Math Tricks

- Keep solution in exact form (*e.g.* in terms of π, e, $\sqrt{2}$, *etc.*), but find floating point expression if useful in checking whether numerical value of answer is reasonable.

  Example: volume of sphere of radius 1 cm should be written $4\pi/3\,\mathrm{cm}^3$, but one might check that this is approximately $4.2\,\mathrm{cm}^3$, a reasonable answer since it is moderately larger than the volume of a cube of side 1 cm.

- Beware not to give too many digits in solution.

  Example: If area of square is $2.0\,\mathrm{cm}^2$, then side of square is approximately 1.4 cm. 1.4142 cm is wrong! It misleads one into believing the area was measured more accurately than in fact it was.

- Do not substitute symbols with numbers until absolutely necessary. If problem is given with numerical dimensions, assign them symbols and then proceed.

  Example: If asked to find the volume of a right circular cylinder of length 10 cm and radius 3 cm, start by letting L be the length and R the radius. Solve the problem (*e.g.* by integration) to get $V = \pi R^2 L$. Only then plug in $L = 10$ and $R = 3$ to get $V = 90\pi\,\mathrm{cm}^3$.

  Working with symbols has many advantages:

  1. one can check dimensional correctness of each step of solution (see below)
  2. algebra is often simplified: $(R^2 - R)/R = R - 1 \to 17$ for $R = 18$ is straightforward whereas simplifying 306/18 is less so.
  3. One can see how result is generalised: volume of cylinder is not just 90π, but one can see it depends on the square of the radius.
  4. Confusing calculations with mixed units are avoided: $V = 30\pi R$ with $R = 3$ is technically correct in top example, but misleads one into believing the volume depends linearly upon R.

- Always check your algebra first by ensuring that all dimensions of variables summed in an equation are the same.

  Example: If U is velocity, L is length, T is time and A is acceleration, then

  $$L/U + T + U/A$$

  is a meaningful expression since each term has units of time.

  $$L + T + U$$

  is a nonsensical expression. An algebraic mistake was made in deriving it.

- A binomial expression is $(1+\epsilon)^p$, where p is any real number. If ϵ is small a Taylor series expansion can be used to approximate the binomial by a polynomial. More easily, use the binomial theorem:

  $$\begin{aligned} (1+\epsilon)^p &= 1 + \begin{pmatrix} p \\ 1 \end{pmatrix} \epsilon + \begin{pmatrix} p \\ 2 \end{pmatrix} \epsilon^2 + \ldots \\ &= 1 + p\epsilon + \frac{p(p-1)}{2}\epsilon^2 + \ldots \end{aligned}$$

  In particular, $\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2}\epsilon$.

- $\int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} \cos^2\theta + \sin^2\theta d\theta = \frac{1}{2}\int_0^{2\pi} 1 d\theta = \pi$

- MATHEMATICAL INDUCTION:

  Example: show $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

  Solution:

  Step 1: Show true for $n = 1$

  Step 2: Assume true for $n = N-1$

  Step 3: Then show true for $n = N$. *E.g.*

  $$\begin{aligned} \sum_{i=1}^N i &= \left[\sum_{i=1}^{N-1} i\right] + N \\ &= \left[\frac{N(N-1)}{2}\right] + N \\ &= \left[\frac{N(N+1)}{2}\right] \end{aligned}$$

- Find $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

  Solution:

  Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Then

  $$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy \\ &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,d\theta\,dr \\ &= 2\pi\int_0^{\infty} e^{-r^2} r\,dr \\ &= \pi \end{aligned}$$

  So $I = \sqrt{\pi}$.

- Integrating by differentiating through the integral sign.

  Example: Find $I(a) = \int_0^\infty e^{-x^2}\cos(ax)\,dx$.

  Solution:

  Note $I(0) = \sqrt{\pi}/2$.

  $$\begin{aligned} I'(a) &= \int_0^\infty -xe^{-x^2}\sin(ax)\,dx. \\ &= -\int_0^\infty \frac{a}{2}e^{-x^2}\cos(ax)\,dx. \\ &= -\frac{a}{2}I(a) \end{aligned}$$

  So $I' = -bI/2$ with $I(0) = \sqrt{\pi}/2$ is a (separable) initial value problem with solution $I(b) = \frac{\sqrt{\pi}}{2}e^{-b^2/4}$.

- Application of eigenvalues and eigenvectors.

  The equations describing the motion of (small amplitude) waves can be written as an "eigenvalue problem". The eigenvalues give the dispersion relation of the waves (relating the frequency to the wavelength), and the eigenfunctions relate velocity, displacement, and other fields to each other (are they in phase? out of phase?).