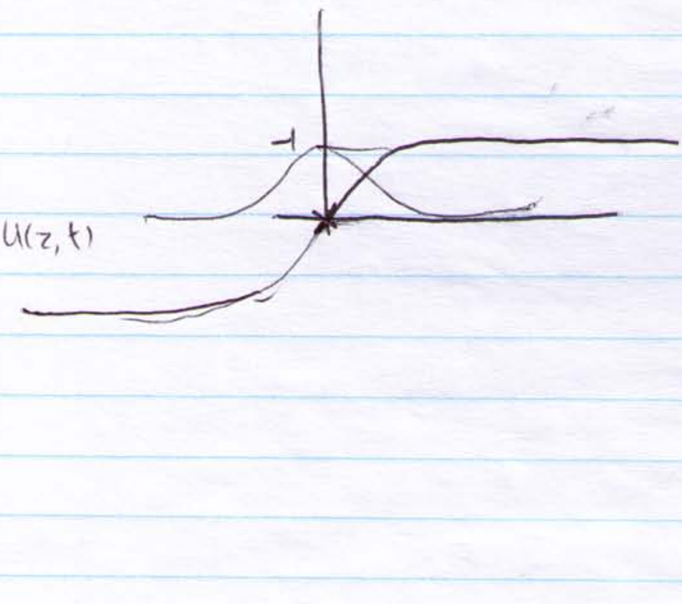
$$\Pi_1 = \frac{u}{U_0}, \quad \Pi_2 = \frac{z}{\sqrt{\nu t}}, \quad \Pi_3 = \frac{z/t}{U_0}$$

ASSUME $\Pi_3 = \frac{z/t}{U_0}$ IS PHYSICALLY UNIMPORT

$$F(\Pi_1, \Pi_2) = 0 \quad \Rightarrow \quad \Pi_1 = F_1(\Pi_2)$$

$$\Rightarrow \quad u(z,t) = U_0 F_1\left(\frac{z}{\sqrt{\nu t}}\right)$$





EXAMPLE B) $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$; $u(0,t) = A\omega \cos \omega t$, u - BOUNDED, STEADY STATE

a) PARAMETERS : u, z, t, ν, A, ω

b) NON-DIMENSIONAL PARAMETERS : $\omega t, \frac{u}{A\omega}, \frac{z}{A}, \frac{z}{\sqrt{\nu t}} \rightarrow \frac{z}{\sqrt{\nu/\omega}}$

c) FOR OSCILLATORY MOTION EVERYWHERE, EXPECT

$$u(z,t) = A\omega f\left(\frac{z}{\delta}\right) g(\omega t)$$

WHERE g IS SOME OSCILLATORY FUNCTION (IT IS CONVENIENT TO TAKE $g(\theta) = e^{i\theta}$ WHERE IT IS UNDERSTOOD THAT u IS THE REAL PART OF THE RESULT.)

AND δ IS A LENGTH SCALE INDEPENDENT OF TIME. (COULD HAVE $\delta = A$ OR $\delta = \sqrt{\nu/\omega}$. THE LATTER IS BETTER SINCE INFORMATION ABOUT THE OSCILLATING PLATE IS TRANSMITTED VIA VISCOSITY INDEPENDENT OF A)

$$\Rightarrow \boxed{u(z,t) = A\omega\, f\left(\frac{z}{\sqrt{\nu/\omega}}\right) e^{i\omega t}}$$ ← (TAKE REAL PART)

LET $\eta = \frac{z}{\sqrt{\nu/\omega}}$

$$\text{PDE} \Rightarrow i\omega\left[A\omega f(\eta) e^{i\omega t}\right] = \nu\left(\frac{1}{\sqrt{\nu/\omega}}\right)^2 f''(\eta)\, A\omega\, e^{i\omega t}$$

$$\Rightarrow \boxed{f''(\eta) = i f(\eta)}$$

B) (cont'd)

d) GENERAL SOLUTION

$$f(\eta) = C_+ e^{+\sqrt{i}\,\eta} + C_- e^{-\sqrt{i}\,\eta} \qquad C_+, C_- \text{ CONSTANTS}$$

$$= C_+ e^{\frac{1}{\sqrt{2}}(i+1)\eta} + C_- e^{-\frac{1}{\sqrt{2}}(i+1)\eta}$$

EXPECT $\lim_{z\to\infty} u(z,t)$ BOUNDED

$\Rightarrow \lim_{\eta\to\infty} f(\eta)$ BOUNDED

SO REQUIRE $C_+ = 0$

LOWER BOUNDARY CONDITION $u(0,t) = A\omega\cos\omega t = \mathrm{Re}\{A\omega e^{i\omega t}\}$

$\Rightarrow f(0) = 1$

SO $C_- = 1$

$$\Rightarrow f(\eta) = e^{-\frac{1}{\sqrt{2}}(i+1)\eta}$$

$$\Rightarrow u(z,t) = \mathrm{Re}\left\{A\omega\, e^{-\frac{1}{\sqrt{2}}(i+1)\frac{z}{\sqrt{\nu/\omega}}}\, e^{i\omega t}\right\}$$

$$\Rightarrow \boxed{u(z,t) = A\omega\, e^{-z/\sqrt{\frac{2\nu}{\omega}}} \cos\left(\omega t - \frac{z}{\sqrt{2\nu/\omega}}\right)}$$